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HETEROSTATIC LOADING AND CRITICAL ASTATIC LOADS

A Generalization of Southwell's Method for the Analysis of Experimental Observations in Problems of Elastic Instability

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ABSTRACT

Southwell has shown how, in some cases, it is possible to compute the critical astatic load, that is, the elastic buckling load, of a structure from measurements of its heterostatic deflections at lower loads. The history of the theory of heterostatic loading and Southwell's method is briefly reviewed. Westergaard's general theory is then applied to the problem. It is shown that Southwell's method and Lundquist's modification of it are theoretically accurate for results of measurements which are proportional to the value of any one astatic parameter. These measurements need not be deflection measurements but may be strain measurements or, theoretically, measurements of any effect linearly dependent upon the deformation. Further, the parameter need not be the parameter corresponding to the lowest critical load but, theoretically, may correspond to any higher critical load.

Southwell's method is thus useful in cases where measurements within the elastic range can be made to depend primarily upon the change of a single astatic parameter. The theory is valid only for cases in which the buckling loads are lower than the load at which appreciable plastic deformation of the material or appreciable deviation from Hooke's law would occur. If, even within the elastic range, the measurements are affected appreciably by changes in other parameters, the critical elastic load computed by Southwell's method or Lundquist's modification may still be considerably in error. A combined numerical and graphical method of computation is outlined which by successive approximations, gives more accurate results in such cases. Finally, experimental results are given in which the second and third critical loads of a "round-end" Euler column are computed from strain-gage measurements taken at loads below the first critical load.

CONTENTS

I. INTRODUCTION

1. IDEAL CASES OF ELASTIC INSTABILITY

In theoretically ideal cases of elastic instability, structures are considered whose individual elements obey Hooke's law and which, under the action of a system of two or more proportional loads in equilibrium, called, as a system, the "load P," initially deform in a mode determined by the distribution of the individual components of P and are in stable equilibrium up to a certain critical load, Q, or, more generally, a discrete series of critical loads, $Q_1, Q_2, \ldots Q_h, \ldots; Q_{h+1} > Q_h$. Under the action of any one of these critical loads, the equilibrium, although remaining stable with respect to further deformation in the initial mode, becomes neutral with respect to some different (buckling) mode of deformation.

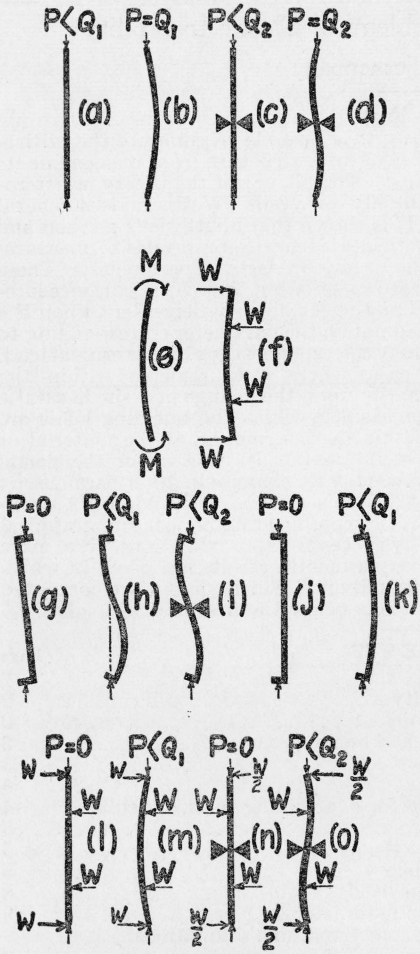

FIGURE 1.—*Examples of astatic loading, a, b, c, d; orthostatic loading, e, f; and heterostatic loading, g, h, i, j, k, l, m, n, o.*

The classical example of this type of behavior is the ideal straight "round-end" Euler[1] column. See figure 1. Under an exactly axial load its initial mode of deformation is pure axial compression proportional to the load, but under any one of a series of critical loads it is in neutral equilibrium in another mode of deformation, a lateral deflection in the shape of a single or multiple-lobed sine wave.

A comprehensive summary of the theory of these ideal cases with important additions was given by Southwell.[2]

2. PRACTICAL CASES OF MIXED ACTIONS

In actual structures these ideal cases are never attained, but, in general, buckling deformations in modes corresponding to the different Q_h are present even under small loads. They increase at first slowly under increasing load, but the mode corresponding to Q_1 increases with greatly increasing rapidity as Q_1 is approached. If the mode of buckling deformation corresponding to this load is prevented by outside constraints (fig. 1), the mode corresponding to Q_2 will increase more and more rapidly as that

[1] Leonhard Euler, Methodus inveniendi Lineas Curvas Maximi Minimive proprietate gaudentes, Bousquet, Lausanne (1744).

[2] R. V. Southwell, *On the general theory of elastic stability*, Phil. Trans. Roy. Soc. (London), **213** [A], 187-244 (1913).

load is approached, etc. The theory of these mixed actions of structures has been studied in much detail by many writers in special cases, more especially in the case of columns of inhomogeneous material, initially deformed, eccentrically loaded, and/or subjected to combined transverse and axial loads. These theoretical developments, even in this case, are practically all confined to purely elastic action and are not valid for loads at which appreciable plastic deformation occurs.

3. WESTERGAARD'S GENERAL THEORY AND NOMENCLATURE

Finally, Westergaard [3] gave a very general theory of buckling which covers all cases in which, although Hooke's law applies to the individual elements of the structure and the deformations are small enough so that second-order terms may be neglected, the deformations and stresses are, in general, not proportional to the load. These assumptions cover the elastic behavior of practically all the usual cases of structural instability, but they also are not valid for loads at which plastic deformation occurs.

Westergaard introduced a convenient nomenclature. He defined as "orthostatic" (fig. 1), quantities and actions which are proportional to the applied load, such as occur in a beam bent under transverse loads; as "astatic" (fig. 1), quantities and actions in the theoretically ideal cases of elastic instability, such as the ideal Euler column; and as "heterostatic" (fig. 1), combinations of astatic and orthostatic quantities and actions.

4. SOUTHWELL'S METHOD OF ANALYSIS

Unavoidable heterostatic action under practical test conditions has been one of the major difficulties in the experiments on elastic instability. In an interesting paper, Ayrton and Perry [4] in 1886 showed how it was theoretically possible to calculate the Euler load approximately from the lateral deflections under lower loads, of an initially bent inhomogeneous and/or eccentrically loaded column, and gave a graphic method for determining it. Ayrton and Perry were primarily interested in verifying their theory of bent or eccentrically loaded columns and only incidentally noted this possibility.

In 1932 Southwell [5] noted that within the elastic range before plastic deformation occurred the load, P, deflection, δ, curve of an initially bent or eccentrically loaded column approximated a rectangular hyperbola passing through the origin and asymptotic to the line $P=Q_1$. If this were rigorously so, plotting δ, the lateral deflection, measured from the position of zero load as ordinate and δ/P as abscissa would give a straight line whose slope was equal to Q_1. Examination of the data published by Von Kármán [6] and the unpublished data of A. Robertson showed that the slopes of lines so plotted were, in a number of the tests, in excellent agreement with the measured critical load.

