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METHODS FOR THE DERIVATION AND EXPANSION OF FORMULAS FOR THE MUTUAL INDUCTANCE OF COAXIAL CIRCLES AND FOR THE INDUCTANCE OF SINGLE-LAYER SOLENOIDS

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ABSTRACT

This paper gives a classification of existing inductance formulas for the general cases indicated, and discusses the possiblity of additional formulas of each class. A number of new formulas are developed which can be used to advantage for certain cases. Still other formulas could be obtained, but on account of complexity or poor convergence they are not likely to be useful. It is shown by numerical examples that the inductance in any given case can be calculated by more than one formula and to a degree or precision far beyond practical requirements.

CONTENTS

I. INTRODUCTION

The two most important cases for which formulas for the calculation of inductance have been derived are those of single-layer coils, or solenoids, and coaxial circular filaments; the former because of the extended use of such coils in practice, and the latter because of its importance for the derivation of formulas for more complicated circuits.

Exact formulas in elliptic integrals have long been known for both cases, but to avoid the necessity of recourse to tables of integrals and to attain greater accuracy in numerical calculations, numerous series expansions have been developed by different investigators from time to time. More than a score of formulas for the mutual inductance of coaxial circles are included in the Bureau of Standards collection (2) [1] of inductance formulas and nearly as many for the inductance of solenoids, and others have been recently added to the list. In fact, it is possible to obtain numerical values with any desired accuracy by more than one formula in any given case.

However, the very abundance of formulas has been a source of confusion, rather than an advantage, to one unfamiliar with the subject. This difficulty has been very completely overcome by the tables of Curtis and Sparks (6), Grover (7), and of Nagaoka and Sakurai (10), in all of which the calculation of the inductance is reduced to simple arithmetic, making use of factors which may be taken from the tables by simple interpolation.

But, although for practical purposes the matter is thus simplified, nevertheless the series formulas are useful for the purpose of obtaining formulas for more complex cases by integration, and are of interest because of the variety of methods of expansion which they represent.

It is the purpose of the present paper to attempt a classification of existing formulas, to present useful formulas which have been heretofore overlooked, and to discuss the question of further possibilities. Since the same fundamental theory underlies the treatment of both coaxial circles and solenoids, both cases are included here. The case of coaxial circles will be treated with some detail, so that it will be necessary to refer only to points of difference in the section which deals with the solenoid.

The bibliography appended to the article can hardly claim to include all the work which has been published, but it is hoped that it is sufficiently complete to prove useful.

II. MUTUAL INDUCTANCE OF COAXIAL CIRCLES

1. ELLIPTIC INTEGRAL FORMULAS

Maxwell (16) derived a formula for the mutual inductance of two coaxial circles of radii A and a (fig. 1) with their planes separated by a distance D, by obtaining the Neumann integral taken around the circumferences of both circles. His expression involved complete elliptic integrals to a modulus k, defined in equation (1) below. On this

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[1] The superior given in parentheses here and throughout the text relate to the reference numbers in the bibliography given at the end of this paper.

formula are based nearly all of the known series formulas. A second formula was obtained from this by Maxwell (16) by applying the well-known Landen (11) transformation to reduce the modulus k to a smaller value k_1. By repeated application of the Landen transformation, an unlimited number of elliptic integral formulas might be

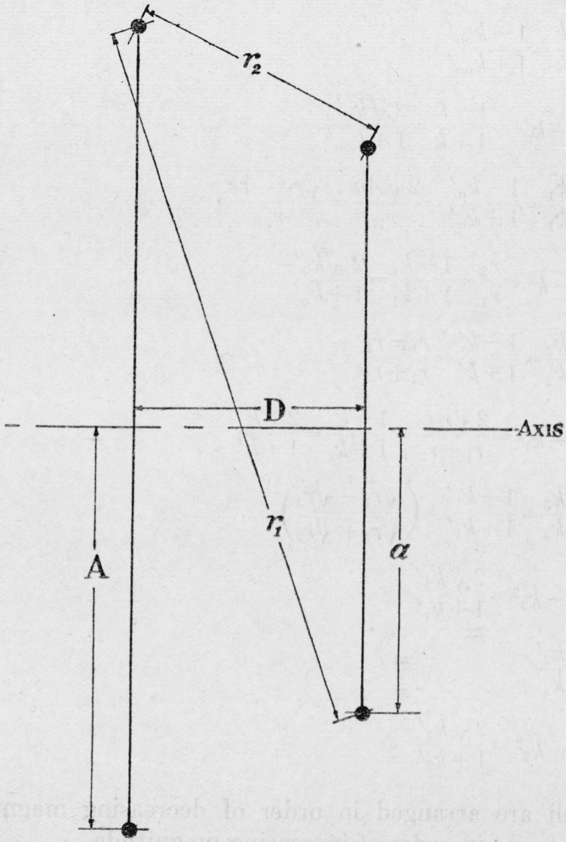

FIG. 1.—*Diagrammatic cross section of two coaxial circles showing dimensions*

obtained with moduli both smaller and greater than those of the Maxwell formulas. The following are the equations connecting the series of moduli obtained from Maxwell's k by use of the Landen transformation. The complementary moduli are, as usual, denoted by primes.

With $r_1=\sqrt{(A+a)^2+D^2}$, $r_2=\sqrt{(A-a)^2+D^2}$,

$$
\begin{aligned}
k_{oo} &= \frac{2\sqrt{k_o}}{1+k_o} \\
k_{oo}' &= \sqrt{1-k_{oo}{}^2} = \frac{1-k_o}{1+k_o} \\
k_o &= \frac{2\sqrt{k}}{1+k} = \frac{1-k_{oo}'}{1+k_{oo}'} \\
k_o' &= \sqrt{1-k_o{}^2} = \frac{1-k}{1+k} = \frac{2\sqrt{k_o{}'_o}}{1+k_{oo}'} \\
k &= \frac{2\sqrt{k_1}}{1+k_1} = \frac{1-k_o'}{1+k_o'} = \frac{2\sqrt{Aa}}{r_1} = \frac{\sqrt{r_1{}^2-r_2{}^2}}{r_1}, \\
k' &= \sqrt{1-k^2} = \frac{r_2}{r_1} = \frac{1-k_1}{1+k_1} = \frac{2\sqrt{k_o'}}{1+k_o'} \qquad (1) \\
k_1 &= \frac{2\sqrt{k_2}}{1+k_2} = \frac{1-k'}{1+k'} = \frac{r_1-r_2}{r_1+r_2} \\
k_1' &= \sqrt{1-k_1{}^2} = \frac{2\sqrt{r_1r_2}}{r_1+r_2} = \frac{1-k_2}{1+k_2} = \frac{2\sqrt{k'}}{1+k'} \\
k_2 &= \frac{2\sqrt{k_3}}{1+k_3} = \frac{1-k_1'}{1+k_1'} = \left(\frac{\sqrt{r_1}-\sqrt{r_2}}{\sqrt{r_1}+\sqrt{r_2}}\right)^2 \\
k_2' &= \sqrt{1-k_2{}^2} = \frac{2\sqrt{k_1'}}{1+k_1'} \\
k_3 &= \frac{1-k_2'}{1+k_2'} \\
k_3' &= \sqrt{1-k_3{}^2} = \frac{2\sqrt{k_2'}}{1+k_2'}
\end{aligned}
$$

The moduli are arranged in order of decreasing magnitude, their complementaries in order of increasing magnitude.

