MUTUAL INDUCTANCE AND TORQUE BETWEEN TWO CONCENTRIC SOLENOIDS

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ABSTRACT

By integrating a certain form of expression for the magnetic field due to a circular current, an expression equation (25) involving zonal harmonics is obtained for the r-component of magnetic field within a solenoid, where r is the radius vector from an origin lying on the axis of the solenoid. By means of a theorem derived in Research Paper No. 18 this component alone enables us to obtain the flux of this field through a circular element perpendicular to r, and, thence, by integration the mutual inductance of the two solenoids is obtained in a rapidly converging series. The axes of the solenoids make an angle θ, and by differentiating with respect to this angle the torque between the two is obtained when each carries an electric current. The effect of the discrete nature of the windings of both solenoids is investigated, and correction terms obtained for this effect.

CONTENTS

I. INTRODUCTION

In a certain type of current balance for absolute measurements a large solenoid incloses a smaller one, the two being concentric and having their axes perpendicular. The torque on the inner one when they both carry electric currents may be found by first finding their mutual inductance when their axes make an angle θ, and then differentiating this with respect to θ. In the derivation of an accurate formula for this mutual inductance it will be assumed that the two solenoids are each current sheets. Correction terms will then be found which take account of the discrete nature of the windings in each solenoid.

A section of the two solenoidal current sheets by a plane containing the axes of both is shown in Figure 1.

II. THE MAGNETIC FIELD WITHIN A SOLENOIDAL CURRENT SHEET

The magnetic potential $\Delta\Omega\ (x,\ y)$ at $P\ (x,\ y)$ (fig. 1), due to the circular current of amount $n_1\ dx_1$ in the circle whose center is at $(x_1,\ o)$ and whose axis is the x-axis, its radius being a_1, is obtained in Research Paper No. 18, equation (12) as

$$\Delta\Omega\ (x,\ y) = 2\pi n_1 a_1 dx_1 \int_0^\infty e^{-s\,(x-x_1)}\ J_1\ (a_1 s)\ J_o\ (ys)\ ds \text{ if } x_1 < x$$

$$= -2\pi n_1 a_1 dx_1 \int_0^\infty e^{s\,(x-x_1)}\ J_1\ (a_1\ s)\ J_o\ (ys)\ ds \text{ if } x_1 > x \qquad (1)$$

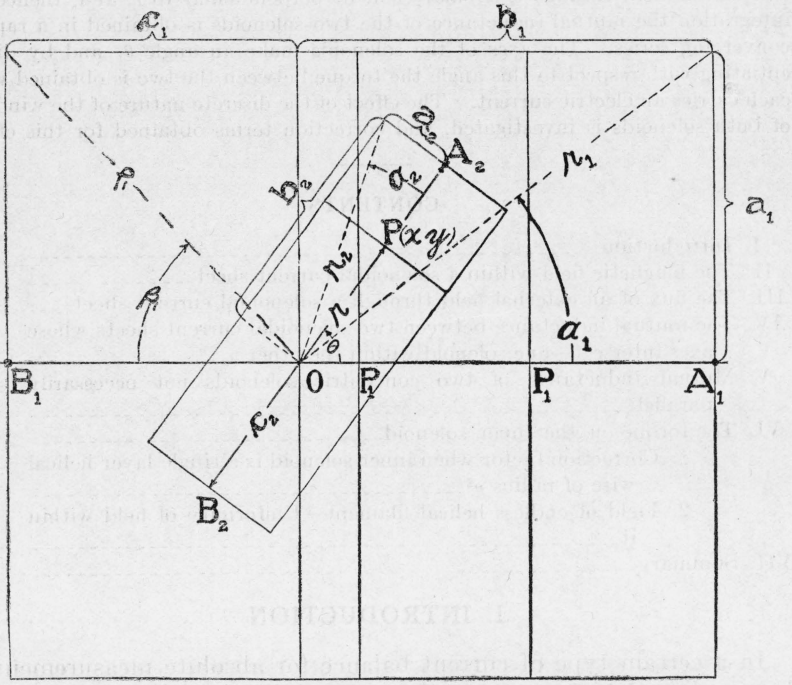

FIG. 1.—*A principal section of two solenoids whose axes intersect*

where J_o and J_1 are Bessel's functions. First differentiating with respect to x and y, respectively, and then integrating with respect to x_1, from $-c_1$ to b_1, gives for the field components at P, due to a unit current circulating around the large solenoid (a cylindrical current sheet),

$$H_x = -\frac{\delta\Omega\ (x,\ y)}{\delta x} = 2\pi n_1 a_1 \int_0^\infty ds\ J_1\ (a_1 s)\ J_o\ (ys)\ \{2 - e^{-s\,(b_1 - x)} - e^{-s\,(c_1 + x)}\} \qquad (2)$$

$$H_y = -\frac{\delta\Omega\ (x,\ y)}{\delta x} = 2\pi n_1 a_1 \int_0^\infty ds\ J_1\ (a_1 s)\ J_1\ (ys)\ \{e^{-s\,(b_1 - x)} - e^{-s\,(c_1 + x)}\} \qquad (3)$$