Southwell gave a detailed discussion of the theory of this method of plotting in the case of the initially bent centrally loaded round-end column.

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[3] H. M. Westergaard, *Buckling of elastic structures*, Trans. Am. Soc. Civil Engrs. **85**, 576-654 (1922). This contains a comprehensive bibliography.

[4] W. E. Ayrton and John Perry, *On struts*, Engineer **62**, 464-465, 513-515 (1886).

[5] R. V. Southwell, *On the analysis of experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London) **135** [A] 601-616 (1932).

[6] Th. von Kármán, Untersuchungen über Knickfestigkeit, Forsch. Gebiete Ingenieurw. Heft 81 (1910).

Gough and Cox [7] applied this method of plotting to the shear buckling of thin strip, finding good agreement between theory and experiment.

Fisher [8] extended Southwell's reasoning to the case of a spar under combined axial and transverse loading, finding agreement between theory and experiment.

Recently, Ramberg, McPherson, and Levy [9] have applied the method with some success to the study of sheet-stringer combinations.

Donnell [10] has recently presented a variation of Southwell's method by plotting the load against the load divided by the deflection. The critical load then appears as the intercept on the P-axis. He also discusses the special cases of elastically supported struts, plates supported on three or four sides, cylinders under axial compression, and struts in the plastic range, finding that Southwell's method can be applied to all these cases in which the buckling does not introduce appreciable second-order stresses.

Hill [11] has applied the method to measurements of the lateral deflection of aluminum-alloy columns tested with "flat ends" and to the buckling of the stems of T-sections similarly tested. He found good agreement with the theory up to loads at which plastic yielding becomes appreciable.

5. DIFFICULTIES IN APPLICATION

In the application of this method it has been found that the lower portion of the (δ), (δ/P) graph was frequently curved and irregular, even when the upper portion was reasonably straight, and further, that straighter graphs could frequently be obtained by plotting $(\delta-\delta_0)$ as ordinate and $(\delta-\delta_0)/P$ as abscissa, where δ_0 represented a zero-point "correction" so chosen as to make the upper portion of the graph approximate most closely to a straight line. Unavoidable irregularities of deflection and strain readings under low loads is a well known difficulty in structural testing, ascribable partly to the practical impossibility of securely seating sensitive measuring instruments until they have been "worked in," and partly to inelastic settling of the structure itself, such as by slipping of rivets, under low loads.

Southwell [12] pointed out that the zero-point correction in an initially bent column would necessarily be present unless the initial bend was exactly a sine wave corresponding to the lateral deflection under the first Euler load, and generalized the statement to apply to other types of instability for which analogous equations applied.

6. METHODS PREVIOUSLY SUGGESTED FOR OVERCOMING THESE DIFFICULTIES

Various methods have been used to determine the zero-point correction and free the method from the uncertainties arising from initial irregularities. Southwell [13] suggested making a series of plots with

[7] H. J. Gough and H. L. Cox, *Some tests on the stability of thin strip material under shearing forces in the plane of the strip*, Proc. Roy. Soc. (London) **137** [A] 145–157 (1932).

[8] H. R. Fisher, *An extension of Southwell's method of analysing experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London) **144** [A] 609–630 (1934).

[9] W. Ramberg, A. E. McPherson, and S. Levy, Experimental Study of Deformation and Effective Width in Axially Loaded Sheet-Stringer Panels, NACA Tech. Note 684, (January 1939.)

[10] L. H. Donnell, On the Application of Southwell's Method for the Analysis of Buckling Tests, Contributions to the Mechanics of Solids—Stephen Timoshenko 60th Anniversary Volume, p. 27–38 (Macmillan Co., New York, N. Y., 1938).

[11] H. N. Hill, Note on the Analytical Treatment of Lateral Deflection Measurements from Tests Involving Stability Problems (Aluminum Company of America) P. T. Report 38–71.

[12] R. V. Southwell, *On the analysis of experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London) **135** [A] 601–616 (1932).

[13] See footnote 12.

different zero-point corrections and choosing the one which gave the straightest line in its upper portion. Gough and Cox [14] used the method of least squares and Fisher [15] a graphic method based on Pascal's hexagram theorem.

A simpler method of eliminating the effect of irregularities under low load has been given by Lundquist.[16] This consists in plotting $(\delta-\delta')$ as ordinate and $(\delta-\delta'),(P-P')$ as abscissa, where δ' is the deflection under some definite initial load P'/which is chosen somewhere in the middle of the observations where the P, δ curve is smooth.

None of these methods can be expected to give straight-line graphs for high loads at which plastic yielding has become appreciable, and for which the theory is no longer applicable.

7. LIMITATIONS OF PREVIOUS WORK

In all of this work following Southwell's 1932 paper,[17] except the very recent paper by Donnell, the detailed theoretical development has been limited to the theory of columns. The extension to other types of instability is only indicated by Southwell [18] by the statement:

"In all ordinary examples of elastic instability, an equation of the same form as (7)[19] governs the deflection as controlled by its initial value, provided that both are small. Corresponding with Euler's theory of the initially straight strut we have a series of 'critical loadings', each associated with a particular 'normal' type of displacement, and by expressing both the initial and final displacement in a series of normal components, we can show that the relation (11)[20] will hold between the original and final amplitudes of the n'th normal component." Donnell, as noted above, extends the theory of the method to other special cases.

In none of the papers we have seen has it been noted that the theory of this type of plotting, and the range and limitation of its applicability, can simply and readily be deduced in very general form from Westergaard's[21] general theory.

II. APPLICATION OF WESTERGAARD'S GENERAL THEORY

1. SUMMARY STATEMENT OF THE THEORY

Westergaard shows that if Hooke's law applies to the elements of a structure under a certain type of astatic loading, P, whose critical values are Q_1, Q_2, . . . , Q_h, . . . ; $Q_{h+1}>Q_h$, then there can be defined a corresponding series of astatic parameters, u_{hj}, that is u_{11}, u_{12}, . . . u_{1j}, . . . u_{1n_1}, corresponding to Q_1; u_{21}, u_{22}, . . . u_{2j}, . . . u_{2n_2}, corresponding to Q_2, etc., an orthostatic parameter, v, and independent parameters, $t_1, t_2, \ldots t_k, \ldots t_m$.

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[14] H. J. Gough and H. L. Cox, *Some tests on the stability of thin strip material under shearing forces in the plane of the strip*, Proc. Roy. Soc. (London) **137** [A] 145–157 (1932).

[15] H. R. Fisher, *An extension of Southwell's method of analysing experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London) **144** [A], 609–630 (1934).

[16] Eugene E. Lundquist, Generalized Analysis of Experimental Observations in Problems of Elastic Stability, NACA Note 658 (July 1938).

[17] R. V. Southwell, *On the analysis of experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London) **135** [A] 601–616 (1932).

[18] See footnote 17.

[19] $EI(y''-y''_0)+Py=0.$

[20] $\frac{\overline{\overline{w}}_n}{\overline{w}_n}=\frac{1.}{1-\frac{P}{P_n}}$

[21] H. M. Westergaard, *Buckling of elastic structures*, Trans. Am. Soc. Civil Engrs. **85**, 576–654 (1922).

