Volume 99, Number 4, July–August 1994
Journal of Research of the National Institute of Standards and Technology

[J. Res. Natl. Inst. Stand. Technol. **99**, 551 (1994)]

Domains of Attraction of Multivariate Extreme Value Distributions

Volume 99 Number 4 July–August 1994

Rinya Takahashi

Kobe University of Mercantile Marine,
Kobe, Hyogo 658, Japan

Some necessary and sufficient conditions for domains of attraction of multivariate extreme value distributions are shown by using dependence functions. The joint asymptotic distribution of multivariate extreme statistics is also shown.

Key words: asymptotic joint distribution; dependence function; multivariate extreme statistics.

1. Introduction

Multivariate extreme value distributions have been studied by many authors, and their contributions are summarized by Galambos [1] and Resnick [2]. The purpose of this paper is to obtain some necessary and sufficient conditions for domains of attraction of the multivariate extreme value distributions. The joint asymptotic distribution of multivariate extreme statistics is also obtained. To study multivariate extreme value distributions and their domains of attraction, Sibuya [3] introduces the notion of a dependence function which is also used by Galambos [1]. A dependence function or copula is a useful notion to construct a family of joint distributions.

In this paper, basic arithmetical operations are always meant componentwise (see Galambos [1], Chapt. 5).

Let $(X_{1j}, X_{2j}, \ldots, X_{kj})$, $j=1,2,\ldots,n$, be a sample of size n, of a k-dimensional random vector with a distribution function $F(\boldsymbol{x})$. The i-dimensional distribution function of the components $X_{j_1}, X_{j_2}, \ldots, X_{j_i}$ will be denoted $F_{j_1 j_2 \ldots j_i}(x_{j_1}, x_{j_2}, \ldots, x_{j_i}) = F_{J(i)}(\boldsymbol{x}_{J(i)})$. We shall also use the notation $\bar{F}_{j_1 \ldots j_i}(x_{j_1}, \ldots, x_{j_i}) = \bar{F}_{J(i)}(\boldsymbol{x}_{J(i)}) = P(X_{j_1} > x_{j_1}, \ldots, X_{j_i} > x_{j_i})$. For $k=1$ and $p \in (0,1)$, let $F^{-1}(p) = \inf\{x : F(x) \geq p\}$.

Let $\boldsymbol{Z}_n = (Z_{1n}, \ldots, Z_{kn})$, where $Z_{in} = \max\{X_{i1}, \ldots, X_{in}\}$, $i=1,2,\ldots,k$, and let us call $\boldsymbol{Z}_n$ a multivariate extreme statistic.

If there exist $\boldsymbol{a}_n > \boldsymbol{0}, \boldsymbol{b}_n \in R^k, n=1,2,\ldots (\boldsymbol{a}_n > \boldsymbol{0}$ means $a_{in} > 0, i=1,\ldots,k)$ such that $(\boldsymbol{Z}_n - \boldsymbol{b}_n)/\boldsymbol{a}_n$ converges in distribution to a random vector $\boldsymbol{U}$ with a nondegenerate distribution H (i.e., all univariate marginals of H are nondegenerate), then F is said to be in the domain of attraction of H, $F \in D(H)$ by symbol, and H is said to be a multivariate extreme value distribution. The convergence in distribution is equivalent to the condition

$$\lim_{n\to\infty} F^n(\boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n) = H(\boldsymbol{x}) \tag{1}$$

for all $\boldsymbol{x}$, because multivariate extreme value distributions are continuous.

We shall need the following lemma to prove a proposition in Sec. 2.

Lemma 1.1 *Equation (1) is equivalent to*

$$\lim_{n\to\infty} n[1 - F(\boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n)] = -\log H(\boldsymbol{x})$$

for all $\boldsymbol{x}$ *such that* $0 < H(\boldsymbol{x}) < 1$. (See Marshall and Olkin [4].)