These are the magnetic field components (due to unit current in the large solenoid with n_1 turns per cm) at any point x, y inside or outside this solenoid provided $-c_1<x<b_1$. For the case where $P(x, y)$ lies within the large solenoid $y^2<a_1^2$. It is known that

$$\int_0^\infty e^{-sx} J_n(ys)\, ds = \frac{[(x^2+y^2)^{\frac{1}{2}}-x]^n}{y^n (x^2+y^2)^{\frac{1}{2}}} \text{ if } x>0 \tag{4}$$

and

$$a_1\int_0^\infty ds\, J_1(a_1 s)\, J_o(ys) = 1 \text{ if } y<a_1 \\ = 0 \text{ if } y>a_1 \tag{5}$$

Hence, for internal point the equation (2) may be replaced by the following when $y<a_1$ and $-c_1<x<b_1$.

$$H_x(x, y) = 4\pi n_1 - 2\pi n_1 a_1 \int_0^\infty ds\, J_1(a_1 s)\, J_o(ys)\, \{e^{-s(b_1-x)} - e^{-s(c_1+x)}\}$$
$$= 4\pi n_1 + 2\pi n_1 a_1 \left[J_1(a_1 D_{b_1}) \int_0^\infty e^{-s(b_1-x)} J_o(ys)\, ds \right. \tag{6}$$
$$\left. + J_1(a_1 D_{c_1}) \int_0^\infty e^{-s(c_1+x)} J_o(ys)\, ds \right]$$

or, making use of equation (4) for $n=0$.

$$H_x(x, y) = 4\pi n_1 + 2\pi n_1 a_1 \left[J_1(a_1 D_{b_1}) \frac{1}{\sqrt{(b_1-x)^2+y^2}} \right.$$
$$\left. + J_1(a_1 D_{c_1}) \frac{1}{\sqrt{(c_1+x)^2+y^2}} \right] \tag{7}$$

Similarly equation (3) leads to

$$H_y(x, y) = -2\pi n_1 a_1 \left[J_1(a_1 D_{b_1}) \int_0^\infty e^{-s(b_1-x)} J_1(ys)\, ds \right.$$
$$\left. - J_1(a_1 D_{c_1}) \int_0^\infty e^{-s(c_1+x)} J_1(ys)\, ds \right]$$

or, by equation (4) for $n=1$.

$$H_y(x, y) = \frac{2\pi n_1 a_1}{y} \left[J_1(a_1 D_{b_1}) \frac{b_1-x}{\sqrt{(b_1-x)^2+y^2}} - J_1(a_1 D_{c_1}) \frac{c_1+x}{\sqrt{(c_1+x)^2+y^2}} \right] \tag{8}$$

Here J_1 is a Bessel's function in the symbolic sense, its argument being a differentiating operator.

If $H_r(r, \theta)$ denotes that component of field at any point in the plane whose polar coordinates are (r, θ), which is reckoned in the direction of increasing r, then from (7) and (8) (placing $x=r\cos\theta$ and $y=r\sin\theta$)

$$H_r(r,\theta)=H_x\cos\theta+H_y\sin\theta$$

$$=4\pi n_1\cos\theta+2\pi n_1 a_1\left[J_1(a_1D_{b_1})\frac{1}{r\sqrt{1-2\frac{r}{b_1}\cos\theta+\frac{r^2}{b_1^2}}}\right.$$

$$\left.-J_1(a_1D_{c_1})\frac{1}{r\sqrt{1+2\frac{r}{c_1}\cos\theta+\frac{r_1^2}{c^2}}}\right] \tag{9}$$

One form of expansion for this may be obtained by using the series of zonal harmonics

$$\frac{1}{r\sqrt{1-2\frac{r}{b_1}\cos\theta+\frac{r^2}{b_1^2}}}=\frac{1}{r}+\sum_{s=0}^{\infty}\frac{r^s}{b_1^{s+1}}P_{s+1}(\cos\theta)\text{ when } r<b_1 \tag{10}$$

Operating on this with

$$J_1(a_1D_{b_1})\equiv\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{a_1}{2}\right)^{2k+1}}{k!\,(k+1)!}D_{b_1}^{2k+1} \tag{11}$$

and noting that

$$D_{b_1}^{2k+1}\frac{1}{b_1^{s+1}}=-\frac{\Gamma(s+2k+2)}{\Gamma(s+1)}\frac{1}{b_1^{s+2k+2}} \tag{12}$$

one finds

$$J_1(a_1D_{b_1})\frac{1}{r\sqrt{1-2\frac{r}{b_1}\cos\theta+\frac{r^2}{b_1^2}}}$$

$$=-\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k\,\Gamma(s+2k+2)\,P_{s+1}(\cos\theta)}{\Gamma(k+1)\,\Gamma(k+2)\,\Gamma(s+1)}\left(\frac{a_1}{2b_1}\right)^{2k+1}\frac{r^s}{b_1^{s+1}} \tag{13}$$

Similarly if r is also less than c_1

$$-J_1(a_1D_{c_1})\frac{1}{r\sqrt{1+2\frac{r}{c_1}\cos\theta+\frac{r^2}{c_1^2}}}$$

$$=-\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k\,\Gamma(s+2k+2)\,P_{s+1}(\cos\theta)}{\Gamma(k+1)\,\Gamma(k+2)\,\Gamma(s+1)}\left(\frac{a_1}{2c_1}\right)^{2k+1}(-1)^s\frac{r^s}{c_1^{s+1}} \tag{14}$$