There follows a list of elliptic integral formulas for the mutual inductance thus derived in which are included for completeness the Maxwell formulas and the formula in k_o also previously published. It is believed that there are included here all of the formulas which are likely to be of use for purposes of expansion. Here, as well as in what follows, formulas occurring in the Bureau of Standards collection will be designated by the numbers there given them with the prefix I for Scientific Paper 169 (2) and II for Scientific Paper 320 (3). Thus the Maxwell formulas are I (1) and I (2). Formulas believed to be new will be designated by an asterisk.

$$M=\pi\sqrt{Aa}\,\frac{(1+\sqrt{k_{oo}'})^2}{(1-k_{oo}')^2}\left[\{(1+\sqrt{k_{oo}'})^4-8k_{oo}'\}K_{oo}'-8E_{oo}\right] \qquad * (2)$$

$$M=4\pi\sqrt{Aa}\left[\frac{K_o}{k(1+k)}-\frac{(1+k)}{k}E_o\right] \qquad \mathrm{I}\ (4)$$

$$=\frac{2\pi\sqrt{Aa}}{1-k_o'}\left[(1+k_o')^2K_o-4E_o\right] \qquad \mathrm{II}\ (7A)$$

$$M=4\pi\sqrt{Aa}\left[\left(\frac{2}{k}-k\right)K-\frac{2}{k}E\right] \qquad \mathrm{I}\ (1)$$

$$M=\frac{8\pi\sqrt{Aa}}{\sqrt{k_1}}\left[K_1-E_1\right] \qquad \mathrm{I}\ (2)$$

$$M=\frac{16\pi\sqrt{Aa}}{\sqrt{2}k_2^{1/4}\sqrt{1+k_2}}\left[(1+k_2)K_2-E_2\right] \qquad * (3)$$

$$M=\frac{16\pi\sqrt{Aa}(1+k_3)^{3/4}}{2\frac{3}{4}k_3^{1/8}(1+\sqrt{k^3})}\left[(1+\sqrt{k_3})K_3-\frac{E_3}{1+k_3}\right] \qquad * (4)$$

In each formula K and E denote, respectively, the complete elliptic integrals, the modulus being in each case indicated by the subscript.

Unfortunately, the elliptic integral formulas are not well suited to numerical calculations, even when accurate tables of elliptic integrals are at hand. For small moduli, the mutual inductance is obtained as the small difference of two much larger terms. For large moduli, it is impossible to obtain the value of K accurately by interpolation from the tables. This is illustrated by the following example for two circles such that $k=\frac{\sqrt{2}}{2}$.

The two elliptic integral terms of the forumula are given for the modulus k and for the larger and smaller moduli k_o and k_1.

Modulus k_o	5. 08976
	−4. 87061
	. 21915
Modulus k	2. 781112
	−2. 701288
	. 079824
Modulus k_1	1. 582566
	−1. 559160
	. 023406

For the still smaller modulus, k_2 the difference is still more difficult to determine. With the modulus k_{oo}, which is larger than k_o, the difference between the terms is more favorable than in any of the cases shown, but the elliptic integral K_{oo} has to be found for the angle 89° 34′ 20″, and can not, therefore, be obtained by interpolation from the tables.

These difficulties may be avoided by series expansions of the elliptic integral formulas, and of the possible types of expansion three are important, viz, hypergeometric, arithmetico-geometric mean series, and the q series of Jacobi.

2. HYPERGEOMETRIC SERIES EXPANSIONS

Although existing formulas of this type have been derived by various methods, two principal methods may be regarded as especially useful.

1. The elliptic integrals satisfy differential equations of the hypergeometric type, so that each of the above elliptic integral expressions yields a different hypergeometric differential equation for the mutual inductance. By thus obtaining the equation corresponding to the Maxwell equation I (1), Butterworth [20] obtained five series expansions for the mutual inductance, four of them being essentially different. In general four such series could be found for each of the elliptic integral formulas by the use of this method.

2. A simpler procedure is to substitute the known hypergeometric series expansions for the elliptic integrals [15] directly into the formulas for the mutual inductance. This procedure leads to the same formulas as the previous method. The four expansions derived from each elliptic integral formula are in powers of the corresponding modulus, the complementary modulus, their ratio, and its reciprocal.

As already pointed out the expansions of I (1) are all known. The series in k is I (5), which was derived by the writer in 1910 and was later found by Butterworth [20]. The writer has discovered only recently that this formula was first given by Weinstein [17], a fact which seems to have been generally overlooked. That in k' was first obtained [17] by Weinstein I (7) and was given by Butterworth in the somewhat different form II (4A). The formula II (3A), also given by Butterworth, is readily obtained from the former by a slight transformation. The series in $\frac{k'}{k}$ was first derived [18] by Havelock I (16) and later by Butterworth, in its general form II (5A), while the expression in $\frac{k}{k'}$, due to Butterworth, is given as II (1A).

Only one of the expansions of the second Maxwell formula I (2) has been published, that which appears as I (6) and involves powers of k_1. This was derived by the writer in 1910 and by Nagaoka [28] at about the same time. The other three of the possible expansions which follow are of practical interest and were derived by the use of the second method.

Writing $\mu_1 = \frac{k_1}{k_1'} = \frac{r_1 - r_2}{2\sqrt{r_1 r_2}}$,

$$M = \frac{2\pi^2 \mu_1^{\frac{3}{2}} \sqrt{Aa}}{(1+\mu_1^2)^{\frac{1}{4}}} \Big[1 - \frac{\mu_1^2}{8} + \frac{3}{64}\mu_1^4 - \frac{25}{1024}\mu_1^6 + \frac{245}{16384}\mu_1^8 \cdots$$

$$+ (-)^{n-1} \left\{ \frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)} \right\}^2 \frac{\mu_1^{2n}}{n} \Big] \qquad *(5)$$

This formula converges only for $k_1^2 < \frac{1}{2}$

The expression in terms of $k_1' = \frac{2\sqrt{r_1 r_2}}{r_1 + r_2}$ is

$$M = \frac{8\pi\sqrt{Aa}}{\sqrt{k_1}} \Big[\left(\log\frac{4}{k_1'} - 1 \right) - \frac{1}{4} k_1'^2 \log\frac{4}{k_1'} - \frac{3}{64} k_1'^4 \left(\log\frac{4}{k_1'} - \frac{5}{6} \right)$$

$$- \frac{5}{256} k_1'^6 \left(\log\frac{4}{k_1'} - \frac{31}{30} \right) - \frac{175}{16384} k_1'^8 \left(\log\frac{4}{k_1'} - \frac{473}{420} \right) \cdots \Big] \qquad *(6)$$

The general term is $-\left[\frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n} \right]^2 \frac{k_1'^{2n}}{2n-1} \varphi_n$

where

$$\varphi_0 = \log\frac{4}{k_1'} - 1, \ \varphi_1 - \varphi_0 = 1, \ \varphi_2 - \varphi_1 = -\frac{5}{6}$$

$$\varphi_n - \varphi_{n-1} = \frac{1}{n} - \frac{1}{2n-3} - \frac{1}{2n-1} = -\frac{(4n-3)}{n(2n-3)(2n-1)}.$$

The formula (6) converges for circles at all distances, but especially well for small values of $\frac{r_2}{r_1}$; that is, for circles near together.