These parameters are "generalized coordinates" in the Lagrangian sense, which define the displacements of the structure. In an axially symmetric Euler column, for example, the u_{11} and u_{12} might be chosen as the two components obtained by resolving the deflection at the midpoint in two mutually perpendicular directions, when the column is bent in a single-lobed sine wave, the type of deflection which corresponds to the lowest critical load. u_{21} and u_{22} might be chosen as the components of the deflection at the quarter points when the column is bent in a two-lobed sine wave, the type of deflection which corresponds to the second critical load, etc. In general, the u_{h1} and u_{h2} might be chosen as the coefficients of the Fourier series expansion of the lateral deflection in two orthogonal planes, the v as the axial shortening of the column and the t_k as coordinates determining inter-

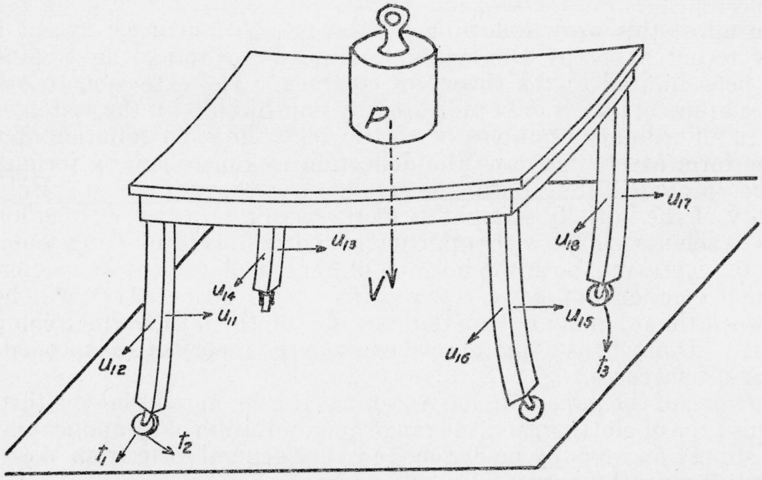

FIGURE 2.—*Types of parameters.*

Eight astatic parameters, u_{hj}, correspond to each critical load, Q_h; one orthostatic parameter, v, corresponds to deflection of the top of the table; and three independent parameters, t_k, correspond to motion of the table as a body.

nal strains or motions of the column as a whole, which involve no deformation.

It is only exceptionally that more than one astatic parameter corresponds to any one critical astatic load. If the Euler column were not axially symmetric and the two principal moments of inertia bore an irrational ratio to each other, there would be two series of critical loads, of which no two could be equal and each would have only a single corresponding astatic parameter. However, it is easy to devise structures in which many astatic parameters would be necessary. For instance, a table (fig. 2) with equal axially symmetric legs centrally loaded would require eight astatic parameters to define the possible magnitudes and directions of the sine-wave components of the deflection of its four legs corresponding to each of its critical loads.

When these parameters are sufficient to define all the significant small displacements and deformations of the structure under some type of heterostatic loading composed of the astatic loading, P, and an orthostatic loading, W, Westergaard proves that the value of each of these parameters is given by

$$\bar{u}_{hj}=u_{hj}+\frac{P}{Q_h-P}u_{hj}=\frac{Q_h}{Q_h-P}u_{hj}, \tag{1}$$

where $\bar{u}_{hj}$ is the value of the parameter under heterostatic loading and u_{hj} is its value under the orthostatic loading, W, acting alone. This is of the same form as Ayrton and Perry's [22] eq 15 and Southwell's [23] eq 11, but refers to the astatic parameters instead of the deflections, and is proved to be applicable under much more general conditions.

Westergaard further shows that any effect, F, such as a stress, strain, deflection, curvature, etc., depending upon the deformation of the structure under these conditions, which under orthostatic loading, can be expressed in the form

$$F=F_0(t_1, \ldots t_\kappa \ldots t_m)+\sum_{h=1}^{h=\infty}\sum_{j=1}^{j=n_h}f_{hj}u_{hj}, \tag{2}$$

under heterostatic loading is subject to an equation of the form

$$\overline{F}=F_0+f_p pP+\sum_{h=1}^{h=\infty}\frac{Q_h}{Q_h-P}\sum_{j=1}^{j=n_h}f_{hj}u_{hj}, \tag{3}$$

where $\overline{F}$ is the value of the effect under the heterostatic loading and F_0 is its value under the action of the independent parameters alone. The f_{hj}, f_p, and p are constants independent of both the orthostatic loading, W, and the astatic loading, P, but, in general, are functions of the location in the structure, determined by the nature of the effect, the geometry of the structure, and the type of astatic loading, P. The constant, p, is so chosen that the orthostatic parameter $v=pP$.

If F, for example, represented the strain on one side at the quarter point of an Euler column, the parameter, u_{11}, corresponding to the single-lobed sine-wave deflection characteristic of the lowest critical load would enter the summation with a coefficient whose relative value $f_{11}=\sin 45°=1/\sqrt{2}$, while the parameter u_{21} corresponding to the double-lobed deflection characteristic of the second critical load would enter the summation with a coefficient whose relative value $f_{21}=\sin 90°=1$. If the deflections were measured at the sixth points the corresponding coefficients would have the relative values, $f_{11}=\sin 30°=1/2$ and $f_{21}=\sin 60°=\sqrt{3}/2$, etc., f_p would determine the average axial strain under the load, P, and be inversely proportional to its length, while F_0 would represent any internal strains present in the column under no load.

Southwell's eq 9[24] is a special case of this general expression.

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[22] W. E. Ayrton and John Perry, *On struts*, Engineer **62**, 464–465, 513–515 (1886).

[23] R. V. Southwell, *On the analysis of experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London), **135** [A], 601–616 (1932).

[24] R. V. South well, *On the analysis of experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London), **135** [A] 601–616 (1932).

$$y=\sum_{n=1}^{\infty}\left[w_n \sin\frac{n\pi x}{2}\right]$$

For these equations to hold, all of the components of the astatic loading, P, must change proportionally to each other, but the orthostatic loading, W, may have components which are independent. These equations could be applied to any case of loading of the structure, but their practical value lies largely in cases in which the orthostatic loading consists of a constant component and a component proportional to the astatic loading, P. That would be the case, for example, in a girder under a combination of a constant transverse load and an eccentrically applied axial load, as in Fisher's [25] experiments. The u_{hj} are linear functions of the linearly independent components of W. In the ideal case in which the loading, P, is exactly astatic, the u_{hj} vanish with W, but in the practical case, there will be initial eccentricities of shape and loading, which may be treated as constant additive terms in the u_{hj}.

It is desirable to emphasize the implications of the assumptions upon which these equations are based.

The equations are strictly applicable only to materials which obey Hooke's law. Experience has shown, however, that they can be usefully applied to materials such as some alloy steels or high-strength aluminum alloys whose stress-strain graphs depart appreciably from a straight line, but they cannot be expected to give more than rough approximations if the stresses are so high that plastic yielding of the material becomes comparable to the elastic deformation.