Using the equations (13) and (14) in (9) gives

$$H_r(r,\theta)=4\pi n_1\cos\theta$$

$$-\pi n_1 a_1^2\sum_{k=o}^{\infty}\sum_{s=o}^{\infty}\frac{(-1)^k\Gamma(s+2k+2)P_{s+1}(\cos\theta)}{\Gamma(k+1)\Gamma(k+2)\Gamma(s+1)}$$

$$\left(\frac{a_1}{2}\right)^{2k}\left[\frac{1}{b_1^{s+2k+2}}+\frac{(-1)^s}{c_1^{s+2k+2}}\right] \tag{15}$$

If in this formula we sum first with respect to k making use of the formula

$$\Gamma(s+2k+2)=\frac{2^{2k+s+1}}{\sqrt{\pi}}\Gamma\left(k+\frac{s+2}{2}\right)\Gamma\left(k+\frac{s+3}{2}\right) \tag{16}$$

it becomes, letting $\mu=\cos\theta$

$$H_r(r_1\theta)=4\pi n_1\mu-n_1\pi a_1^2 D_r[\psi(r,\mu,b_1)-\psi(r,-\mu,c_1)] \tag{17}$$

where

$$\psi(r,\mu,b_1)\equiv\sum_{k=1}^{\infty}\frac{r^k}{b_1^{k+1}}P_k(\mu)\,F\left(\frac{k+1}{2},\frac{k+2}{2},2,-\frac{a_1^2}{b_1^2}\right) \tag{18}$$

where F denotes the hypergeometric function, which has a meaning only if $\frac{a_1}{b_1}$ and $\frac{a_1}{c_1}$ are less than unity, which is the case for the applications to be made here. By applying Euler's transformation

$$F\left(\frac{k+1}{2},\frac{k+2}{2},2,-\frac{a_1^2}{b_1^2}\right)=\left(\frac{b_1^2}{a_1^2+b_1^2}\right)^{1+\frac{k}{2}}F\left(\frac{2+k}{2},\frac{3-k}{2},2,\frac{a_1^2}{a_1^2+b_1^2}\right) \tag{19}$$

This reduces equation (18) to

$$\psi(r,\mu,b_1)=\cos\alpha_1\sum_{k=1}^{\infty}\frac{r^k P_k(\mu)\,F\left(\frac{2+k}{2},\frac{3-k}{2},2,\sin^2\alpha_1\right)}{r_1^{k+1}} \tag{20}$$

where

$$r_1\equiv\sqrt{a_1^2+b_1^2}\text{ and }\tan\alpha_1=\frac{a_1}{b_1}\quad\text{and }\alpha_1\text{ is a positive} \tag{21}$$

acute angle shown in Figure 1. Similarly, if we let

$$\rho_1\equiv\sqrt{a_1^2+c_1^2}\text{ and }\tan\beta_1=\frac{a_1}{c_1}$$

where β_1 is the positive acute angle shown in Figure 1, then

$$\psi(r,-\mu,c_1)=\cos\beta_1\sum_{k=1}^{\infty}\frac{r^k P_k(-\mu)\,F\left(\frac{2+k}{2},\frac{3-k}{2},2,\sin^2\beta_1\right)}{\rho_1^{k+1}} \tag{22}$$

By the use of (20) and (22), the expression (17) becomes

$$H_r(r, \mu) = 4\pi n_1 \mu$$

$$-n_1 \pi a_1^2 D_{r_1} \sum_{k=1}^{\infty} r^k \left\{ P_k(\mu) \frac{\cos\alpha_1 F\left(\frac{2+k}{2}, \frac{3-k}{2}, 2, \sin^2\alpha_1\right)}{r_1^{k+1}} \right. \tag{23}$$

$$\left. -P_k(-\mu) \frac{\cos\beta_1 F\left(\frac{2+k}{2}, \frac{3-k}{2}, 2, \sin^2\beta_1\right)}{\rho_1^{k+1}} \right\}$$

If Gauss's transformation be applied to these hypergeometric functions, one finds that

$$F\left(\frac{1+k}{2}, \frac{2-k}{2}, 2, \sin^2\alpha_1\right)$$

$$= \cos\alpha_1 F\left(\frac{2+k}{2}, \frac{3-k}{2}, 2, \sin^2\alpha_1\right) = \frac{2(1-\cos\alpha_1)}{\sin^2\alpha_1} \text{ if } k=1 \tag{24}$$

$$= \frac{2P'_{k-1}(\cos\alpha_1)}{k(k-1)} \text{ if } k>1$$

where P'_k denotes the derivative of P_k with respect to its argument. Hence, the expression (23) for $H_r(r, \mu)$ becomes

$$H_r(r, \mu) = 2\pi n_1 \left\{ (\cos\alpha_1 + \cos\alpha_2)\mu \right.$$

$$-D_r \sin^2\alpha_1 \sum_{k=1}^{\infty} \frac{r^{k+1} P_{k+1}(\mu) P'_k(\cos\alpha_1)}{r_1^k k(k+1)} \tag{25}$$

$$\left. +D_r \sin^2\beta_1 \sum_{k=1}^{\infty} \frac{r^{k+1} P_{k+1}(-\mu) P'_k(\cos\beta_1)}{\rho_1^k k(k+1)} \right\}$$

This r-component of the field is all that will be required in what follows. If the θ-component is desired it may be found by projection of H_x and H_y which are given in equations (7) and (8).

