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DETERMINATION OF PRINCIPAL STRESSES FROM STRAINS ON FOUR INTERSECTING GAGE LINES 45° APART

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ABSTRACT

The analytical solution of the particular problem of determining the principal stresses from measurements of strains on four intersecting gage lines 45° apart is so simple that it is to be preferred to the general graphical solution published in a previous paper. The analytical solution is given here.

Since the original paper on the determination of stresses from strains on three intersecting gage lines was written,[1] an investigation has been undertaken at the National Bureau of Standards in which strains were measured on four intersecting gage lines 45° apart for the purpose of determining the principal stresses. The analytical solution of this particular problem is so simple that it is to be preferred in general to the graphical solution, and it seems worth while to call attention to it. A solution has been published [2] in another connection, which embodies the present one in all essentials but is not directly applicable.

In the case of plane stress the equations expressing the relation between the components of strain at a point referred to any pair of rectangular axes x' and y' parallel to the plane of stress and the components of the same strain referred to any other pair of rectangular axes x and y parallel to the plane of stress are [3]

$$\epsilon_x' = \frac{1}{2}(\epsilon_x+\epsilon_y)+\frac{1}{2}(\epsilon_x-\epsilon_y)\cos 2\psi+\frac{1}{2}\gamma_{xy}\sin 2\psi, \tag{1}$$

$$\epsilon_y' = \frac{1}{2}(\epsilon_x+\epsilon_y)-\frac{1}{2}(\epsilon_x-\epsilon_y)\cos 2\psi-\frac{1}{2}\gamma_{xy}\sin 2\psi, \tag{2}$$

$$\gamma_{xy}' = -(\epsilon_x-\epsilon_y)\sin 2\psi+\gamma_{xy}\cos 2\psi, \tag{3}$$

where

- ϵ_x and ϵ_y are the components of strain in the directions x and y, respectively,
- γ_{xy} is the shearing strain with respect to the axes of x and y,
- ψ is the angle measured positive counterclockwise from the positive axis of x to the positive axis of x',
- ϵ_x' and ϵ_y' are the components of strain in the directions x' and y', respectively, and
- γ_{xy}' is the shearing strain with respect to the axes of x' and y'.

[1] Wm. R. Osgood and Rolland G. Sturm, *The Determination of stresses from strains on three intersecting gage lines and its application to actual tests*, B S J. Research **10**, 685 (1933) RP559.
[2] L. B. Tuckerman, G. H. Keulegan, and H. N. Eaton, *A fabric tension meter for use on aircraft*, Tech. Pap. BS **20**, 581 (1926) T320.
[3] Fuller and Johnston, Applied Mechanics, **2**, Art. 42 (John Wiley & Sons, New York).

Suppose that strains have been measured (tensile strain as positive) along four intersecting gage lines, 1 and 3, 90° apart, and 2 and 4, 90° apart and making angles of 45°, respectively, with 1 and 3. Then the maximum and minimum principal strains, ϵ_u and ϵ_v, respectively, in the plane of stress and the directions of these strains may be

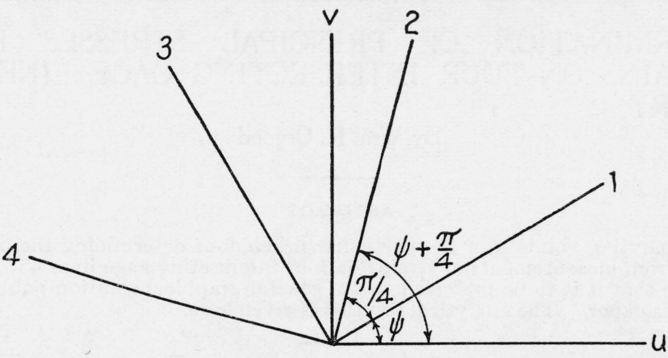

FIGURE 1.—*Four intersecting gage lines (1, 2, 3, 4) and directions (u, v) of principal strains.*

obtained as follows. If we take the directions of x' and y' in the directions of the maximum and minimum principal strains, ϵ_u and ϵ_v, respectively, we obtain from equations 1 and 2

$$\epsilon_1=\frac{1}{2}(\epsilon_u+\epsilon_v)+\frac{1}{2}(\epsilon_u-\epsilon_v)\cos 2\psi, \tag{4}$$

$$\epsilon_2=\frac{1}{2}(\epsilon_u+\epsilon_v)-\frac{1}{2}(\epsilon_u-\epsilon_v)\sin 2\psi, \tag{5}$$

$$\epsilon_3=\frac{1}{2}(\epsilon_u+\epsilon_v)-\frac{1}{2}(\epsilon_u-\epsilon_v)\cos 2\psi, \tag{6}$$

$$\epsilon_4=\frac{1}{2}(\epsilon_u+\epsilon_v)+\frac{1}{2}(\epsilon_u-\epsilon_v)\sin 2\psi. \tag{7}$$

The four simultaneous equations 4, 5, 6, and 7 contain only three unknowns, ϵ_u, ϵ_v, and ψ. The best solution can be shown by the method of least squares to be

$$\epsilon_u=\frac{1}{2}\left[\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)+\sqrt{(\epsilon_1-\epsilon_3)^2+(\epsilon_2-\epsilon_4)^2}\right], \tag{8}$$

$$\epsilon_v=\frac{1}{2}\left[\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)-\sqrt{(\epsilon_1-\epsilon_3)^2+(\epsilon_2-\epsilon_4)^2}\right], \tag{9}$$

$$\tan 2\psi=-\frac{\epsilon_2-\epsilon_4}{\epsilon_1-\epsilon_3}. \tag{10}$$

The maximum and minimum principal stresses, σ_u and σ_v, respectively, in the plane of stress are then obtained as

$$\sigma_u=\frac{E}{1-m^2}(\epsilon_u+m\epsilon_v), \tag{11}$$

$$\sigma_v=\frac{E}{1-m^2}(\epsilon_v+m\epsilon_u), \tag{12}$$

where E is the modulus of elasticity, and m is Poisson's ratio. By substituting the values of ϵ_u and ϵ_v from equations 8 and 9 in equations 11 and 12, the principal stresses σ_u and σ_v may be expressed directly in terms of the observed components of strain ϵ_1, ϵ_2, ϵ_3, and ϵ_4:

$$\sigma_u=\frac{E}{2(1+m)}\left[\frac{1}{2}\frac{1+m}{1-m}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)+\sqrt{(\epsilon_1-\epsilon_3)^2+(\epsilon_2-\epsilon_4)^2}\right], \tag{13}$$

$$\sigma_v=\frac{E}{2(1+m)}\left[\frac{1}{2}\frac{1+m}{1-m}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)-\sqrt{(\epsilon_1-\epsilon_3)^2+(\epsilon_2-\epsilon_4)^2}\right]. \tag{14}$$

For most purposes it is probably adequate to assume $m=\frac{1}{3}$, in which case the expressions for the principal stresses reduce to the simpler form

$$\sigma_u=\frac{3E}{8}\left[\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\sqrt{(\epsilon_1-\epsilon_3)^2+(\epsilon_2-\epsilon_4)^2}\right], \tag{15}$$

$$\sigma_v=\frac{3E}{8}\left[\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4-\sqrt{(\epsilon_1-\epsilon_3)^2+(\epsilon_2-\epsilon_4)^2}\right]. \tag{16}$$

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