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A METHOD FOR FINDING THE ROOTS OF THE EQUATION $f(x)=0$ WHERE f IS ANALYTIC

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ABSTRACT

An expression is derived giving the roots of equations involving only functions which may be expressed in power series. The expression takes the form $x_n = x_o - \frac{U_{n-1}}{U_n}$, where $\sum_{i=0}^{n} (-1)^i a_i U_{n-i} = 0$ and $a_r \equiv \frac{f^{(r)}(x_o)}{r!f(x_o)}$, and x_n approaches the value of the root as n increases without bound. Methods are given for the treatment of certain special forms. An example is given of the application of the method to the calculation of real roots.

CONTENTS

I. INTRODUCTION

The use of recursion formulas in evaluating roots of equations was described by Euler.[1] Since that time the formulas and the related determinants have been presented by various authors,[2] the most recent example being the series involving determinants developed by E. T. Whittaker.[3] However, three of the quantities necessary to the evaluation of the r^{th} term of this series are sufficient to determine the sum of the first r terms, so that the series form is not the most convenient one for calculation. In the present paper the recursion formulas are developed in an elementary way, and put in a form which is convenient for calculation.

[1] Introductio in Analysii Infinitorum, **1**, chap. 17 (1748), or the translation by J. B. Labey, p. 257 (Paris, 1796).

[2] F. Cajori, A History of Mathematics, p. 227. (McMillan, London, 1919.)

[3] Proc. Edinburgh Math. Soc. **36**, 103 (1918), or Whittaker and Robinson, Calculus of Observations, p. 120. (Blackie and Son, London, 1924.)

II. DERIVATION OF EQUATIONS

Required the value of x, which is a root of the equation $f(x)=0$, where $f(x)$ is analytic in the neighborhood of x. Then if an arbitrary approximation to x, x_0, lies in the neighborhood of x the function $f(x)$ is given by the series

$$f(x)=a_0+a_1(\Delta x)+\frac{a_2}{2!}(\Delta x)^2+\;.\;.\;.\;.\;+\frac{a_r}{r!}(\Delta x)^r+\;.\;.\;.\;.\;.\;. \tag{1}$$

where the coefficients $a_r\equiv f^{(r)}(x_0)$ are given and $\Delta x\equiv x-x_0$.

Define

$$g_n(x)\equiv 1+b_{n,1}(\Delta x)+\frac{b_{n,2}}{2!}(\Delta x)^2+\;.\;.\;.\;.\;.\;+\frac{b_{n,n-1}}{(n-1)!}(\Delta x)^{n-1} \tag{2}$$

where the $n-1$ constants $b_{n,1,}b_{n,2,}$——$,b_{n,n-1}$ are so chosen that n–1 consecutive derivatives (beginning with the second) of the function $\varphi_n(x)$ vanish at x_0, where

$$\varphi_n(x)\equiv f(x)g_n(x)=\varphi_n(x_0)+(\Delta x)\varphi_n{}'(x_0)+\frac{(\Delta x)^{n+1}}{(n+1)!}\varphi_n{}^{(n+1)}(x_0)+\frac{(\Delta x)^{n+2}}{(n+2)!}\varphi_n{}^{(n+2)}(x_0)+\;.\;.\;.\;. \tag{3}$$

Then

$$\left.\begin{array}{l}
\varphi_n'(x_0)=a_1+a_0b_{n,1}\\
\varphi_n''(x_0)=0=a_2+2a_1b_{n,1}+a_0b_{n,2}\\
\varphi_n'''(x_0)=0=a_3+3a_2b_{n,1}+3a_1b_{n,2}+a_0b_{n,3}\\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\varphi_n^{(r)}(x_0)=0=a_r+\binom{r}{1}a_{r-1}b_{n,1}+\;.\;.\;.\;.\;.\;.\;.\;+\binom{r}{r-1}a_1b_{n,r-1}+a_ob_{n,r}\\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\varphi_n^{(n-1)}(x_0)=0=a_{n-1}+\binom{n-1}{1}a_{n-2}b_{n,1}+\;.\;.\;.+\binom{n-1}{n-2}a_1b_{n,n-2}+a_0b_{n,n-1}\\
\varphi_n^{(n)}(x_0)=0=a_n+\binom{n}{1}a_{n-1}b_{n,1}+\cdot\;\cdot\;+\binom{n}{n-2}a_2b_{n,n-2}+\binom{n}{n-1}a_1b_{n,n-1}
\end{array}\right\} \tag{4}$$

The symbol $\binom{r}{i}$ is used to represent the number of combinations of r things taken i at a time.