2. Domains of Attraction

For any k-dimensional distribution F,

$$D_F(\mathbf{y})=F(F_1^{-1}(y_1),\ldots,F_k^{-1}(yk)),\ \mathbf{y}=(y_1,\ldots,y_k)\in(0,1)^k$$

is called the dependence function of F. In this section, we derive necessary and sufficient conditions for domains of attraction in terms of the dependence function.

Proposition 2.1 *Let F be a k-dimensional distribution and let H be a multivariate extreme value distribution with univariate marginals H_i, $i=1,\ldots,k$. Then the following statements are equivalent:*

1) $F\in D(H)$.
2) $F_i\in D(H_i)$, $i=1,\ldots,k$, and

$$\lim_{s\to\infty} s[1-D_F(\mathbf{y}^{1/s})]=-\log D_H(\mathbf{y}) \text{ for all } \mathbf{y}\in(0,1)^k.$$

3) $F_i\in D(H_i)$, $i=1,\ldots,k$, and

$$\lim_{x\uparrow 1}\frac{1-D_F(\mathbf{y}^{1-x})}{1-x}=-\log D_H(\mathbf{y}) \text{ for all } \mathbf{y}\in(0,1)^k.$$

4) $F_i\in D(H_i)$, $i=1,\ldots,k$, and

$$\lim_{t\downarrow 0}\frac{1-D_F(\mathbf{y}^t)}{1-D_H(\mathbf{y}^t)}=1 \text{ for all } \mathbf{y}\in(0,1)^k.$$

Proof. The proof is straightforward from Lemma 1.1, Theorem 5.2.3 and Lemma 5.4.1 of Galambos [1]. □

Proposition 2.2 *Let F be a k-dimensional distribution and let H be a multivariate extreme value distribution with univariate marginals H_i, $i=1,\ldots,k$.*

(A) $F\in D(H)$ if and only if $F_i\in D(H_i)$, $i=1,\ldots,k$, and the functions

$$d_{J(i)}(\mathbf{y}_{J(i)})=\lim_{n\to\infty} n\bar{D}_{F_{J(i)}}((\mathbf{y}_{J(i)})^{1/n})$$

for each fixed vector $J(i)(i>1)$ and for all $\mathbf{y}\in(0,1)^k$ are finite, and the function

$$D_H(\mathbf{y};r)=y_1\cdots y_k\exp\{\sum_{i=2}^{k}(-1)^i\sum_{1\le j_1<\cdots<j_i\le k} d_{J(i)}(\mathbf{y}_{J(i)})\}$$

is a dependence function of H.

(B) The following inequalities hold.

$$D_H(\mathbf{y};2r+1)\le D_H(\mathbf{y})\le D_H(\mathbf{y};2r),$$

for a nonnegative integer r, where

$$D_H(\mathbf{y};r)=y_1\cdots y_k\exp\{\sum_{i=2}^{r}(-1)^i\sum_{1\le j_1<\cdots<j_i\le k} d_{J(i)}(\mathbf{y}_{J(i)})\}$$

and $D_H(\mathbf{y};r)$ is a dependence function of a multivariate extreme value distribution.

Proof. It is easily seen that for all $s>0$,

$$sd_{J(i)}((\mathbf{y}_{J(i)})^{1/s})=d_{J(i)}(\mathbf{y}_{J(i)}).$$

From Theorems 5.3.1 and 5.2.4 of Galambos [1], we have the result. □

Example 2.1 (See Examples 5.2.2 and 5.2.3 of Galambos [1].) For a Mardia's distribution

$$F(x_1,x_2)=1-e^{-x_1}-e^{-x_2}+(e^{x_1}+e^{x_2}-1)^{-1},$$

$$D_F(y_1,y_2)=y_1+y_2-1+\left[\frac{1}{1-y_1}+\frac{1}{1-y_2}-1\right]^{-1},$$

and

$$n\bar{D}_F(y_1^{1/n},y_2^{1/n})=n\left[\frac{1}{1-y_1^{1/n}}+\frac{1}{1-y_2^{1/n}}-1\right]^{-1}$$