The equations are limited to small deflections. If the deflections should become too large, an effect, F, might cease to be even an approximately linear function of the parameters, and eq 3 would no longer be adequate to represent it. Further, the equations are valid only if the astatic parameters selected are sufficient to define all the significant small displacements and deformations of the structure under the type of heterostatic loading considered. For example, the coefficients of the Fourier series expansion of the lateral deflections give astatic parameters adequate to describe the significant deformations of a sturdy column. They are, however, inadequate to describe the significant deformations of a column which may fail by twisting or by local buckling of its web or outstanding flanges. For such columns, additional astatic parameters are needed to define all the significant small deformations.

2. MATHEMATICAL TRANSFORMATIONS USED

In the further development, repeated use will be made of certain transformations which are a generalization of those used by Southwell and Lundquist. If

$$z=\frac{Q(a+bP)}{Q-P}-f(P)=\frac{Q(a+bQ)}{Q-P}-bQ-f(P), \tag{4}$$

where z and P are variables, $f(P)$ is any arbitrary function of P alone, and Q, a, and b are constants, let

$$w=z+bQ+f(P). \tag{5}$$

Then

$$w=\frac{Q(a+bQ)}{Q-P}. \tag{6}$$

[25] H. R. Fisher, An *extension of Southwell's method of analysing experimental observations in problems of lasticstability*, Proc. Roy. Soc. (London) **144** [A] 609–630 (1934).

Designating by primes quantities corresponding to a particular constant value of P, P';

$$w'=\frac{Q(a+bQ)}{Q-P'} \tag{7}$$

from which,

$$\frac{w}{Q-P'}=\frac{w'}{Q-P}=\frac{w-w'}{P-P'} \tag{8}$$

or,

$$(w-w')=(Q-P')\frac{w-w'}{P-P'}-w'. \tag{9}$$

By substituting values from eq 5 for w and w',

$$(z-z')+f(P)-f(P')=(Q-P')\frac{z-z'+f(P)-f(P')}{P-P'}-w' \tag{10}$$

3. APPLICATION TO A SINGLE ASTATIC PARAMETER

Under the condition of an orthostatic loading consisting of a constant component and a component proportional to the astatic loading, P, arising from initial eccentricity either of shape or elastic inhomogeneity of the material in a structure, eq 1 will become

$$\bar{u}_{hj}=\frac{Q_h}{Q_h-P}(u_{hj}+p_{hj}P), \tag{11}$$

where u_{hj} is the value of the astatic parameter under the constant component of the orthostatic loading acting alone and p_{hj} is a constant determining the contribution of the orthostatic component of P to the value of the corresponding astatic parameter. This is of the same form as eq 4 with $f(P)=0$ and may be written

$$\bar{u}_{hj}+p_{hj}Q_h=\frac{Q_h}{Q_h-P}(u_{hj}+p_{hj}Q_h) \tag{12}$$

This is in the same form as eq 1, except for the displacement of the zero of the u_{hj} by a constant amount $p_{hj}Q_h$. Equation 10 then gives

$$\bar{u}_{hj}-\bar{u}'_{hj}=(Q_h-P')\frac{\bar{u}_{hj}-\bar{u}'_{nj}}{P-P'}-(\bar{u}'_{hj}+p_{hj}Q_{hj}). \tag{13}$$

If then, under these general conditions, any measurement or measurements can be taken on the structure which will determine a quantity proportional to any one of the astatic parameters, $\bar{u}_{hj}$, for varying astatic loads, P, and if the differences $\bar{u}_{hj}-\bar{u}'_{hj}$ between its value for the varying loads and its value $\bar{u}'_{hj}$ for a fixed load P', be plotted as ordinates with the ratios $(\bar{u}_{hj}-\bar{u}'_{hj})/(P-P')$ as abscissas, the resulting plot should be a straight line whose slope is (Q_h-P').

4. COMPARISON WITH SOUTHWELL'S AND LUNDQUIST'S EQUATIONS

Equation 13 is of the same form as Lundquist's [26] eq 15, and with P' chosen as zero, it is of the same form as Southwell's [27] eq 17.

[26] Eugene E. Lundquist, Generalized Analysis of Experimental Observations in Problems of Elastic Stability, NACA Tech. Note 658 (July 1938).
[27] R. V. Southwell, *On the analysis of experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London) **135** [*A*], 601-616 (1932).

Plotting deflection against deflection divided by load gives curves which approximate straight lines because, as Southwell pointed out, when the load approaches the first critical load, Q_1, the modes of buckling corresponding to that load and represented by the u_{1j}, are large in comparison with the modes of buckling corresponding to higher loads. Accordingly, the deflections become approximately proportional to the u_{1j}.

5. APPLICATION TO THE GENERAL CASE

In practical testing it is impossible to so locate deflection-and/or strain-measuring apparatus that quantities proportional to any one astatic parameter can be determined from their readings. Only such quantities would necessarily be subject to an equation of the form of eq 13. The measurements can only determine some more general heterostatic effect, $\overline{F}$, subject to an equation of the form of eq 3, in which each of the u_{hj} is replaced by a linear function of P, as in eq 11. Each of the terms of the double summation can be transformed, as in eq 12. Equation 3 will then have the form

$$\overline{F}=r+sP+\sum_{h=1}^{h=\infty}\frac{q_h}{Q_h-P}, \tag{14}$$

where r, s, and the q_h are constants. Their exact value, which could be determined from eq 3, 11, and 12, is of no significance in this connection. Practical interest in this equation is largely confined to cases in which one term of the summation is large in comparison with the others. This may occur in two ways. It may be possible to arrange the measurements so that one of the coefficients, q_h, is large in comparison with any of the others. More frequently one of the critical loads, Q_1, is much lower than any of the others. In such cases when the load approaches Q_1, even though q_1 is not large in comparison with the other q_h, the term $q_1/(Q_1-P)$ may become large in comparison with any other $q_h/(Q_h-P)$. That, however, is not necessarily the case, as may be illustrated by the observations of Gough and Cox. Only when their deflection measurements were taken accurately on the crest of the buckle were they able to get satisfactory results with Southwell's method. In either of these cases the largest variable term on the right-hand side of eq 3 containing the factor $q_h/(Q_h-P)$ corresponding to the critical load, Q_h (usually Q_1), may be selected for special consideration and all the other terms treated as a single function of P, $f(P)$. Equation 14 will then be of the same form as eq 4 with $\overline{F}$ substituted for z, so that an equation in $\overline{F}$ of the same form as eq 10 may immediately be written down. In this case

$$-f(P)=r+sP+\sum_{k\neq h}\frac{q_k}{Q_k-P} \tag{15}$$

and

$$f(P)-f(P')=-s(P-P')-\sum_{k\neq h}\frac{q_k}{Q_k-P'}\cdot\frac{P-P'}{Q_k-P}, \tag{16}$$

so that eq 10 gives

$$\overline{F}-\overline{F}'+[w'+Q_h s]-sP+\sum_{k\neq h}\frac{q_k}{Q_k-P'}\cdot\frac{Q_h-P}{Q_k-P}=(Q_h-P')\frac{\overline{F}-\overline{F}'}{P-P'}. \tag{17}$$

6. APPROXIMATION BY EXPANSION IN A POWER SERIES

If all the (Q_k-P') are greater in absolute magnitude than $(P-P')$, the summation can be expanded in a convergent power series in $(P-P')$ with positive integral exponents, so that

$$\overline{F}-\overline{F}'-\sum_{k=0}^{k=\infty} c_k(P-P')^k=(Q_h-P')\frac{\overline{F}-\overline{F}'}{P-P'}, \tag{18}$$

where the c_k are constants. Their exact value which could be determined from eq 17 is of no significance in this connection. This equation will be practically useful in the analysis of experimental data only in the case where the series is very rapidly convergent.