III. THE FLUX OF AN EXTERNAL FIELD THROUGH A SOLENOIDAL CURRENT SHEET

The magnetic flux through a circle of radius a_2 whose center is at $P(x, y)$ and whose plane is perpendicular to r has been obtained in Research Paper No. 18, equation (10) as

$$\Delta M = -2\pi a_2 J_1(a_2 D_r) \Omega(x, y)$$

$$= 2\pi a_2 \left\{ \frac{a_2}{2} H_r(r, \theta) + \sum_{n=1}^{\infty} (-1)^n \frac{\left(\frac{a_2}{2}\right)^{2n+1}}{\Gamma(n+1)\Gamma(n+2)} D_r^{2n} H_r(r, \theta) \right\} \tag{26}$$

The mutual energy between this magnetic field and a solenoid or cylindrical current sheet made up of a series of continuously distributed circles, all parallel, coaxial, and of equal radius a_2, the axis entending from $r=c_2$ to $r=b_2>c_2$ $\left(0\leqq\theta\leqq\frac{\pi}{2}\right)$ and carrying a circulating current n_2 per unit length (which is constant) may be derived from equation (26) by multiplying by n_2dr and integrating with respect to r from c_2 to b_2 (holding θ constant at some value between zero and $\frac{\pi}{2}\Big)$. This energy W_1 is given by

$$W_1=2\pi n_2 a_2\left\{\frac{a_2}{2}\int_{c_2}^{b_2} H_r dr+\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{a_2}{2}\right)^{2n+1}}{\Gamma(n+1)\Gamma(n+2)}\left[(D_r^{2n-1}H_r)_{r=b_2}\right.\right.$$
$$\left.\left.-(D_r^{2n-1}H_r)_{r=c_2}\right]\right\} \qquad (27)$$

It is important to notice that this formula, although it makes no assumption as to the origin of the magnetic field H, does not apply to the case shown in Figure 1, where the axis of the small solenoid extends beyond the origin. It only applies to the case where both ends of the axis of this solenoid lie in the direction of the acute angle θ from the origin. On account of the peculiar character of the polar coordinates it must be modified as follows to fit the case of a solenoid shown in Figure 1. In the first place, it is evident that the integral $\int_{c_2}^{b_2} H_r dr$ must be replaced by $\int_{B_2}^{A_2} H_s ds$ where s is the distance from B_2 to any point on the axis of the solenoid and H_s is the field component in the direction *from* B_2 *to* A_2. Its value will of course be independent of the path from B_2 to A_2. In the second place, if it is agreed that the polar coordinates of the point A_2 are (b_2, θ) where θ is the positive angle between zero and π (shown in the fig. 1 as acute), then the polar coordinates of B_2 will be $(c_2, \theta-\pi)$ making the convention that all polar angles shall lie between $-\pi$ and π. With this understanding the term in (27), namely, $(D_r^{2n-1}H_r)_{r=c_2}$ for $n=1, 2, 3$ (which is the derivative of odd order in the direction B_2A_2 of the field component in this same direction, taken at the point whose polar coordinates are c_2, θ where $0<\theta<\pi$) must be replaced by $D_{c_2}^{2n-1}H_{c2}$ $(c_2, \pi-\theta)$ when B_2 lies on the opposite side of the origin from A_2 and where H_{c_2} $(c_2, \pi-\theta)$ denotes the field component at B_2 in the direction of increasing c_2; that is, away from the origin. The modification of (27), which gives the mutual inductance between the unit current with n_2 turns per cm in the cylindrical sheet No. 2 and the external field, for the case shown in Figure 1 where the faces of the cylinder lie on opposite sides of the origin, is (using μ for $\cos\theta$)

$$M = n_2\pi a_2^2\left\{\int_0^{b_2} H_r(r,\mu)\,dr + \sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{a_2}{2}\right)^{2n}}{\Gamma(n+1)\,\Gamma(n+2)}D_{b_2}^{2n-1}H_{b_2}(b_2,\mu)\right.$$

$$\left.-\int_0^{c_2} H_r(r,-\mu)\,dr - \sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{a_2}{2}\right)^{2n}}{\Gamma(n+1)\,\Gamma(n+2)}D_{c_2}^{2n-1}H_{c_2}(c_2,-\mu)\right\} \quad (28)$$

This formulation of the flux through the cylindrical sheet of radius a_2 is independent of the particular source of the field. It is to be understood that b_2 and c_2 are the positive real radii vectors from the origin to the points A_2 and B_2, respectively, the corresponding angles being θ and $\pi-\theta$, and the subscript, b_2 or c_2 indicate that the component of field is reckoned in the direction of increasing b_2 or c_2, respectively.

It is in a form suitable for those applications in which the field at the points A_2 and B_2 is given in terms of the plane polar coordinates of these points, by expressions which hold for all values of r and of θ in the range between $-\pi$ and π.