The equations 4 may be solved for $\varphi_n'(x_0)$, with the result

$$\varphi_n'(x_0)=\frac{D_n}{A_n} \tag{5}$$

where A_n is the cofactor of a_1 in the determinant of the n^{th} order having for the element of the r^{th} row and c^{th} column $\binom{r}{c-1}a_{r-c+1}$. The determinant may be written

$$D_n \equiv \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ a_2 & 2a_1 & a_0 & \cdots & 0 & \cdots & 0 & 0 \\ a_3 & 3a_2 & 3a_1 & \cdots & 0 & \cdots & 0 & 0 \\ a_4 & 4a_3 & 6a_2 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_r & \binom{r}{1}a_{r-1} & \binom{r}{2}a_{r-2} & & \binom{r}{c-1}a_{r-c+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{n-1} & \binom{n-1}{1}a_{n-2} & \binom{n-1}{2}a_{n-3} & \cdots & \binom{n-1}{c-1}a_{n-c} & \cdots & \binom{n-1}{n-2}a_1 & a_0 \\ a_n & \binom{n}{1}a_{n-1} & \binom{n}{2}a_{n-2} & \cdots & \binom{n}{c-1}a_{n-c+1} & \cdots & \binom{n}{n-2}a_2 & \binom{n}{n-1}a_1 \end{vmatrix}$$

Equation 3 may be put in the form

$$\varphi_n(x) = a_0 + (\Delta x)\varphi_n{}'(x_0) + R_n, \tag{6}$$

where R_n represents the sum of the terms containing powers of Δx higher than the n^{th}. Solving equation 6 for Δx and putting $\varphi_n(x)=0$, we have

$$\Delta x = -\frac{a_0 + R_n}{\varphi_n{}'(x_0)} \tag{7}$$

We may write

$$\Delta_n x \equiv -\frac{a_0}{\varphi_n{}'(x_0)} \tag{8}$$

Then, as $\lim_{n\to\infty} R_n = 0$, $\lim_{n\to\infty} \Delta_n x = \Delta x$.

From equations 8 and 5 we have

$$\Delta_n x = -\frac{a_0 A_n}{D_n} \tag{9}$$

It may be shown that $A_n = nD_{n-1}$. Multiplying each element of D_{n-1} by $\frac{r+1}{c}$, the element becomes $\binom{r+1}{c}a_{r-c+1}$, which is the element of A_n. In this process D_{n-1} is multiplied by $\frac{n!}{(n-1)!}$, or n. Thus

$$A_n = nD_{n-1} \tag{10}$$

Then from equations 9 and 10

$$\Delta_n x = -\frac{n a_0 D_{n-1}}{D_n} \tag{11}$$

A recursion formula for D_n may be obtained by expanding it on the elements of the n^{th} row. The result is

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i a_0^{\,i-1} D_{n-i} = 0 \tag{12}$$

This may be simplified by writing $U_r \equiv \frac{D_r}{r!a^r_0}$ and $\alpha_r \equiv \frac{a_r}{r!a_0}$.

Equation 12 then becomes

$$\sum_{i=0}^{n} (-1)^i \alpha_i U_{n-i} = 0 \quad (n=1, 2, 3, \ldots) \tag{13}$$

Now equation 11 may be written

$$\Delta_n x = -\frac{U_{n-1}}{U_n} \tag{14}$$

Clearly $U_0 = D_0 = 1$, in this development, but in general the value of U_0 may be chosen arbitrarily and another value may be more convenient.

The product of k consecutive convergents to Δx is

$$\prod_{r=n+1}^{r=n+k} \Delta_r x = (-1)^k \frac{U_n}{U_{n+k}} \tag{15}$$

The limit of this product as n increases without bound is $(\Delta x)^k$.

III. TREATMENT OF SPECIAL FORMS

In case $f(x) \equiv F(x^2)$, and x_0 is taken equal to zero, the derivatives of odd order are zero, and the functions U_{2m+1} are also zero. Thus, $\Delta_{2m} x = 0$ and $|\Delta_{2m+1} x| = \infty$. Taking n even and $k=2$ in equation 15 gives

$$(\Delta_{2m+1} x)(\Delta_{2m+2} x) = \frac{U_{2m}}{U_{2m+2}} \tag{16}$$

This expression may be evaluated, giving a convergent to $(\Delta x)^2$. For instance, let $f(x) \equiv \cos x = 0$, and take $x_0 = 0$. Then U_{2m} becomes $\frac{E_m}{(2m)!}$, where E_m is Euler's number of order m, or Bernoulli's number of order $2m$, and

$$(\Delta_{2m+1} x)(\Delta_{2m+2} x) = \frac{(2m+1)(2m+2)E_m}{E_{m+1}} \tag{17}$$