7. USE OF THIS APPROXIMATION IN COMPUTATION

Computations based upon Euler's column indicate that in some experimental cases it may be worth-while to take into account second-order terms in P in eq 18. To avoid cumbersome notation it is convenient to change the notation, and to write

$$\begin{aligned} z&=\overline{F}-\overline{F}' \\ Q&=Q_h-P' \end{aligned} \tag{19}$$

and to replace $P-P'$ by P. Equation 18 then becomes

$$z+c_0-Q\frac{z}{P}-c_1P-c_2P^2=0, \tag{20}$$

where z represents the difference of the measured value of some effect such as a displacement, a strain, or some linear combination of displacements and/or strains produced by the change in load, P, from the arbitrary fixed load, P', and Q represents the difference between the particular critical load, Q_h, selected for special treatment and the same fixed load, P'. z/P should be calculated to *two significant figures beyond* [28] the number of significant figures observed in z. When c_1 and c_2 are small, plotting z as ordinate and z/P as abscissa will give a curve which approximates a straight line. The slope of the curve for highest values of the load within the range for which Hooke's law is approximately valid should be fairly close to Q. To determine Q more closely, draw a straight line, $\bar{z}=\overline{A}+\overline{B}(z/P)$, approximating closely the upper portion of the z, (z/P) curve, choosing the nearest convenient round numbers for $\overline{A}$ and $\overline{B}$. For each observed value of z/P calculate the value of $\bar{z}$ to *two significant figures beyond* the observed significant figures in z. Plot the difference $z-\bar{z}$ as ordinates to a scale so large that the estimated error of a single reading is at least 1 mm, with z/P as abscissas. This will make it easier to estimate changes in curvature and to distinguish between average trend and accidental errors. Choose some convenient round number as a trial value of c_1, and plot $z-\bar{z}-c_1P$ as ordinates with z/P as abscissas. Compare the resulting curve with the $z-\bar{z}$, (z/P) curve. The resulting—curve may be:

1. More curved.
2. Curved in the opposite direction.

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[28] Carrying the intermediate computations to *two significant figures beyond* the determinate significant figures is important. If it is not done there may be a cumulation of computational error which will materially affect the accuracy of the results.

3. Less curved in the same direction.
4. Have an inflection.
5. Practically straight for lower values of P and curved for larger values.
6. Straight within the limits of experimental error.

In case 1 change the sign of c_1 and proceed as in 2, 3, 4, or 5.

In case 2 repeat with smaller trial values of c_1, until case 3, 4, or 5 is observed.

In cases 3 and 4 repeat with larger trial values of c_1 until the portion of the curve corresponding to the lower values of P is straight within the limits of experimental error. The last trial value should then be a good approximation to the actual value of c_1. This gives either case 5 or case 6. If it gives case 6, it shows that the term c_2P^2 is negligible and the computation is completed.

In case 5 choose some convenient round number as a trial value of c_2 and plot $z\text{-}\bar{z}\text{-}c_1P\text{-}c_2P^2$ as ordinates with z/P as abscissas, and compare with the $z\text{-}\bar{z}\text{-}c_1P$, (z/P) curve. The resulting curve may be:

1. More curved.
2. Curved in the opposite direction.
3. Less curved in the same direction.
4. Have an inflection.
5. Practically straight within the experimental error.

In case 1, change the sign of c_2 and proceed as in 2, 3, 4, or 5.

In case 2 repeat with smaller trial values of c_2 until case 3, 4, or 5 is observed.

In case 3 repeat with larger trial values of c_2 until case 4 or 5 is observed.

In case 4 start again the sequence of corrections of the first order leading to a new approximation to c_1, and then repeat the sequence of corrections of the second order leading to better approximation to c_2.

In case 5 the computation is completed.

These successive approximations, although complicated in their expression, can, provided data are consistent with eq 20, be made easily and quickly once the method is learned.

8. POSSIBLE ACCURACY ATTAINABLE

To gain some idea of the computational error involved in this type of approximation from data involving appreciable amounts of more than one term of the form $q_k/(Q_k-P)$ of eq15, strains corresponding to different assumed combinations of different types of eccentricity and lack of straightness of an Euler column were computed. The combinations were so chosen that, at the highest loads assumed the contributions other than the major term amounted to 10 percent of the total computed strain. To these theoretically computed strains "accidental errors" taken at random from Shewhart's [29] "drawings from a normal universe," and of the magnitude found in good strain-gage readings were added.

From these synthetic "load-strain" data the critical load was computed by the method outlined. In all cases the critical load so computed agreed with the assumed critical load within less than 2 percent.

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[29] Walter A. Shewhart, Economic Control of Quality, p. 442-445 (D. Van Nostrand Co., Inc., 250 Fourth Avenue, New York, N. Y. (1931)).

9. DISCUSSION OF LIMITATIONS OF THE METHOD

As Southwell [30] has pointed out, it is, at present, impossible in most practical experiments to predict in advance how closely critical astatic loads can be determined from measurements of heterostatic action under lower loads. Only in cases where one term $q_h/(Q_h-P)$ of eq 14 is or becomes large in comparison with the other terms of the summation for loads below those at which the material shows appreciable plastic yielding will the $(\bar{F}-\bar{F}')$, $(\bar{F}-\bar{F}')/(P-P')$, graph be found to be reasonably straight for any considerable range. Only in such cases can any accuracy at all be expected.

The relative magnitude of the different terms of the form $q_h/(Q_h-P)$ will, in general, depend upon small irregularities of shape and material of the structure and small inaccuracies of loading which, in many cases, are difficult, if not impossible, to control. In practically all cases of simple deflection or strain measurements the term corresponding to the lowest critical load will predominate as that load is approached, but this may not occur before plastic yielding has become appreciable.

However, a single term will not necessarily ever become predominant, and, in particular, it is not likely to do so in a structure in which the two lowest critical loads corresponding to two different modes of instability are nearly equal. This may have been the case in one of the experiments reported by Ramberg, McPherson, and Levy.[31] Whether another type of analysis might be successful in such cases has not been investigated. In some relatively simple cases, however, it is possible to alter the conditions of loading and/or the type of measurement so as to ensure that even a term corresponding to a higher critical load will predominate below loads at which plastic yielding becomes appreciable.

Even if the measurements are such that one term predominates sufficiently to enable a close determination of the corresponding theoretical critical astatic load, the stresses corresponding to that load may exceed and even far exceed the stresses at which the material yields plastically as is the case in short and medium length sturdy columns.

In such cases the calculated critical loads will only give an upper limit to the strength of the structure. In spite of these limitations the method can be expected in many cases to give valuable information concerning the stability of structures, but the determination whether it will give valuable information in the case of any given structure under a given type of loading can, for the present at least, only be determined by trial.