IV. THE MUTUAL INDUCTANCE BETWEEN TWO SOLENOIDAL CURRENT SHEETS WHOSE AXES INTERSECT—ONE SOLENOID WITHIN THE OTHER

The two field components which occur in (28); namely, $H_{b_2}(b_2,\mu)$ and $H_{c_2}(c_2,-\mu)$ are obtainable from (25) and are

$$H_{b_2}(b_2,\mu) = 2\pi n_1\left\{(\cos\alpha_1+\cos\beta_1)\,\mu - D_{b_2}\sin^2\alpha_1\sum_{k=1}^{\infty}\frac{b_2^{k+1}P_{k+1}(\mu)P'_k(\cos\alpha_1)}{k(k+1)r_1^k}\right.$$

$$\left.+D_{b_2}\sin^2\beta_1\sum_{k=1}^{\infty}\frac{b_2^{k+1}P_{k+1}(-\mu)P'_k(\cos\beta_1)}{k(k+1)\rho_1^k}\right\} \quad (29)$$

$$H_{c_2}(c_2,-\mu) = 2\pi n_1\left\{-(\cos\alpha_1+\cos\beta)\mu\right.$$

$$-D_{c_2}\sin^2\alpha_1\sum_{k=1}^{\infty}\frac{c_2^{k+1}P_{k+1}(-\mu)P'_k(\cos\alpha_1)}{k(k+1)r_1^k} \quad (30)$$

$$\left.+D_{c_2}\sin^2\beta_1\sum_{k=1}^{\infty}\frac{c_2^{k+1}P_{k+1}(\mu)P'_k(\cos\beta_1)}{k(k+1)\rho_1^k}\right\}$$

Making use of (25), (29), and (30) in (28) gives

$$M=2\pi n_1 n_2 \pi a_2^2\{(b_2+c_2)\ (\cos\alpha_1+\cos\beta_1)\mu$$

$$-u(b_2,\mu,r_1,\alpha_1)+u(b_2,-\mu,\rho_1,\beta_1)+u(c_2,-\mu,r_1,\alpha_1)-u(c_2,\mu,\rho_1,\beta_1)\}\quad(31)$$

where

$$u(b_2,\mu,r_1,\alpha_1)\equiv\sin^2\alpha_1\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{a_2}{2}\right)^{2n}D_{b_2}{}^{2n}}{\Gamma(n+1)\,\Gamma(n+2)}\sum_{k=1}^{\infty}\frac{b_2{}^{k+1}P_{k+1}(\mu)P'_k(\cos\alpha_1)}{k(k+1)r_1{}^k}$$

$$=b_2\sin^2\alpha_1\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{a_2}{2b_2}\right)^2}{\Gamma(n+1)\Gamma(n+2)}\sum_{k=1}^{\infty}\frac{b_2{}^k\Gamma(k)P_{k+1}(\mu)P'_k(\cos\alpha_1)}{r_1{}^k\Gamma(k+2-2n)}\quad(32)$$

$$=b_2\sin^2\alpha_1\sum_{k=1}^{\infty}\left(\frac{b_2}{r_1}\right)^k P_{k+1}(\mu)P_k(\cos\alpha_1)\,\Gamma(k)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{a_2}{2b_2}\right)^{2n}}{\Gamma(n+1)\Gamma(n+2)\Gamma(k+2-2n)}$$

But when k is a positive integer

$$\Gamma(k)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{a_2}{2b_2}\right)^{2n}}{\Gamma(n+1)\,\Gamma(n+2)\,\Gamma(k+2-2n)}=\frac{F\left(\frac{-k}{2},-\frac{k+1}{2},\ 2,-\tan^2\alpha_2\right)}{k(k+1)}$$

$$=\frac{F\left(\frac{5+k}{2},\ \frac{-k}{2},\ 2,\ \sin^2\alpha_2\right)}{k(k+1)\ \cos^k\alpha_2}=\frac{2P'_{k+2}(\cos\alpha_2)}{k(k+1)\,(k+2)\,(k+3)\ \cos^{k+1}\alpha_2}\quad(33)$$

if $r_2{}^2=a_2{}^2+b_2{}^2$ and $\tan\alpha_2=\frac{a_2}{b_2}$. Hence

$$u(b_2,\mu,r_1,\alpha_1)=2r_2\sin^2\alpha_1\sum_{k=1}^{\infty}\left(\frac{r_2}{r_1}\right)^k\frac{P'_k(\cos\alpha_1)P_{k+1}(\mu)P'_{k+2}(\cos\alpha_2)}{k(k+1)\,(k+2)\,(k+3)}$$

$$=\frac{2r_2}{\sin^2\alpha_2}\sum_{k=1}^{\infty}\left(\frac{r_2}{r_1}\right)^k\left[\frac{P_{k-1}(\cos\alpha_1)-P_{k+1}(\cos\alpha_1)}{2k+1}\right]P_{k+1}(\mu)\quad(34)$$

$$\left[\frac{P_{k+1}(\cos\alpha_2)-P_{k+3}(\cos\alpha_2)}{2k+5}\right]$$

For the other functions u one may adopt a similar notation; namely, $\rho_2{}^2=a_2{}^2+c_2{}^2$, $\tan\beta_2=\frac{a_2}{c_2}$.