$$\to-\frac{(\log y_1)(\log y_2)}{\log y_1+\log y_2},\ \text{as } n\to\infty.$$

Thus, by Proposition 2.2 we have $F\in D(H)$, where

$$D_H(y_1,y_2)=y_1\cdot y_2\exp\left[-\frac{(\log y_1)(\log y_2)}{\log y_1+\log y_2}\right],$$

$$H(x_1,x_2)=\Lambda(x_1)\Lambda(x_2)\exp\{1/(e^{x_1}+e^{x_2})\},$$

and $\Lambda(x)=\exp(-e^{-x})$.

Proposition 2.3 *Let F and G be k-dimensional distributions and let H be a multivariate extreme value distribution.*

1) If $F,G\in D(H)$, then

$$\lim_{t\downarrow 0}\frac{1-D_F(\mathbf{y}^t)}{1-D_G(\mathbf{y}^t)}=1 \text{ for all } \mathbf{y}\in(0,1)^k.$$

2) If $F\in D(H)$, $G_i\in D(H_i)$, $i=1,\ldots,k$, and

$$\lim_{\mathbf{y}\uparrow\mathbf{1}}\frac{1-D_F(\mathbf{y})}{1-D_G(\mathbf{y})}=1,\ \text{where } \mathbf{1}=(1,\ldots,1),$$

then $G\in D(H)$.

3. Marginally Independent or Perfect Dependent Multivariate Extreme Value Distributions

Let H be a multivariate extreme value distribution with univariate marginals H_i, $i=1,...,k$. Let $H_*(x)=H_1(x_1)\cdots H_k(x_k)$ and $H^*(x)=\min\{H_i(x_i), i=1,...,k\}$, then it holds

$$H_*(x)\le H(x)\le H^*(x)$$

for all $x\in R^k$. Both bounds, H_* and H^*, are multivariate extreme value distributions. Characterizations of these distributions are obtained by Takahashi [5].

In the bivariate case Sibuya [3] obtains necessary and sufficient conditions for $F\in D(H_*)$ and $F\in D(H^*)$. In this section we generalize his results.

Proposition 3.1 *Let F be a k-dimensional distribution and let H_i be a univariate extreme value distribution, $i=1,...,k$. Then the following statements are equivalent:*
1) $F\in D(H_*)$.
2) $F_i\in D(H_i)$, $i=1,...,k$, *and there exists* $y\in(0,1)^k$ *such that*

$$\lim_{n\to\infty}(D_F(y^{1/n}))^n=y_1y_2\cdots y_k.$$

3) $F_i\in D(H_i)$, $i=1,...,k$, *and*

$$\lim_{y\uparrow 1}\frac{1-D_F(y\mathbf{1})}{1-y}=k.$$

4) $F_i\in D(H_i)$, $i=1,...,k$, *and*

$$\lim_{y\uparrow 1}\frac{1-D_F(y\mathbf{1})}{1-y^k}=1.$$

Proof. The proof is straightforward from Theorems 2.2 and 4.1 and Corollary 2.4 of Takahashi [6]. □

Remark. If $k=2$, we have the same result as Proposition 3.1 by Corollary 2.2 of Takahashi [6].

Example 3.1 (See Example 5.2.3 of Galambos [1].) For the Morgenstern distribution

$$F(x_1,x_2)=1-e^{-x_1}-e^{-x_2}+e^{-x_1-x_2}[1+\alpha(1-e^{-x_1})(1-e^{-x_2})],$$

$$D_F(y_1,y_2)=y_1\cdot y_2[1+\alpha(1-y_1)(1-y_2)]$$

where $-1\le\alpha\le 1$, and

$$\lim_{y\uparrow 1}\frac{1-D_F(y,y)}{1-y^2}=1.$$

By Proposition 3.1 4) we have $F\in D(H_*)$, where $H_*(\cdot,\cdot)=\Lambda(\cdot)\Lambda(\cdot)$.