Now $\lim_{m\to\infty}(\Delta_{2m+1}x)(\Delta_{2m+2}x)=(\Delta x)^2$ and $\lim_{m\to\infty}\frac{(2m+1)(2m+2)E_m}{E_{m+1}}=\frac{\pi^2}{4}$, so that $(\Delta x)^2=\frac{\pi^2}{4}$ or $\Delta x=\pm\frac{\pi}{2}$, which gives the roots $\pm\frac{\pi}{2}$ for $\cos x=0$.

The case in which $f(x)\equiv F[(x-c)^p]$ and $x_0=c$ may be treated in a different way. Put $(x-c)^p\equiv z$, and find Δz for $z_0=0$. Then $\Delta x=(\Delta z)^{1/p}$.

IV. APPLICATION OF RESULTS

The familiar equation $x^3-2x-5=0$ does not afford a good test of the method of this paper. If x_0 is taken as 2, the values of α_1, α_2, and α_3 are -10, -6, and -1. Thus, the successive values of U_n are integers, and may be written down immediately as far as U_4, which is sufficient for the calculation of the root correct to six decimal places.

The advantages of the method are better illustrated by the solution of a transcendental equation, such as

$$f(x)\equiv x+\sin x+\log x+e^x-5=0 \tag{18}$$

Taking $x_0=1.1$, we find

$$a_0=\ 0.090\,683\,6$$
$$a_1=\ 5.366\,853$$
$$a_2=\ 1.286\,51 \qquad a_2/2\ =\ 0.643\,26$$
$$a_3=\ 4.053 \qquad a_3/6\ =\ 0.675\,5$$
$$a_4=-0.2 \qquad a_4/24=-0.01$$

TABLE 1.—*Table of the products necessary for the calculation of $\Delta_4 x$.*

	$\alpha_0=1$	$\alpha_1=\frac{a_1}{a_0}$	$\alpha_2=\frac{a_2}{2a_0}$	$\alpha_3=\frac{a_3}{6a_0}$	$\alpha_4=\frac{a_4}{24a_0}$
U_0	1	59.182 18	7.093 4	7.449	−0.1
U_1	59.182 18	3,502.530	419.80	440.8	
U_2	3,495.437	206,867.6	24,794		
U_3	206,455.2	12,218,460			
U_4	12,194,116				

$U_0=1$

$U_1=\alpha_1 \qquad \Delta_1 x=-\ \ 1/U_1=-0.016\,90$

$U_2=\alpha_1U_1-\alpha_2 \qquad \Delta_2 x=-U_1/U_2=-0.016\,931\,3$

$U_3=\alpha_1U_2-\alpha_2U_1+\alpha_3 \qquad \Delta_3 x=-U_2/U_3=-0.016\,930\,7$

$U_4=\alpha_1U_3-\alpha_2U_2+\alpha_3U_1-\alpha_4 \qquad \Delta_4 x=-U_3/U_4=-0.016\,930\,7$

It may be seen that the value of α_4 was not significant in the calculation of U_4. The accuracy of the calculation is not increased by the last step, as $\Delta_3 x$ contains no error which is significant with respect to the value of a_0 used. Then $x_3=x_4=1.083\,069\,3$. The value of the root correct to eight decimal places is 1.083 069 29.

It should be noted that in solving equation 6 for Δx, $\varphi_n'(x_0)$ was used as a divisor. Thus, if $\varphi_n'(x_0)$ is zero or infinity, as may be the case for certain values of x_0 and n, equation 7 does not hold. Generally this difficulty may be avoided by taking a higher value of n. $\varphi'_n(x_0)$ will not ordinarily be zero, however, if x_0 is a good approximation to a root.

As the derivation of equations 13 and 14 does not involve any assumptions other than that $f(x)$ is analytic, the equations hold for complex values of the coefficients or the roots. In the evaluation of complex roots of real equations it is convenient to use a complex value for x_0, as equation 15 must be used if x_0 is real. This treatment is not satisfactory, as the sequence U_n/U_{n+k} does not usually converge rapidly in such a case, and also because extraneous roots may be introduced.

V. SUMMARY

An expression is derived giving the roots of equations which involve only functions which may be expressed as power series. The expression involves only the values of the function and its derivatives at a given value of the variable.

Methods are given for the treatment of certain special forms.

An example is given of the application of the method to the evaluation of real roots.

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