III. EXPERIMENTAL DATA VERIFYING THE THEORY

1. DETERMINATION OF LOWEST CRITICAL LOAD

Experimental data sufficient to show the value of this method of analysis in determining the lowest critical astatic load of some structures, have, as noted above, been presented by Southwell; Gough and Cox; Fisher; Ramberg, McPherson, and Levy; and Lundquist. Fur-

[30] R. V. Southwell, *On the analysis of experimental observations in problems of elastic stability*, Proc. Roy. Soc. (London) **135** [A] 601-616 (1932).

[31] W. Ramberg, A. E. McPherson, and S. Levy, Experimental Study of Deformation and of Effective Width in Axially Loaded Sheet Stringer Panels, NACA Tech. Note. 684. (Jan. 1939.)

ther data of the same kind will be of interest only in cases where the critical loads of the structures themselves are of interest.

2. THEORETICAL POSSIBILITY OF DETERMINING HIGHER CRITICAL LOADS

The general theory here presented, however, indicates that the method is not necessarily limited to the determination of the lowest critical load, but may, in suitable cases, be used to determine higher critical loads.[32] It was thought worth-while to check this experimentally. To do this it is necessary to arrange the loading conditions and measurements of a structure, so that an effect $\overline{F}$ of the loading may be determined in which the q_k, $k<h$ of eq14 will be very small and q_h relatively large.

3. EXPERIMENTAL ARRANGEMENT FOR SECOND CRITICAL LOAD

Two arrangements of this kind were set up. In the first a commercial straight piece of cold-rolled steel, ¼ by ½ by 13.5 inches (see fig. 3), was loaded as a "round end" column with approximately equal and opposite eccentricities at either end secured by bevelling the ends to form eccentric knife-edges. This introduced relatively large orthostatic components of the load corresponding to all the critical loads of even order but principally of the second order. Three pairs of 2-inch Tuckerman optical strain gages were placed on opposite sides of the specimen, one pair at the midpoint and one at each of the quarter points (see fig. 3). The difference between the readings of each pair of gages is proportional to the average bending strain over the gage length. It is easy to see that changes in the astatic parameters of even order will contribute but little to the bending strain at the midpoint and also to the sum of the bending strains at the quarter points. These will, provided the corresponding term of $\overline{F}$ is sufficiently predominant, depend chiefly upon the variation of the astatic parameter corresponding to the lowest critical load. Application of the analysis to either of these should therefore be expected to determine the first critical load. Further, changes in the astatic parameters of odd order will contribute but little to the difference of the bending strains at the quarter points. This difference will depend chiefly upon the astatic parameter corresponding to the second critical

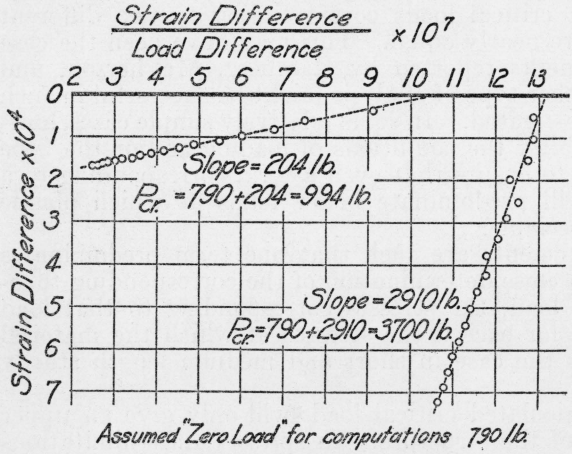

FIGURE 4.—*Southwell analysis of eccentrically loaded column*

[32] Donnell's recent paper also notes this possibility in the case of columns. (See ref. 10.)

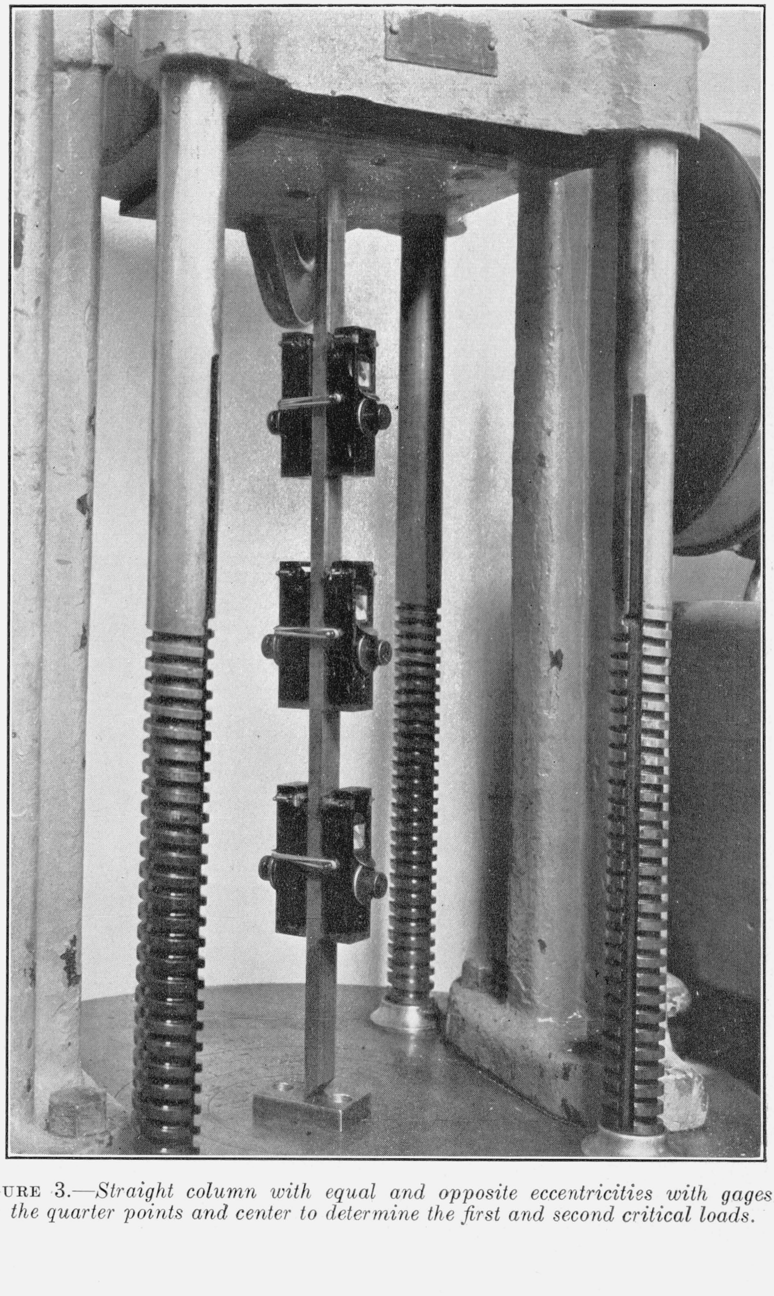

FIGURE 3.—*Straight column with equal and opposite eccentricities with gages at the quarter points and center to determine the first and second critical loads.*