The expansion in powers of $\nu_1 = \frac{1}{\mu_1} = \frac{2\sqrt{r_1 r_2}}{r_1 - r_2}$ is closely related to this. It reads

$$M = \frac{8\pi\sqrt{Aa}}{(1+\nu_1^2)^{\frac{1}{4}}} \Big[\left(\log\frac{4}{\nu_1} - 1 \right) + \frac{\nu_1^2}{4} \log\frac{4}{\nu_1} - \frac{3}{64}\nu_1^4 \left(\log\frac{4}{\nu_1} - \frac{5}{6} \right)$$

$$+ \frac{5}{256}\nu_1^6 \left(\log\frac{4}{\nu_1} - \frac{31}{30} \right) - \cdots \Big] \qquad *(7)$$

the general term being $(-)^{n+1} \left[\frac{1.3.5\cdots(2n-1)}{2.4.6.\cdots 2n} \right]^2 \frac{\nu_1^{2n}}{2n-1} \psi_n$, where ψ_n is the same as φ_n except that ν_1 occurs in place of k_1'.

This formula converges only when $\mu_1 > 1$.

None of the expansions of the elliptic integral formula (3) have been published. All are readily obtained, but from the standpoint of convergence only those in k_2 and $\frac{k_2}{k_2'}$ are likely to be of practical use, and these only will be here given.

The expansion in terms of the modulus $k_2 = \left(\frac{\sqrt{r_1} - \sqrt{r_2}}{\sqrt{r_1} + \sqrt{r_2}}\right)^2$ is

$$M = \frac{8\pi^2 k_2^{\frac{3}{4}} \sqrt{Aa}}{\sqrt{2}\sqrt{1+k_2}} \left[1 + \frac{k_2^2}{4} + \frac{9}{64} k_2^4 + \cdot\cdot\cdot + \left\{ \frac{1.3.5 \cdot\cdot (2n-1)}{2.4.6 \cdots 2n} \right\}^2 k_2^{2n} \right.$$
$$\left. + \frac{k_2}{2} \left(1 + \frac{3}{8} k_2^2 + \frac{15}{64} k_2^4 + \cdot\cdot + \frac{2n+1}{n+1} \left\{ \frac{1.3.5 \cdot\cdot (2n-1)}{2.4.6 \cdots 2n} \right\}^2 k_2^{2n} \right) \right] \quad *(8)$$

and placing $\frac{k_2}{k_2'} = \mu_2$, and remembering that $k_2 = \frac{\mu_2}{\sqrt{1+\mu_2^2}}$,

$$M = \frac{8\pi^2 \mu_2 \sqrt{Aa}}{\sqrt{2}\, k_2^{\frac{1}{4}} \sqrt{1+k_2}} \left[1 - \frac{1}{4} \mu_2^2 + \frac{9}{64} \mu_2^4 - \cdots + (-1)^n A_n \mu_2^{2n} \right.$$
$$\left. + \frac{k_2}{2} \left(1 - \frac{1}{8} \mu_2^2 + \frac{3}{64} \mu_2^4 - \cdots + (-)^n \frac{A_n}{n+1} \mu_2^{2n} \right) \right] \quad *(9)$$

in which

$$A_n = \left[\frac{1.3.5 \cdot\cdot (2n-1)}{2.4.6 \cdot\cdot 2n} \right]^2$$

Formula (9) converges only for $k_2^2 < \frac{1}{2}$, but (8) converges for circles at all distances, but especially for those where r_2 is nearly equal to r_1 (distant circles).

The expansion of the elliptic integral formula II (7A) in terms of k'_o was derived by the author [2] and published as II (8A), but the general term was not given. The complete expression may be written as

$$M = \frac{2\pi\sqrt{Aa}}{k_o^2} \left[\left(\log \frac{4}{k'_o} - 4 \right) + 3k'_o \left(\log \frac{4}{k'_o} - \frac{4}{3} \right) \right.$$
$$+ \frac{5}{4} k_o'^2 \left(\log \frac{4}{k'_o} + \frac{3}{5} \right) - \frac{k_o'^3}{4} \left(\log \frac{4}{k'_o} - 1 \right)$$
$$+ \frac{9}{64} k_o'^4 \left(\log \frac{4}{k'_o} - \frac{13}{18} \right) - \frac{5}{64} k_o'^5 \left(\log \frac{4}{k'_o} - \frac{9}{10} \right) \quad *(10)$$
$$\left. + \frac{13}{256} k_o'^6 \left(\log \frac{4}{k'_o} - \frac{77}{78} \right) - \frac{9}{256} k_o'^7 \left(\log \frac{4}{k'_o} - \frac{19}{18} \right) + \cdots \right]$$

The general term for the even powers of k'_o is

$$\left[\frac{1.3.5\cdots(2n-3)}{2.4.6\cdots 2n}\right]^2 (4n+1)\, k'^{\,2n}_o\, \varphi'_{2n}$$

where

$$\varphi'_o = \log\frac{4}{k'_o} - 4 \quad , \quad \varphi'_2 - \varphi'_o = \frac{23}{5}$$

$$\varphi'_{2n} - \varphi'_{2n-2} = \frac{2}{4n-3} - \frac{2}{4n+1} + \frac{1}{n} - \frac{2}{2n-3}, = -\frac{32n^2-9}{n\,(2n-3)\,(4n-3)\,(4n+1)}$$

and for the odd powers the general term is

$$-(4n-3)\left[\frac{1.3\cdots(2n-3)}{2.4\cdots 2n}\right]^2 k_o'^{\,2n+1}\, \varphi'_{2n+1}$$

in which

$$\varphi'_1 = \log\frac{4}{k'_o} - \frac{4}{3} \quad , \quad \varphi'_3 - \varphi'_1 = \frac{1}{3}$$

$$\varphi'_{2n+1} - \varphi'_{2n-1} = \frac{1}{n} - \frac{2}{2n-3} + \frac{2}{4n-7} - \frac{2}{4n-3} = -\frac{(32n^2-96n+63)}{n\,(2n-3)\,(4n-3)\,(4n-7)}$$

This formula converges for all values of k'_o, but especially well when this quantity is small; that is, when the circles are far apart.

Of the remaining three possible expansions of II (7A) that in $\nu_o = \frac{k'_o}{k_o}$ only is likely to prove itself useful. It reads

$$\begin{aligned} M = 2\pi\sqrt{Aa}\sqrt{1+\nu_o^2}\Bigg[&\left(\log\frac{4}{\nu_o} - 4\right) + \frac{7}{4}\nu_o^2\left(\log\frac{4}{\nu_o} - \frac{3}{7}\right) \\ &- \frac{39}{64}\nu_o^4\left(\log\frac{4}{\nu_o} - \frac{83}{78}\right) + \frac{95}{256}\nu_o^6\left(\log\frac{4}{\nu_o} - \frac{679}{570}\right) - \cdots \\ &+ (-)^{n+1} A_n \frac{(6n+1)}{(2n-1)}\nu_o^{2n}\psi'_{2n} + 3k'_o\Bigg\{\left(\log\frac{4}{\nu_o} - \frac{4}{3}\right) \\ &+ \frac{5}{12}\nu_o^2\left(\log\frac{4}{\nu_o} - \frac{1}{5}\right) - \frac{7}{64}\nu_o^4\left(\log\frac{4}{\nu_o} - \frac{41}{42}\right) \\ &+ \frac{15}{256}\nu_o^6\left(\log\frac{4}{\nu_o} - \frac{103}{90}\right) - \cdots + (-)^{n+1}\frac{A_n}{3}\frac{(2n+3)}{(2n-1)}\nu_o^{2n}\psi''_{2n}\Bigg\}\Bigg] \end{aligned} \qquad *(11)$$

in which A_n has the same significance as in formula (9) and

$$\psi'_o = \log\frac{4}{\nu_o} - 4 \quad , \quad \psi'_2 - \psi'_o = \frac{25}{7}$$

$$\psi'_{2n} - \psi'_{2n-2} = \frac{1}{n} - \frac{1}{2n-3} - \frac{1}{2n-1} - \frac{3}{6n+1} + \frac{3}{6n-5}$$

and

$$\psi''_o = \log\frac{4}{\nu_o} - \frac{4}{3} \quad , \quad \psi''_2 - \psi''_o = \frac{17}{15}$$

$$\psi''_{2n} - \psi''_{2n-2} = \frac{1}{n} - \frac{1}{2n-3} - \frac{1}{2n+3} - \frac{1}{2n-1} + \frac{1}{2n+1} = \frac{1}{n} - \frac{4n}{4n^2-9} - \frac{2}{4n^2-1}$$

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Formula (11) converges only for $k'^2_0 < \frac{1}{2}$.