V. MUTUAL INDUCTANCE OF TWO CONCENTRIC SOLENOIDS NOT NECESSARILY PARALLEL

In this case $c_1=b_1$, $\beta_1=\alpha_1=\tan^{-1}\dfrac{a_1}{b_1}$, $\rho_1=r_1=\sqrt{a_1^2+b_1^2}$, $c_2=b_2$, $\beta_2=\alpha_2=\tan^{-1}\dfrac{a_2}{b_2}$, $\rho_2=r_2=\sqrt{a_2^2+b_2^2}$, $\mu=\cos\theta$, where θ is the angle between their axes. The formula (31) then reduces to

$$M=4\pi n_1 n_2 \pi a_2^2 2b_2\left\{\mu\cos\alpha_1 - 2\sin^2\alpha_1\cos\alpha_2\sum_{s=1}^{\infty}\frac{P_{2s+1}(\mu)\,P'_{2s}(\cos\alpha_1)\,P'_{2s+2}(\cos\alpha_2)}{2s(2s+1)(2s+2)(2s+3)}\left(\frac{r_2}{r_1}\right)^{2s}\right\}$$

$$=8\pi n_1 n_2 \pi a_2^2 b_2\cos\alpha_1\left\{\mu-\frac{\sin^2\alpha_1\cos^2\alpha_2}{2}\cdot\sum_{s=1}^{\infty}C_s(\alpha_1)\,C_{s+1}(\alpha_2)\,P_{2s+1}(\mu)\left(\frac{r_2}{r_1}\right)^{2s}\right\} \tag{35}$$

where

$$C_s(\alpha)=\frac{2P'_{2s}(\cos\alpha)}{2s(2s+1)\cos\alpha}=\frac{2\left[P_{2s-1}(\cos\alpha)-P_{2s+1}(\cos\alpha)\right]}{(4s+1)\sin^2\alpha\cos\alpha}$$

$$=F\left(1-s,\,s+\frac{3}{2},\,2,\,\sin^2\alpha\right)=\frac{\Gamma(s)}{\Gamma\left(s+\frac{3}{2}\right)}\sum_{k=0}^{s-1}\frac{\Gamma\left(k+s+\frac{3}{2}\right)(-\sin^2\alpha)^k}{\Gamma(k+1)\,\Gamma(k+2)\,\Gamma(s-k)}$$

$$=1+\frac{(1-s)}{2}\cdot\frac{\left(s+\frac{3}{2}\right)}{1}\sin^2\alpha+\frac{(1-s)}{2}\cdot\frac{(2-s)}{3}\,\frac{\left(s+\frac{3}{2}\right)}{1}\,\frac{\left(s+\frac{5}{2}\right)}{2}\sin^4\alpha+\cdots \tag{36}$$

For accurate computation the polynomials C_s for low orders of s, ($s=1$, 2, 3, 4) may be computed conveniently by this formla, which gives the special cases

$$C_1(\alpha)=1$$

$$C_2(\alpha)=1-\frac{7}{4}\sin^2\alpha$$

$$C_3(\alpha)=1-\frac{9}{2}\sin^2\alpha+\frac{33}{8}\sin^4\alpha$$

$$C_4(\alpha)=1-\frac{33}{4}\sin^2\alpha+\frac{143}{8}\sin^4\alpha-\frac{715}{64}\sin^6\alpha$$

etc., etc.

For large values of s the values of C_s might be more readily obtained (with sufficient accuracy) by interpolation from a table or curves for zonal harmonics or their derivatives.

VI. THE TORQUE ON THE INNER SOLENOID

If the two solenoids carry currents I_1 and I_2, respectively, the couple or torque T, on the inner one, tending to decrease θ is found from (35) to be

$$T = -I_1 I_2 \frac{\delta M}{\delta\theta} = I_1 I_2 \frac{\delta M}{\delta\mu} \tag{37}$$

or

$$T = 4\pi n_1 I_1 (2b_2 n_2 I_2)\, \pi a_2^2 \cos\alpha_1 \sin\theta$$

$$\left\{1 - \frac{\sin^2\alpha_1 \cos^2\alpha_2}{2} \sum_{s=1}^{\infty} C_s(\alpha_1) C_{s+1}(\alpha_2) P'_{2s+1}(\mu) \left(\frac{r_1}{r_2}\right)^{2s}\right\}$$

This torque is a maximum T_s when $\theta = \frac{\pi}{2}$, and $\mu = 0$, in which case

$$P'_{2s+1}(0) = \frac{2}{\sqrt{\pi}} (-1)^s \frac{\Gamma\left(s+\frac{3}{2}\right)}{\Gamma(s+1)}$$

and

$$T_s = 4\pi n_1 I_1 (2b_2 n_2 I_2)\, \pi a_2^2 \cos\alpha_1$$

$$\left\{1 + \frac{\sin^2\alpha_1 \cos^2\alpha_2}{\sqrt{\pi}} \sum_{s=1}^{\infty} (-1)^{s+1} C_s(\alpha_1) C_{s+1}(\alpha_2) \frac{\Gamma\left(s+\frac{3}{2}\right)}{\Gamma(s+1)} \left(\frac{r_2}{r_1}\right)^{2s}\right\} \tag{38}$$

$$= 4\pi n_1 I_1 N_2 I_2 \pi a_2^2 \cos\alpha_1 (1+S)$$

where $N_2 I_2 = 2b_2 n_2 I_2 =$ total current circulating around the inner solenoid, and where

$$S = \sin^2\alpha_1 \cos\alpha_2 \sum_{s=1}^{\infty} (-1)^{s+1} \left(\frac{r_2}{r_1}\right)^{2s} \tag{39}$$

$$C_s(\alpha_1)\, C_{s+1}(\alpha_2) \left(s+\frac{1}{2}\right) \left(\frac{(1.3.5\ldots(2s-1)}{2.4.6\ldots\ldots(2s)}\right)$$

In equation (38) the factor $4\pi n_1 I_1$ represents the uniform field which would exist if the outer solenoid were infinite in length. Hence, the factor $\cos\alpha_1(1+S)$ is the end-correction factor, which reduces to 1 when $\alpha_1 = 0$; that is, when the outer solenoid is infinitely long.