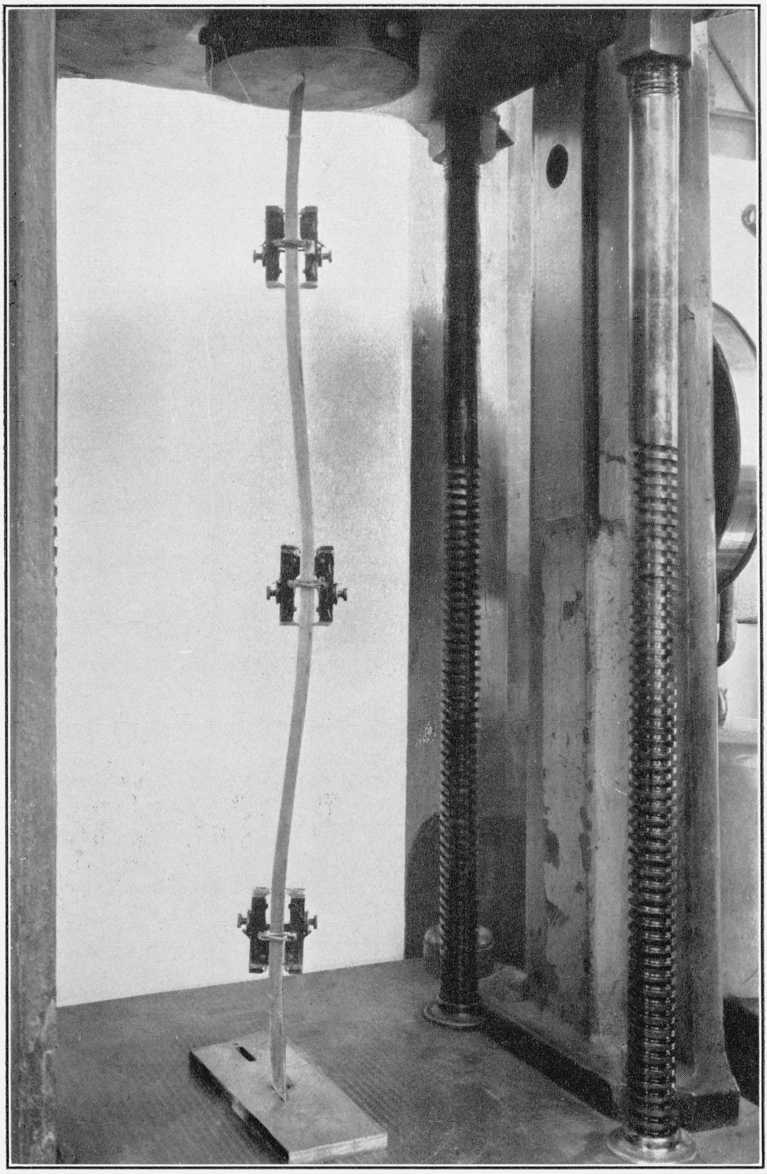

FIGURE 5.—*Eccentrically loaded bent column with gages at the sixth points and center to determine the first and third critical loads.*

load, which is theoretically four times the first critical load. Application of the analysis to this difference should therefore be expected to determine this second critical load, provided the corresponding term of $\overline{F}$ is sufficiently predominant.

4. EXPERIMENTAL RESULTS FOR SECOND CRITICAL LOAD

The experimental data are given in table 1 and the final curves are plotted in figure 4.

In table 2 the theoretical critical loads calculated from Euler's theory are compared with the critical loads computed from the strain-gage readings. The values agree within 0 and 3 percent for the first critical load and 11 percent for the second.

TABLE 1.—*Critical load of 13.5-by-¼ by ½-inch cold-rolled steel column with equal and opposite eccentricities at the ends*

Load	[a] Δ_T	[a] Δ_M	[a] Δ_B	P_1	$\frac{1}{2}d_1$ [b] $=\left(\frac{\Delta_M}{2}\right)$	$\frac{\frac{1}{2}d_1}{P_1}$ [b]	d_2 [b] $=\left(\frac{\Delta_T-\Delta_B}{2}\right)$	$\frac{d_2}{P_1}$ [b]
lb	10^{-4}	10^{-4}	10^{-4}	*lb*	10^{-4}	$\frac{10^{-4}}{lb}$	10^{-4}	$\frac{10^{-4}}{lb}$
790	0	0	0	0	0	----------	0	----------
750	−1.02	−0.73	+0.01	−40	−0.37	+0.00912	−0.51	0.01288
710	−1.93	−1.20	+.13	−80	−.60	.00750	−1.03	.01288
670	−2.73	−1.63	+.33	−120	−.81	.00679	−1.53	.01275
630	−3.40	−1.93	.54	−160	−.96	.00602	−1.97	.01231
590	−4.10	−2.19	.91	−200	−1.09	.00548	−2.50	.01252
550	−4.67	−2.34	1.21	−240	−1.17	.00488	−2.94	.01225
510	−5.23	−2.53	1.52	−280	−1.26	.00452	−3.38	.01205
470	−5.73	−2.66	1.82	−320	−1.33	.00416	−3.78	.01179
430	−6.29	−2.78	2.19	−360	−1.39	.00386	−4.24	.01178
390	−6.74	−2.87	2.52	−400	−1.44	.00359	−4.63	.01158
350	−7.22	−3.01	2.83	−440	−1.50	.00342	−5.02	.01142
310	−7.69	−3.05	3.14	−480	−1.52	.00318	−5.42	.01130
270	−8.14	−3.13	3.47	−520	−1.56	.00301	−5.80	.01115
230	−8.53	−3.19	3.79	−560	−1.60	.00285	−6.16	.01100
190	−9.01	−3.26	4.08	−600	−1.63	.00272	−6.55	.01090
150	−9.40	−3.35	4.40	−640	−1.68	.00262	−6.90	.01080
110	−9.78	−3.36	4.71	−680	−1.68	.00247	−7.24	.01065

[a] Δ is the difference in strain on opposite sides of the specimen (bending strain) arranged to be zero at the arbitrary "zero load" for computations of 790 lb.
[b] d_1 and d_2 are the bending strains corresponding to the first and second critical loads, respectively.

TABLE 2.—*Critical load of 13.5- by ¼- by-½-inch cold-rolled steel column with equal and opposite eccentricities at the ends*

	Pounds
FIRST CRITICAL LOAD	
From strain at middle	994
From strain at quarter points	1,015
Theoretical $\left(\frac{\pi^2 EI}{l^2}\right)$	1,020
(Assumed $E=29\times10^6$ lb/in.²)	
SECOND CRITICAL LOAD	
From strain at quarter points	3,700
Theoretical $\left(\frac{4\pi^2 EI}{l^2}\right)$	4,100
(Assumed $E=29\times10^6$ lb/in.²)	

5. EXPERIMENTAL ARRANGEMENT FOR THIRD CRITICAL LOAD

The fair success obtained in the calculation of the second critical load from heterostatic strains below the first critical load made it seem worth-while to see if still higher critical loads could be similarly

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determined. A piece of cold-rolled steel, ⅜ by ¾ by 27.7 inches, was chosen so as to allow strain gages to be placed one-sixth of its length from each end without coming too close to the ends. To ensure that the term of the summation in eq 14 corresponding to the third critical load should be fairly large, it was loaded as a "round-end" column with approximately equal like eccentricities at either end. This automatically ensured that terms corresponding to the second critical load would be small. To keep the terms corresponding to the first critical load small, the bar was slightly bent symmetrically at the middle in a direction toward the line of application of the eccentric load. It is readily seen that difference between the sum of the bending strains (difference in strain on opposite gages) at the two-sixth points and the bending strain at the middle will depend chiefly upon the variation of the astatic parameter corresponding to the third critical load, while their sum plus twice the bending strain at the middle will depend chiefly upon the astatic parameter corresponding to the first critical load. Two tests were made, one in which the bending strains corresponding to the first critical load were predominant and, a second, after bending to the shape shown in figure 5, in which the bending strains corresponding to the third critical load were predominant.