3. SERIES INVOLVING THE ARITHMETICO-GEOMETRICAL MEAN

The use of scales of arithmetico-geometric means for the calculation of the elliptic integrals was established by the work of Lagrange, Legendre, and Gauss. This method has been recently employed by L. V. King [25] for the derivation of formulas for the mutual inductance of coaxial circles. King's paper gives full references to the historical development of the subject.

The formation of the series of arithmetical and geometric means is accomplished according to the following scheme:

$$\begin{array}{lll}
a_0 & b_0 & c_0 = \sqrt{a_0^2 - b_0^2} \\
a_1 = \frac{1}{2}(a_0 + b_0) & b_1 = \sqrt{a_0 b_0} & c_1 = \frac{1}{2}(a_0 - b_0) \\
a_2 = \frac{1}{2}(a_1 + b_1) & b_2 = \sqrt{a_1 b_1} & c_2 = \frac{1}{2}(a_1 - b_1)
\end{array}$$

It is found that, even when a_0 and b_0 are far from equal, the successive values of the a's and b's converge with extreme rapidity toward the same value, so that for small values of n, a_n, and b_n differ by an amount which is negligible in numerical calculations, and the quantity c_n differs from zero by an amount that is insensible. In what follows we will denote by a_n the limit which is approached by the arithmetical and geometric means, as was done by King.

If, as special values we choose $a_0 = 1$, and $b_0 = k'$, the relation $c_n = \sqrt{a_n^2 - b_n^2}$ shows that $c_0 = k$; that is, c_0 and b_0 are complementary moduli. It may be shown (13) that the elliptic integrals are given by the relations.

$$\begin{aligned}
K(k) &= \frac{\pi}{2}\frac{1}{a_n} \\
\frac{K(k) - E(k)}{K(k)} &= \frac{1}{2}(c_0^2 + 2c_1^2 + 4c_2^2 + \cdots + 2^n c_n^2 + \cdots)
\end{aligned} \tag{12}$$

King has shown that if, further, the convergent of the complementary arithmetico-geometric mean (a. g. m.) series, $a'_0 = 1$, $b'_0 = k$, be denoted by a'_n, the equations for the elliptical integrals are also given by the formulas

$$\begin{aligned}
K(k) &= \frac{Q}{2a'_n} \\
E(k) &= \frac{Q}{2a'_n}\left(\frac{1}{2}c'^2_0 + c'^2_1 + 2c'^2_2 + \cdots + 2^{n-1}c'^2_n\right)
\end{aligned} \tag{13}$$

in which

$$Q = \log\frac{4a'_1}{c'_1} - \log\frac{a'_1}{a'_2} - \frac{1}{2}\log\frac{a'_2}{a'_3} - \cdots - \frac{1}{2^{n-1}}\log\frac{a'_n}{a'_{n+1}} \tag{13a}$$

Making use of the relations (12) and (13), two distinct formulas may be derived for the mutual inductance corresponding to each of the elliptic integral formulas. In general, those derived from (13) are less suited to numerical calculations than what may be termed the direct series, derived from (12). Since, in practical cases, the a. g. m. series converge so rapidly that only three or four steps of the series have to be calculated for a six-figure accuracy, the advantage of a change of modulus to gain convergence is not here of such importance as is the case with hypergeometric developments. It does not, therefore, seem worth while to present the complete list of formulas which have been found by this method. However, by a different choice of a. g. m. series King has developed formulas for the mutual inductance which are of great interest.

Adopting the nomenclature (a_o, b_o) to denote an a. g. m. series, it is readily seen that if ϵ be a constant, the series $(\epsilon a_o, \epsilon b_o)$ converges to a value ϵ times as great as the first series. Remembering that for coaxial circles the modulus $k'=\frac{r_2}{r_1}$ it is evident that the a. g. m. series (r_1, r_2) gives a convergent which is r_1 times as great as the value a_n corresponding to the series $(1, k')$. Thus from (12) King derived [25] for the mutual inductance the elegant formula

$$M=\frac{2\pi^2}{a_n}(c_1{}^2+2c_2{}^2+4c_3{}^2+\cdots+2^{n-1}c_n{}^2) \tag{14}$$

based on the series (r_1, r_2).

For purposes of tabulation it is an advantage to write formula (14) so that the modulus appears explicitly. The equation then reads

$$\frac{M}{4\pi\sqrt{Aa}}=\frac{\pi}{ka_n}(c_1{}^2+2c_2{}^2+4c_3{}^2+\cdots+2^{n-1}c_n{}^2+\cdots) \qquad {}^*(15)$$

where the c's and a_n are now based on the a. g. m. series $(1, k')$.

King pointed out that formula (14) may be derived from the second Maxwell formula I (2) as well as from I (1), and it is to be expected that it can be obtained from each of the elliptic integral formulas for the mutual inductance, a fact which it is not difficult to verify. This suggests however, that the a. g. m. series for the different moduli must be closely related. That this is so may be shown as follows:

Suppose that for the a. g. m. series $(1, k')$ we write $a_o=1$, $b_o=k'$ $=\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$ and, therefore, $c_o=k=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$. Then choosing the multiplier $\epsilon=\sqrt{\alpha^2+\beta^2}$, we have another a. g. m. series

$$A_o=\sqrt{\alpha^2+\beta^2} \qquad B_o=\beta \qquad C_o=\alpha$$

$$A_1=\frac{\sqrt{\alpha^2+\beta^2}+\beta}{2} \qquad B_1=\sqrt{\beta\sqrt{\alpha^2+\beta^2}} \qquad C_1=\frac{\sqrt{\alpha^2+\beta^2}-\beta}{2}, \text{ etc.}$$

whose limit A_n is related to that of the first series a_n by the equation $A_n = a_n\sqrt{\alpha^2+\beta^2}$.