In case this solenoid is 100 cm long and 30 cm in diameter, $\cos\alpha_1 = 0.957826$. If the length and diameter of the inner solenoid is 10 cm, then $S = 0.000054$, so that the end correction factor $\cos\alpha_1(1+S) = 0.957879$, showing that the torque on the inner solenoid is about 4 per cent less than if the outer one were infinitely long. Evidently the small solenoid lies in a field which is practically uniform in this case.

1. CORRECTION FACTOR WHEN INNER SOLENOID IS A SINGLE LAYER HELICAL WIRE OF RADIUS ρ_2

If the inner solenoid (lying in a uniform field $4\pi n_1 I_1$) be not a continuous-current sheet as previously assumed, but consists rather of any number of strips or rings of no cross section, the torque upon it remains the same as before if their radii remain the same and the total current $N_2 I_2$ be the same. The torque on each ring is unaltered by giving it a parallel displacement. Hence, it is evident that in those cases where the inner solenoid lies in an approximately uniform field, but consists of N_2 circular turns of mean radius a_2, whose circular cross section is ρ_2, the factor πa_2^2 which occurs in (38) must be replaced by $\frac{1}{\pi\rho_2^2}\int\int \pi a^2 dS$ where the integration is taken over the circular section of the wire of radius ρ_2 whose center is a distance a_2 from the axis of the solenoid, and where a is the distance of any point in this section from that axis. (This assumes uniform current distribution over the section of the wire.) For a circular section

$$\frac{\pi}{\pi\rho_2^2}\int\int a^2 dS = \pi a_2^2\left(1+\frac{\rho_2^2}{4a_2^2}\right)$$

Hence, when the inner solenoid consists of N_2 parallel circular rings of wire, of mean radius a_2, with wire section of radius ρ_2, it experiences a torque

$$T = 4\pi n_1 I_1 \pi a_2^2 (N_2 I_2)\left(1+\frac{\rho_2^2}{4a_2^2}\right)\cos\alpha_1\,(1+S) \qquad (40)$$

The N_2 parallel turns might be spaced irregularly in any manner without affecting the validity of this formula if the field $4\pi n_1 I_1$ is uniform as assumed. Moreover, since this field is perpendicular to the axis of the small solenoid, this formula for N_2 parallel circles could also be used for a helix of N_2 turns, since the torque due to the axial component of current in the helix could have no component of the type represented by T. Furthermore, if the return lead wires of this helix lie in the vertical plane containing the knife-edge upon which the helix is balanced, they would contribute nothing to the component of torque about that knife-edge.

2. FIELD OF AN ENDLESS HELICAL FILAMENT—UNIFORMITY OF FIELD WITHIN IT

Although the lack of uniformity of the field of the outer solenoid over the space occupied by the inner one, may be small due to the end-correction factor $\cos\alpha_1\,(1+S)$ being very nearly unity, there will be in any actual case another source of nonuniformity; namely, that due to the fact that the outer solenoid is not a current sheet but a helix. It is evident that if the number of turns per cm is large the field may be made very uniform over the region in which the inner coil lies, provided all parts of the latter are sufficiently distant from the windings of the former. What assurance have we that this condition is realized in any actual case, and to such an extent that (40) may be regarded as a precision formula?

The answer may be found in an examination of the field within and near the first solenoid, when it is considered endless though not a current sheet but a helical current filament wound upon a cylinder of radius a_1, whose axis is the x axis. If any point in space P has the cylindrical coordinates x, r, θ (where r is now its distance from the x axis) then the magnetic field at P is the curl of a vector potential A, and its cylindrical components are given by

$$H_x = \frac{1}{r}\left[\frac{\delta}{\delta r}(rA_\theta) - \frac{\delta A_r}{\delta\theta}\right], H_r = \frac{1}{r}\frac{\delta A_x}{\delta\theta} - \frac{\delta A_\theta}{\delta_x}, H_\theta = \frac{\delta A_r}{\delta x} - \frac{\delta A_x}{\delta r} \tag{41}$$