6. EXPERIMENTAL RESULTS FOR THIRD CRITICAL LOAD

The data are given in table 3 and the final curves are plotted in figure 6.

In table 4 the theoretical critical loads calculated by Euler's theory are compared with the critical loads computed from the strain-gage readings. The values agree within 1 percent for the first critical load and 12 percent for the third critical load.

TABLE 3.—*Critical load of 27.7-by ⅜-by ¾-inch cold-rolled steel column with equal eccentricities at the ends*

P	$^a\Delta_T$	$^a\Delta_M$	$^a\Delta_B$	$d_1{}^b=\left(\frac{2\Delta_M+\Delta_B+\Delta_T}{3}\right)$	$P_1{}^a$	$\frac{d_1}{P_1}{}^b$	$^c\Delta_T$	$^c\Delta_M$	$^c\Delta_B$	$d_3{}^b=\left(\frac{\Delta_M-\Delta_T-\Delta_B}{3}\right)$	$P_3{}^c$	$\frac{d_3}{P_3}{}^b$
lb	10^{-4}	10^{-4}	10^{-4}	10^{-4}	*lb*	$\frac{10^{-4}}{lb}$	10^{-4}	10^{-4}	10^{-4}	10^{-4}	*lb*	$\frac{10^{-4}}{lb}$
100	1.81	−6.03	1.16	−3.03	−900	0.00337	0	0	--------	0	0	--------
200	1.53	−5.60	0.96	−2.90	−800	.00363	−1.10	+1.32	−1.59	+1.34	100	0.01337
300	1.22	−5.24	.74	−2.84	−700	.00406	−2.55	+2.57	−3.07	2.73	200	.01365
400	0.92	−4.81	.51	−2.73	−600	.00455	−3.81	+3.96	−4.60	4.12	300	.01373
500	.66	−4.31	+.31	−2.55	−500	.00510	−5.13	+5.34	−6.21	5.56	400	.01390
600	.41	−3.78	+.10	−2.35	−400	.00587	−6.44	+6.75	−7.80	7.00	500	.01399
700	.17	−3.20	−.06	−2.10	−300	.00699	−7.87	+8.23	−9.55	8.55	600	.01425
800	+.01	−2.40	−.17	−1.65	−200	.00960	−9.26	+9.77	−11.25	10.09	700	.01442
900	−.11	−1.49	−.20	−1.10	−100	.01097	−10.70	+11.35	−13.01	11.69	800	.01461
1,000	0	0	0	0	0	--------	−12.09	+12.79	−14.75	13.21	900	.01469

[a] Δ is the difference in strain on opposite sides of the specimen (bending strain) arranged to be zero at the arbitrary "zero load" for computations of 1,000 lb.

[b] d_1 and d_3 are the bending strains corresponding to the first and third critical loads, respectively.

[c] For these readings the column was bent as shown in figure 5 to emphasize the third harmonic. For these the arbitrary "zero load" for computations is 100 lb.

TABLE 4.—*Critical load of a 27.7- by ⅜- by ¾-inch cold-rolled steel column*

FIRST CRITICAL LOAD	
	Pounds
From strain gages	1,251
Theoretical $\left(\frac{\pi^2 EI}{l^2}\right)$	1,240
(Assumed, E=29x10⁶ lb/in.²)	
THIRD CRITICAL LOAD	
From strain gages	9,825
Theoretical $\left(\frac{9\pi^2 EI}{l^2}\right)$	11,100
(Assumed, E=29x10⁶ lb/in.²)	

7. AGREEMENT WITH EULER COLUMN THEORY

The agreement in the case of the first critical load (in all cases within 3 percent) is materially better than in the case of the second and third critical loads, (within 11 percent and 12 percent, respectively). This was to be expected since in spite of the fact that the experimental conditions were adjusted to make q_1 as small as possible, the denominator, (Q_1-P), decreased rapidly with increasing P, so that $q_1/(Q_1-P)$, may have been appreciable in comparison with $q_2/(Q_2-P)$, and $q_3/(Q_3-P)$. Further, it was impossible in the experimental arrangement to eliminate wholly terms corresponding to still higher loads. In spite of careful work the experimental errors were so great that it was not feasible to secure better values by means of the successive approximations outlined in section II, 7. The calculation of these higher critical loads is in effect an extrapolation of loads over a range of nearly *5 to 1* in the case of the second critical load and of nearly *10 to 1* in the case of the third critical load. In view of the experimental difficulties involved, the errors are less than might have been expected for extrapolations over such great ranges.

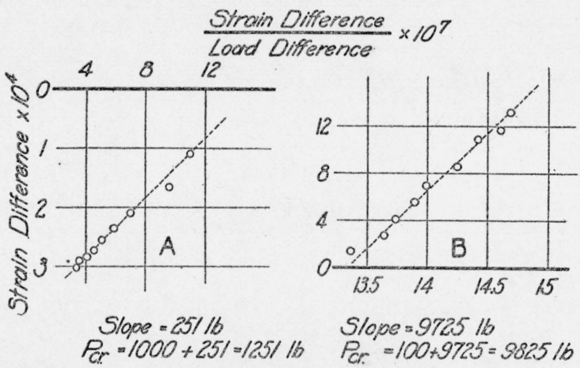

FIGURE 6.—*Southwell analysis of eccentrically loaded column.*

Curve *A*, column bent to accentuate fundamental mode. Curve *B*, same column bent to accentuate third harmonic mode. Assumed "zero load" for computations *A*, 1,000 pounds; *B*, 100 pounds.

IV. SUMMARY

1. The history of Southwell's method for the analysis of experimental observations in problems of elastic instability is reviewed.
2. Westergaard's general theory is applied to the problem.
3. It is shown that Southwell's method and Lundquist's modification of it are theoretically accurate for results of measurements which are proportional to the value of any one astatic parameter.

4. If the measurements are affected appreciably by changes in other parameters, the critical load computed by Southwell's method or Lundquist's modification may be considerably in error. A combined numerical and graphical method of computation is outlined which by successive approximations gives more accurate results in some such cases.

5. Finally, experimental results are given in which the second and third critical loads of a "round-end" Euler column are computed from strain-gage measurements taken at loads below the first critical load.

The author is much indebted to A. E. McPherson and S. Levy for making the experimental measurements reported and for valuable suggestions and assistance in preparing the manuscript.

WASHINGTON, October 15, 1938.