The Landen transformation to the modulus $k_1 < k$ is accomplished by means of the equations

$$k_1 = \frac{1-k'}{1+k'},\ k'_1 = \frac{2\sqrt{k'}}{1+k'}.$$

We may, however, write $k_1 = \dfrac{\alpha_1}{\sqrt{\alpha_1{}^2+\beta_1{}^2}}$ and $k'_1 = \dfrac{\beta_1}{\sqrt{\alpha_1{}^2+\beta_1{}^2}}$ and it appears that

$$\alpha_1 = (\sqrt{\alpha^2+\beta^2}-\beta),\quad \beta_1 = 2\sqrt{\beta\sqrt{\alpha^2+\alpha^2}},\quad \sqrt{\alpha_1{}^2+\beta_1{}^2} = (\sqrt{\alpha^2+\beta^2}+\beta).$$

The multiplier $\dfrac{\sqrt{\alpha_1{}^2+\beta_1{}^2}}{2}$ changes the series $(1, k'_1)$ into

$${}_1A_o = \frac{\sqrt{\alpha_1{}^2+\beta_1{}^2}}{2} = \frac{\sqrt{\alpha^2+\beta^2}+\beta}{2},\ {}_1\mathrm{B}_o = \frac{k'_1}{2}\sqrt{\alpha_1{}^2+\beta_1{}^2}, = \sqrt{\beta\sqrt{\alpha^2+\beta^2}},$$

$${}_1C_o = k_1\frac{\sqrt{\alpha_1{}^2+\beta_1{}^2}}{2} = \frac{\sqrt{\alpha^2+\beta^2}-\beta}{2}$$

whose limit is $\dfrac{\sqrt{\alpha_1{}^2+\beta_1{}^2}}{2}$ times that of the series $(1, k'_1)$.

Comparing the two series for the different moduli, we note that ${}_1A_o = A_1$, ${}_1B_o = B_1$, ${}_1C_o = C_1$, so that they will converge to the same limit. Thus it is proved that *the a. g. m. series for two successive moduli in the Landen series differ only by a constant multiplier in their limiting values,* although by the transformation the whole series is displaced one step, so that the initial terms of one series are the second terms of the other, and so on.

Formulas (14) and (15) cover adequately the cases of all coaxial circles except those which are very close together. For such cases King obtained on substituting (13) into the Maxwell formula I (2), the following formula based on the a. g. m. series $[(r_1+r_2), (r_1-r_2)]$

$$M = 2\pi a'_n\left[\frac{(r_1{}^2 - 2c'_2{}^2 - 4c'_3{}^2 - \cdots)\,Q}{a_n'^2} - 2\right] \tag{16}$$

which, written in terms of the modulus, reads

$$\frac{M}{4\pi\sqrt{Aa}} = \frac{a'_n}{k}\left[\frac{Q}{a'_n{}^2}\left(\frac{1}{2} - c'_1{}^2 - 2c'_2{}^2 - 4c'_3{}^2 - \cdots\right) - 2\right] \tag{17}$$

the a. g. m. series being $(1, k)$.

It is to be noted that the series (r_1, r_2) employed in (14) and the series $[(r_1+r_2), (r_1-r_2)]$ of (16) converge to the same limit.

The formula (16) may be obtained from any of the elliptic integral formulas, and it is easy to prove that in stepping down to a smaller Landen *complementary* modulus the same simple relations are found to exist between the complementary a. g. m. series as was found for the Landen transformation to a smaller modulus. In addition, the quantity Q, for the smaller complementary modulus, may be shown to be exactly twice as great as the corresponding quantity for the next larger complementary modulus of the Landen series.

In the light of these facts it is thus evident that the two King formulas completely cover the calculation of the mutual inductance of coaxial circles, and that any other a. g. m. series formulas which may be derived by Landen's transformation are not essentially different.

Example 1.—The most difficult cases of coaxial circles to calculate are those where the relation between the radii and the distance between the planes of the circles is such as to make the modulus $k=\dfrac{2\sqrt{Aa}}{\sqrt{(A+a)^2+D^2}}$ and its complementary modulus $k'=\dfrac{\sqrt{(A-a)^2+D^2}}{\sqrt{(A+a)^2+D^2}}$ nearly equal. For the case $k=k'=\dfrac{\sqrt{2}}{2}$ formula II (4A) in k' requires 11 terms of the series to be included in order to give a seven-figure accuracy, and the formula I (5) in k is more unfavorable still.

For this case the expansions in terms of other moduli are, therefore, useful. The values of other moduli of the Landen series are as follows:

$$k_o=0.985171=k'_1$$
$$k=0.707107=k'$$
$$k_1=0.171573=k'_o$$
$$k_2=0.0074697$$

and in addition

$$\mu_1=0.174155=\nu_o$$
$$\mu_2=0.0074699$$

The following summary shows the results by the available formulas exclusive of those in k and k'.

Formula	Modulus	Number of terms	$\dfrac{M}{4\pi\sqrt{Aa}}$
I (6)	k_1	4	0.1128885_6
(5)	μ_1	4	$.1128885_2$
(8)	k_2	2	$.1128885_8$
(9)	μ_2	2	$.1128885_6$
(10)	k'_o	9	$.1128883_9$
(11)	ν_o	12	$.1128883_9$

The formulas involving the complementary moduli are here not so favorable as what may be called the direct formulas. The agreement of the latter among themselves and with the known value [28] 0.112888542 is as good as the accuracy of seven-place logarithms will allow.

The a. g. m. series formula (15) gives for this case $a_n = a_3 = 0.8472131$, $(c_1{}^2 + 2c_2{}^2 + 4c_3{}^2 + \cdots) = 0.02152671$, and $\dfrac{M}{4\pi\sqrt{Aa}} = 0.11288854$.

Example 2.—For the circles $a = A = 25$, $D = 1$, only the formulas involving complementary moduli are sufficiently convergent. The following summary shows the results obtained with the available formulas.

Formula	Modulus	Value of modulus	Number of terms	M
(10)	k'_o	0. 000099980	2	1, 036. 664_9
11)	ν_o	. 000099980	2	1, 036. 664_9
II (4A)	k'	. 0199960	2	1, 036. 664_9
II (5A)	ν	. 02	3	1, 036. 664_9
(6)	k'_1	. 277270	6	1, 036. 664_6
(7)	ν_1	. 288585	6	1, 036. 664_9

King has solved this problem by his complementary formula, a calculating machine having been used. The result, correct to eight significant figures, he finds to be $M = 1{,}036.6648_6$.

4. EXPANSIONS IN q SERIES

The expansions of the elliptic integrals in q series are well known. In their amazingly rapid convergence they resemble the a. g. m. series formulas, with which they are closely connected. Two formulas for the mutual inductance of coaxial circles in q series were obtained by Nagaoka [26] in 1903, by expressing the integral which gives the mutual inductance, in terms of the Weierstrass elliptic functions, and thence in terms of θ functions, which are readily expanded in q series. The same formulas are obtained if the q series expansions for the elliptic integrals [14] are substituted in the Maxwell formula I (1).

No essentially new expressions are obtained, if the same methods are applied to the other elliptic integral formulas for the mutual inductance. Each change of modulus, according to the Landen transformation merely gives a new value of q, which for the smaller modulus is the square of the value corresponding to the next larger modulus. This may be proved from the relation between q and the modulus, or by noting that

$$\log\frac{1}{q} = Q \quad \text{a. g. m. series } (1, k)$$

where Q is defined by equation (13a). As already stated, the Landen transformation to the next larger modulus doubles Q; that is, it changes q to a new value q_o, such that $q_o = \sqrt{q}$. The formula derived by Nagaoka [27], in 1911, from the Maxwell expression I (2), (modulus k_1) clearly illustrates this point, which is also discussed on page 188 of Scientific Paper, No. 169.

Thus the two q series expressions of Nagaoka, formulas I (8) and I (9), are the only ones possible for the mutual inductance of coaxial circles.

5. EXPANSIONS IN WHICH THE RADII OF THE CIRCLES AND THEIR SPACING APPEAR DIRECTLY

Series for the mutual inductance of coaxial circles in which their radii and the distance between their planes appear directly, although not as convergent as those in terms of the moduli of the elliptic integrals, are especially useful for purposes of integration. Those which have been published have been obtained by various methods. It is, however, possible to obtain them directly and with no particular difficulty, by substituting in some one of the hypergeometric series expansions the relation between the modulus which there appears and the radii and spacing. The same expression may, of course, be obtained from a number of the formulas, but the work is much simpler in some cases than in others.