Let the number of turns of the helix be n_1 (not necessarily integral) and let the trace of this helical filament on the upper part of the plane $z=0$, for which $y=+a_1$, be the points whose x coordinates are $-\infty, \ldots\ldots -\frac{2}{n}, -\frac{1}{n}, 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n} \ldots\ldots +\infty$. It may be shown that if $r<a_1$

$$A_x = 2I_1\left\{\log\frac{1}{a_1} + \pi i \sum_{k=1}^{\infty} J_k(2\pi n_1 rki) H_k(2\pi n_1 a_1 ki) \cos k(2\pi n_1 x - \theta)\right\}$$

$$A_r = 4\pi n_1 a_1 I_1\left\{\frac{\pi i}{2}\sum_{k=1}^{\infty}[J_{k+1}(2\pi n_1 rki) H_{k+1}(2\pi n_1 a_1 ki)\right.$$

$$\left. - J_{k-1}(2\pi n_1 rki) H_{k-1}(2\pi n_1 a_1 ki)\ ]\sin k(2\pi n_1 x - \theta)\right\} \tag{42}$$

$$A_\theta = 4\pi n_1 a_1 I_1\left\{\frac{r}{2a_1} + \frac{\pi i}{2}\sum_{k=1}^{\infty}[J_{k+1}(2\pi n_1 rki) H_{k+1}(2\pi n_1 a_1 ki)\right.$$

$$\left. - J_{k-1}(2\pi n_1 rki) H_{k-1}(2\pi n_1 a_1 ki)]\cos k\,(2\pi n_1 x - \theta)\right\}$$

where H_k denotes Hankel's function of the first kind whose asymptotic expansion when x is a large positive real is

$$H_k(ix) \simeq \sqrt{\frac{2}{\pi x}}\, e^{-x-i\frac{\pi}{2}(k+1)}$$

The corresponding expansion for the Bessel's function is

$$iJ_k(ix) \simeq \frac{1}{\sqrt{2\pi x}} e^{x+i\frac{\pi}{2}(k+1)}$$

When the point is outside the helix, $r>a_1$, the corresponding values of A_x, A_r, A_θ are obtainable from (42) by interchanging a_1 and r where they occur inside the braces. These exact expressions may be simplified, and, in fact, the series may be summed, in case the number of turns n_1 per cm of the winding is sufficiently great, and the radius a_1 of the winding so large that $2\pi n_1 a_1$ is large enough to make the foregoing asymptotic expansions valid. In this case (42) reduces in case r is slightly less than a_1 to

$$\begin{aligned} A_x &= 2I_1\left\{\log\frac{1}{a_1} - \frac{\log\psi}{4\pi n_1\sqrt{a_1 r}}\right\} \\ A_r &= 0 \\ A_\theta &= 4\pi n_1 a_1 I_1\left\{\frac{r}{2a_1} - \frac{\log\psi}{4\pi n_1\sqrt{a_1 r}}\right\} \end{aligned} \tag{43}$$

where

$$\psi = 1 - 2e^{-2\pi n_1(a_1-r)}\cos(2\pi n_1 x - \theta) + e^{-4\pi n_1(a_1-r)} \tag{44}$$

For outside points $(r>a_1)$ the quantities a_1 and r must be interchanged in (44) and within the braces of (43). If $e^{-2\pi n_1(a_1-r)}$ is very small one finds from (43) and (44) the approximations for $r<a_1$

$$\begin{aligned} A_x &= 2I_1\left\{\log\frac{1}{a_1} + \frac{e^{-2\pi n_1(a_1-r)}\cos(2\pi n_1 x-\theta)}{2\pi n_1 a_1}\right\} \\ A_r &= 0 \\ A_\theta &= 4\pi n_1 a_1 I_1\left\{\frac{r}{2a_1} + \frac{e^{-2\pi n_1(a_1-r)}\cos(2\pi n_1 x-\theta)}{2\pi n_1 a_1}\right\} \end{aligned}$$

This gives for the field components

$$\begin{aligned} H_x &= 4\pi n_1 I_1\{1 + e^{-2\pi n_1(a_1-r)}\cos(2\pi n_1 x-\theta)\} \\ H_r &= 4\pi n_1 I_1\, e^{-2\pi n_1(a_1-r)}\sin(2\pi n_1 x-\theta) \\ H_\theta &= -\frac{2I_1}{a_1}\, e^{-2\pi n_1(a_1-r)}\cos(2\pi n_1 x-\theta) \end{aligned} \tag{45}$$

In case $n_1 \geqq 3$, $a_1 \geqq 5$ cm, then $2\pi n_1 a_1$ is approximately 100, and the asymptotic expansion may be used to obtain a good approximation. An inspection of (45) shows that the nonuniform component field will be less than one-millionth of the uniform value provided $2\pi n_1(a_1-r) \geqq 15$, which would be the case if $a_1 - r = 1$ cm, so that in this case the assumption that the field is uniform is amply justified.

VII. SUMMARY

The formula (35) gives the mutual inductance of the two concentric solenoids shown in Figure 1, where n_1 and n_2 are the number of turns per cm and μ is $\cos\theta$, θ being the angle between their axes. The formula presumes a strip winding so that the solenoids constitute current sheets in which the axial component of current is neglected. a_1 is the radius of the outer solenoid, $2\ b_1$ is length, and $2\ \alpha_1$ the angle subtended at the center by its end diameter, and $2\ r_1$ the diagonal of a principal section. The inner solenoid is characterized by the similar quantities a_2, b_2, α_2, and r_2.

The torque in the inner solenoid is given by equation (37) in general, and by equation (38) when the axes are perpendicular. The latter expression is modified in equation (40) to take account of the finite cross section and discrete nature of the windings of the inner solenoid. The effect of the discrete nature of the windings of the outer solenoid is investigated in VI, 2, and is found to be negligible for certain relative dimensions of the two solenoids.

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