The Maxwell [21] expression I (10), and its extensions by Rosa and Cohen [23], formula I (14), and by Coffin [24], formula II (6A) give the mutual inductance in terms of $\frac{a}{A}$ and $\frac{c}{A}=\frac{A-a}{A}$. This formula may be obtained from any of the hypergeometric series involving the complementary moduli k'_0, k', or k'_1.

The Havelock formula I (17) in $\frac{a}{A}$ and $\frac{A}{D}$, may most readily be obtained from the formula II (1A) which involves $\mu^2=\frac{4Aa}{(A-a)^2+D^2}$.

In the Havelock [18] formula I (16), which was generalized by Butterworth [20], II (5A), the three parameters $\frac{a}{A}$, $\frac{c}{A}$, and $\frac{D}{a}$ all appear directly in the variable $\nu^2=\frac{k'^2}{k^2}=\frac{1}{\mu^2}$.

A further possibility of expansion in terms of $\frac{a}{A}$ and $\frac{D}{A}$ seems to have been previously overlooked. We find without difficulty from either the formula I (5) in k, or I (6) in k_1, the expression

$$M=\frac{2\pi^2a^2}{A}\left[1+\frac{3}{8}\frac{a^2}{A^2}+\frac{15}{64}\frac{a^4}{A^4}+\frac{175}{1024}\frac{a^6}{A^6}+\cdots+\frac{(\underline{|2n})^2\,(2n+1)\left(\frac{a}{A}\right)^{2n}}{2^{4n}\,(\underline{|n})^4\,(n+1)}\right.$$
$$-\frac{3}{2}\frac{D^2}{A^2}+\frac{15}{8}\frac{D^4}{A^4}-\frac{35}{16}\frac{D^6}{A^6}+\cdots$$
$$\left.-\frac{45}{16}\frac{a^2D^2}{A^4}-\frac{525}{128}\frac{a^4D^2}{A^6}+\frac{525}{64}\frac{a^2D^4}{A^6}-\cdots\right] \quad *(18)$$

This formula converges well only in case $\frac{D}{A}$ is small, (distant circles), unless one radius is very much smaller than the other. For the special case of coplanar circles, $D=0$, and there results the remarkably simple expression found by Curtis [6].

An expansion of the mutual inductance of circles whose axes meet at an angle was given by Maxwell in a series of zonal harmonic terms. This can readily be adapted to the coaxial case, but on account of its poor convergence, in general, it is not suitable for numerical calculations. For the coplanar case the convergence is better, and it may be shown that the formula is the same as that of Curtis.

III. INDUCTANCE OF SINGLE-LAYER SOLENOIDS

Formulas for the calculation of the inductance of a cylindrical current sheet will be here treated. Methods for correcting this case to obtain the inductance of an actual single-layer winding of round cross section are well known.[37] The recent formula of Snow [38] stands alone in that it takes into account the helicity of an actual coil. His formula is of greater accuracy than any other known formula.

The classification of formulas for the inductance of a cylindrical current sheet follows quite the same lines as that for the mutual inductance of coaxial circles, so that what follows may well be read in conjunction with the corresponding portions of the preceding portion of the paper. For routine calculations the tables of Grover,[9] or Nagaoka and Sakurai [10] should be used.

1. ELLIPTIC INTEGRAL FORMULAS FOR SOLENOIDS

In what follows let

a = the radius of the solenoid,

n = the number of turns,

b = the axial length = n times the pitch of the winding.

The only elliptic integral formula for the inductance of a cylindrical current sheet which has been published was obtained by Lorenz [31] in 1879. Others may be derived using the Landen transformation. The following list includes with Lorenz's formula I (72) others which have been found useful. Writing $L_1 = 4\pi^2 n^2 \frac{a^2}{b}$, the value of the inductance of an infinitely long solenoid, the formulas for finite coils may be written to show the expression for the factor which takes into account the effect of the ends.

Arranged in order of the values of the moduli, the formula in terms of

$$k_1 = \frac{4a^2}{(\sqrt{4a^2+b^2}+b)^2} = \frac{(\sqrt{4a^2+b^2}-b)^2}{4a^2} \qquad * \,(19)$$

is

$$L = L_1 \cdot \frac{4}{3\pi}\left(\frac{2a}{b}\right)\left[\frac{(1-k_1^2)(1-3k_1)}{4k_1\sqrt{k_1}}K(k_1) - \frac{(1-6k_1+k_1^2)}{4k_1\sqrt{k_1}}E(k_1) - 1\right]$$

The modulus k of Lorenz's formula is $k=\frac{2a}{\sqrt{4a^2+b^2}}$, and the inductance is given by

$$L=L_1\cdot\frac{4}{3\pi}\left(\frac{2a}{b}\right)\left[\frac{2k^2-1}{k^3}E(k)+\frac{1-k^2}{k^3}K(k)-1\right] \qquad \text{I (72)}$$

The modulus $k_o>k$ is best obtained from the relation $k_o=\frac{2\sqrt{k}}{1+k}$, or from its complementary

$$k'_o=\frac{(\sqrt{4a^2+b^2}-2a)^2}{b^2}=\frac{b^2}{(\sqrt{4a^2+b^2}+2a)^2}.$$

The inductance formula corresponding is

$$L=L_1\cdot\frac{4}{3\pi}\left(\frac{2a}{b}\right)\left[\frac{(1-6k'_o+k'^2_o)}{(1-k'_o)^3}E(k_o)\right.$$
$$\left.+\frac{(3k'_o-2k'^2_o+3k'^3_o)}{(1-k'_o)^3}K(k_o)-1\right] \qquad *\ (20)$$

There are, of course, an indefinite number of integral formulas which could be derived for the inductance, each with a different modulus, but, with the exception of those already given, they are not simple. The elliptic integral formulas, in general, as already pointed out in the case of circles, are not convenient for numerical calculations, which are more simply, and often more accurately, carried out using the series expansions which follow.

2. HYPERGEOMETRIC SERIES EXPANSIONS FOR SOLENOIDS

Butterworth [20] obtained eight series expansions of Lorenz's formula. These arrange themselves in pairs whose members are closely related, so that only four of the formulas are essentially different. The following summary refers to those which are the more convergent.

Variable	Source	B. S. formula number
k	Butterworth [20]	II (20A)
$\mu=\frac{k}{k'}$	Webster-Havelock [33]	I (79)
	Butterworth	II (21A)
k'	Butterworth	II (22A)
$\nu=\frac{k'}{k}$	Rayleigh [35] and Niven	I (69)
	Coffin [36] (extension)	I (71)
	Butterworth (general)	II (23A)

No expansions in terms of other moduli have heretofore been published. The following expressions were found from the elliptic integral formulas (19) and (20). Although four expansions of each elliptic integral formula could readily be derived, those given here are believed to be sufficient for practical purposes.

From (19) the expansion in k_1 is derived.

$$L = L_1\left(\frac{2a}{b}\right)\left[\frac{1}{2\sqrt{k_1}}\left\{1-\frac{k_1}{2}+\frac{k_1^2}{4}+\frac{k_1^3}{16}+\frac{k_1^4}{64}+\frac{k_1^5}{128}+\frac{k_1^6}{256}+\cdots\right.\right.$$

$$+\left(\frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n)}\right)^2 k_1^{2n}+\cdots \qquad *(21)$$

$$\left.\left.+\left(\frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n)}\right)^2\frac{2n}{2n-3}k_1^{2n-1}+\cdots\right\}-\frac{4}{3\pi}\right]$$

the value of k_1 being the same as in (19).

This formula converges for all values of k_1, but most rapidly when k_1 is small; that is, for relatively long solenoids.

The expansion in $\mu_1 = \frac{k_1}{k_1'}$ is also obtained from (19)

$$L = L_1\left(\frac{2a}{b}\right)\left[\frac{1}{2\sqrt{k_1}\sqrt{1+\mu_1^2}}\left\{1+\frac{3}{4}\mu_1^2-\frac{15}{64}\mu_1^4+\frac{35}{256}\mu_1^6-\cdots\right.\right.$$

$$-(-)^n A_n\frac{(2n+1)}{(2n-1)}\mu_1^{2n}-\cdots-\frac{k_1}{2}\left(1+\frac{3}{8}\mu_1^2-\frac{5}{64}\mu_1^4\right. \qquad *(22)$$

$$\left.\left.\left.+\frac{35}{1024}\mu_1^6-\cdots-(-)^n\frac{A_n}{(n+1)}\frac{(2n+1)}{(2n-1)}\mu_1^{2n}-\cdots\right)\right\}-\frac{4}{3\pi}\right]$$

where

$$A_n = \left[\frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n}\right]^2$$

Formula (22) converges only for $k_1^2 < 1/2$.

The elliptic integral formula (20), when expanded in terms of the complementary modulus k_o', yields

$$L = L_1 \cdot \frac{4}{3\pi}\left(\frac{2a}{b}\right)\frac{3k'_o}{(1-k'_o)^2}\left[\left(\log\frac{4}{k'_o}-1\right)-\frac{k'_o}{2}\left(\log\frac{4}{k'_o}+\frac{3}{2}\right)\right.$$

$$+\frac{k'^2_o}{4}\left(\log\frac{4}{k'_o}+\frac{7}{3}\right)+\frac{k'^3_o}{16}\left(\log\frac{4}{k'_o}+\frac{1}{4}\right)+\frac{k'^4_o}{64}\left(\log\frac{4}{k'_o}-\frac{1}{2}\right) \qquad *(23)$$

$$\left.+\frac{k'^5_o}{128}\left(\log\frac{4}{k'_o}-\frac{2}{3}\right)+\sum_3^\infty\frac{A_n k'^{2n}_o}{(2n-1)^2}\varphi_{2n}+\sum_4^\infty\frac{A_n k'^{2n-1}_o\, 2n}{(2n-1)^2\,(2n-3)}\varphi_{2n-1}\right]$$

in which

$$A_n = \left[\frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n}\right]^2$$

$$\varphi_6 = \log\frac{4}{k'_o} - \frac{5}{6},\ \varphi_{2n} - \varphi_{2n-2} = \frac{1}{n} - \frac{2}{2n-3}$$

$$\varphi_5 = \log\frac{4}{k'_o} - \frac{2}{3},\ \varphi_{2n-1} - \varphi_{2n-3} = \frac{1}{2n} - \frac{1}{2n-3} + \frac{1}{2n-2} - \frac{1}{2n-5}.$$

Although (23) converges for all values of k'_o, the formula is useful especially where k'_o is small; that is for short coils.

The formula in terms of $\nu_o = \frac{k'_o}{k_o}$ is found to be

$$L = L_1 . \frac{4}{3\pi}\left(\frac{2a}{b}\right)\left\{\frac{1}{(1-k'_o)^3}\frac{3\nu_o}{1+\nu_o^2}\left[\left(\log\frac{4}{\nu_o} - 2\right) + \frac{3}{4}\nu_o^2\left(\log\frac{4}{\nu_o} - \frac{1}{3}\right)\right.\right.$$

$$\left. - \frac{15}{64}\nu_o^4\left(\log\frac{4}{\nu_o} - \frac{31}{30}\right)\sum_3^\infty (-)^n\frac{(2n+1)}{(2n-1)}A_n\nu_o^{2n}\psi_{2n}\right] \qquad *(24)$$

$$-\frac{k'_o}{2}\left\{\left(\log\frac{4}{\nu_o} - \frac{3}{2}\right) + \frac{3}{8}\nu_o^2\left(\log\frac{4}{\nu_o} - \frac{1}{12}\right) - \frac{5}{64}\nu_o^4\left(\log\frac{4}{\nu_o} - \frac{7}{10}\right)\right.$$

$$\left.\left. - \sum_3^\infty (-)^n\frac{(2n+1)}{(2n-1)}\frac{A_n\nu_o^{2n}}{(n+1)}\omega_{2n}\right\}\right] + \left[\frac{1}{(1-k'_o)^3(1+\nu_o^2)^{\frac{3}{2}}} - 1\right]\right\}$$

with

$$\psi_4 = \log\frac{4}{\nu_o} - \frac{31}{30},\ \psi_{2n} - \psi_{2n-2} = \frac{1}{n} - \frac{1}{2n+1} - \frac{1}{2n-3}$$

$$\omega_6 = \log\frac{4}{\nu_o} - \frac{883}{840},\ \omega_{2n} - \omega_{2n-2} = \frac{1}{2n+2} - \frac{1}{2n+1} - \frac{1}{2n-3} + \frac{1}{2n}$$

and A_n has the same value as in the preceding equation.

Formula (24) converges only when $k'^2_o < \frac{1}{2}$.

3. A. G. M. SERIES FORMULAS FOR SOLENOIDS

Just as explained for coaxial circles, each elliptical integral formula for the inductance of a solenoid will yield two a. g. m. series formula, the series being $a_o = 1$, and $b_o =$ the modulus in the one case, and $a_o = 1$, $b_o =$ the complementary modulus in the other. However, by making use of the simple relations between the a. g. m. series for different moduli, which have already been pointed out in section 3, two formulas may be derived which, between them, cover all cases with any desired accuracy. These formulas, which may be derived starting with any one of the elliptic integral formulas (19), (20), or I (72), are as follows:

$$L=L_1\left\{\frac{1}{a_n b}\left[\frac{4}{3}a^2+b^2-\frac{2}{3}\left(\frac{4a^2-b^2}{4a^2}\right)(c_1{}^2+2c_2{}^2+4c_3{}^2+\cdots+2^{n-1}c_n{}^2)\right]-\frac{4}{3\pi}\left(\frac{2a}{b}\right)\right\} \qquad *(25)$$

based on the a. g. m. series $(\sqrt{4a^2+b^2}, b)$, and the complementary series

$$L=L_1\cdot\frac{4}{3\pi}\left(\frac{2a}{b}\right)\left[\frac{(4a^2-b^2)}{8a^3}a'_n-1+\frac{Q}{2a'_n}\left\{\frac{b^2\left(6a^2+\frac{b^2}{2}\right)}{8a^3}+\frac{(4a^2-b^2)}{8a^3}(c'_1{}^2+2c'_2{}^2+\cdots+2^{n-1}c'_n{}^2)\right\}\right] \qquad *(26)$$

where a'_n and c'_n are based on the complementary a. g. m. series $(\sqrt{4a^2+b^2}, 2a)$. The quantity Q is defined by formula (13a), referred to the same a. g. m. series.

For purposes of tabulation it is convenient to have expressions which involve the parameter $\frac{2a}{b}$. Such formulas are most simply obtained from the Lorenz formula.

Placing $\frac{2a}{b}=\gamma$, and forming the series $(1, k')$, where $k'=\frac{1}{\sqrt{1+\gamma^2}}$, the inductance is

$$L=L_1\left\{\frac{1}{3a_n k'}\left[\frac{\gamma^2+3}{\gamma^2+1}-2\frac{(\gamma^2-1)}{\gamma^2}(c_1{}^2+2c_2{}^2+\cdots+2^{n-1}c_n{}^2)\right]-\frac{4}{3\pi}\gamma\right\} \qquad *(27)$$

while the complementary formula, series $(1, k)$, $k=\frac{\gamma}{\sqrt{1+\gamma^2}}$ is

$$L=L_1\cdot\frac{4}{3\pi}\gamma\left[\frac{\gamma^2-1}{\gamma^2}\cdot\frac{a'_n}{k}-1+\frac{Q}{4a'_n\gamma^2 k}\left\{\frac{3\gamma^2+1}{\gamma^2+1}+2(\gamma^2-1)(c'_1{}^2+2c'_2{}^2+\cdots+2^{n-1}c'_n{}^2)\right\}\right] \qquad *(28)$$

The direct formulas (25) and (27) suffice for all cases except for short coils. For such an extreme case as $\gamma=50$, both complementary and direct formulas give the inductance as the difference of two large terms, and series formulas such as illustrated in example 5 are more convenient in those cases.

Example 3.—For a coil whose diameter is equal to its length the modulus is $k=k'=\frac{\sqrt{2}}{2}$, and to obtain the inductance with an accuracy equal to that of seven-place logarithms it is necessary to include 14 and 13 terms, respectively, in the series formulas II (20A) and II (22A), which involve these moduli. The following table shows that the use of other moduli is advantageous.

Formula	Modulus	Value	Number of terms included	$\frac{L}{L_1}$
(21)	k_1	0. 1715729	7	0. 6884226
(22)	μ_1	. 1741554	7	. 6884223
(23)	k'_o	. 1715729	7	. 6884226
(24)	ν_o	. 1741554	7	. 6884226

The value obtained from (27) by the use of a calculating machine with eight places is 0.68842264. The agreement of the various formulas gives a check on their correctness.

Example 4.—For the coil of example 60, Bureau of Standards Scientific Paper No. 169, $a = 27.0862$, $b = 30.5510$. Using the a. g. m. series formula (25)

$$a_o = \sqrt{4a^2 + b^2} = 62.19335,\ b_o = 30.5510$$

$$a_3 = b_3 = a_n = 44.97022$$

$$c_1^2 + 2c_2^2 + 4c_3^2 + \cdot\cdot = 250.3095 + 3.8708 + 0.0005 = 254.1808$$

$$\frac{4}{3}a^2 + b^2 = 1911.5798,\ \frac{4}{3\pi}\left(\frac{2a}{b}\right) = 0.7525607$$

$$\frac{2}{3}\left(\frac{4a^2 - b^2}{4a^2}\right)(c_1^2 + 2c_2^2 + 4c_3^2 + \cdot\cdot) = 114.5591$$

$$\frac{L}{L_1} = 1.3072567 - 0.7525607 = 0.5546960$$

Solving the same problem by the complementary formula (26)

$$a'_o = 62.19335,\ \ b_o = 2a = 54.1724$$

$$a'_3 = b'_3 = 58.11366 = a'_n$$

$$Q = 4.059785,\ c'_1 = 4.01047_8$$

$$\frac{4a^2 - b^2}{8a^3} \cdot a'_n = 0.7315658,\ c'_2 = 0.06919_2$$

$$c'_3 = 0.00002$$

$$\frac{b^2\left(6a^2 + \frac{b^2}{2}\right)}{8a^3} = 28.58416,\ \frac{4a^2 - b^2}{8a^3}(c'^2_1 + 2c'^2_2 + \cdot\cdot) = 0.20259$$

Terms in

$$Q = 1.0055124$$

$$\frac{L}{L_1} = 0.5546960$$

The values found by a calculating machine using other formulas are:

Formula	Modulus	Number of terms included	$\frac{L}{L^1}$
(21)	k_1	11	0. 55469621
(23)	k'_o	6	. 55469622
(27)			. 55469626
(28)			. 55469621

Example 5.—If the a. g. m. series formula (26) be used to make the calculation for the very short coil $a=25$, $b=1$, the series to be formed is $(\sqrt{2501}, 1)$, and the limiting value a'_n lies very close to a'_1. However, the difficulties are met (*a*) that c'_1 can not be obtained directly with accuracy, and (*b*) that the first two terms of (26) nearly cancel, and that their small difference is of the same magnitude as the main term.

The calculation may be made accurately by expanding the formula in terms of the small quantity $\delta=\frac{b}{2a}\cdot$ For the problem in question higher powers than δ^4 do not need to be retained, and the series are

$$a'_1=2a\left[1+\frac{\delta^2}{4}-\frac{\delta^4}{16}\right],\quad c'_1=\frac{a\delta^2}{2}\left(1-\frac{\delta^2}{4}+\frac{\delta^4}{8}\right)$$

$$a'_n=2a\left[1+\frac{\delta^2}{4}-\frac{5}{64}\delta^4\right],\quad Q=2\log\frac{4}{\delta}+\frac{\delta^2}{2}-\frac{13}{64}\delta^4$$

$$\left(\frac{4a^2-b^2}{8a^3}\right)a'_n-1=-\frac{3}{4}\delta^2\left(1+\frac{7}{16}\delta^2\right),\quad \left(\frac{4a^2-b^2}{16a^3a'_n}\right)c'^2_1=\frac{\delta^4}{32}\left(1-\frac{7}{4}\delta^2\right)$$

$$\frac{b^2\left(6a^2+\frac{b^2}{2}\right)}{16a^3a'_n}=\frac{3}{4}\delta^2\left(1+\frac{\delta^2}{12}+\frac{11}{192}\delta^4\right)$$

Placing $\delta=\frac{1}{50}$, the terms in Q are 0.003179208 and $\left(\frac{4a^2-b^2}{8a^2}\right)a'_n-1=$ -0.000300052, so that $\frac{L}{L_1}=0.06910758$. The term in c'_1 is here nearly negligible, so that the value of c'_1 is of importance only in the calculation of Q.

The value for the same case, using the more convenient formula I (71) or II (23A), in terms of the modulus $\frac{k'}{k}=\frac{b}{2a}$, is 0.06910754, two terms in the series being sufficient to give this accuracy.

4. q SERIES FORMULAS FOR SOLENOIDS

Two formulas for the inductance of a solenoid in *q* series were derived by Nagaoka (8); these are formulas I (76) and I (77). For very long coils the latter may be put into the very convenient form I (78). Together these cover the whole range of solenoids, and have the advantage of very rapid convergence. As has already been pointed out in section 4, no other essentially different expansions in q series are possible.

5. SOLENOID FORMULAS INVOLVING THE DIMENSIONS DIRECTLY

The only parameter involving the dimensions as a simple ratio is $\frac{2a}{b}$ or its reciprocal. Since $\frac{k}{k'}=\frac{2a}{b}$, the Webster-Havelock (33) formula I (79) gives the series in $\frac{2a}{b}$, and the Rayleigh and Niven (35) formula I (69) and its extensions I (71) and II (23A) the expansion in $\frac{b}{2a}$.

IV. SUMMARY

Each of the different types of formula for expressing the mutual inductance of coaxial circles and the self inductance of solenoids has been reviewed with the object of obtaining further formulas where such are possible. The new formulas presented above are believed to include all the previously overlooked formulas for these cases which are likely to prove useful.

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