Proposition 3.2 *Let F be a k-dimensional distribution and let H_i be a univariate extreme value distribution, $i=1,...,k$. Then the following statements are equivalent:*
1) $F\in D(H^*)$.
2) $F_i\in D(H_i)$, $i=1,...,k$, *and there exists* $y\in(0,1)$ *such that*

$$\lim_{n\to\infty}(D_F(y^{1/n}\mathbf{1}))^n=y.$$

3) $F_i\in D(H_i)$, $i=1,...,k$, *and*

$$\lim_{y\uparrow 1}\frac{1-D_F(y\mathbf{1})}{1-y}=1.$$

Proof. The proof is straightforward from Theorem 3.1 and Corollary 3.1 of Takahashi [6]. □

4. Joint Asymptotic Distribution of the Multivariate Extreme Statistics

In this section, we show the joint asymptotic distribution of several multivariate extreme statistics along the arguments in Sec. 2.3 of Leadbetter et al. [7]. For simplicity we shall consider the bivariate case.

Let $(X_1,Y_1),...,(X_n,Y_n)$ be a sequence of independent random vectors with common distribution F. The order statistics of the components will be denoted by

$$X_{1:n}\le X_{2:n}\le\cdots\le X_{n:n};\ and\ Y_{1:n}\le Y_{2:n}\le\cdots\le Y_{n:n}.$$

For $i=0,1,...,r-1$, define

$$Z_{n-i}=(X_{n-i:n},Y_{n-i:n})$$

and let us call Z_{n-i} an $(i+1)$-th multivariate extreme statistic.

Proposition 4.1 *Suppose that*

$$P\{(Z_n-b_n)/a_n\le x\}\xrightarrow{w}H(x)$$

for some nondegenerate distribution H. Then, for $x_1=(x_1,y_1)>x_2=(x_2,y_2)$,

$$P\{(Z_n-b_n)/a_n\le x_1,\ (Z_{n-1}-b_n)/a_n\le x_2\}\xrightarrow{w}H_j(x_1,x_2)$$

where

$$\begin{aligned}H_j(x_1,x_2)=H(x_2)\{&1+\log H(x_1)-\log H(x_2)\\&+[\log H_1(x_1)-\log H_1(x_2)+(h(x_1,y_2)-h(x_2))]\\&\times[\log H_2(y_1)-\log H_2(y_2)+(h(x_2,y_1)-h(x_2))]\}\end{aligned}$$

and $h(x)=\lim_{n\to\infty}\overline{F}(a_n x+b_n)$.

Proof. Define

$$S_0^n=\#\{j \mid X_j>a_{1n}x_1+b_{1n} \text{ or } Y_j>a_{2n}y_1+b_{2n}, j=1,2,\ldots,n.\},$$
$$S_1^n=\#\{j \mid a_{1n}x_2+b_{1n}<X_j\le a_{1n}x_1+b_{1n} \text{ and } Y_j\le a_{2n}y_2+b_{2n}, j=1,2,\ldots,n.\},$$
$$S_2^n=\#\{j \mid X_j\le a_{1n}x_2+b_{1n} \text{ and } a_{2n}y_2+b_{2n}<Y_j\le a_{2n}y_1+b_{2n}, j=1,2,\ldots,n.\},$$
$$S_{12}^n=\#\{j \mid a_n x_2+b_n<(X_j,Y_j)\le a_n x_1+b_n, j=1,2,\ldots,n.\}$$

then, we have

$$P\{(Z_n-b_n)/a_n\le x_1, (Z_{n-1}-b_n)/a_n\le x_2\}$$
$$=P\{S_0^n=0, S_{12}^n=0, S_1^n\le 1, S_2^n\le 1\}$$
$$+P\{S_0^n=0, S_{12}^n=1, S_1^n=0, S_2^n=0\}.$$

On the other hand, by using Theorem 5.3.1 of Galambos [1], we can evaluate the asymptotic probabilities of the evens

$$\{S_0^n=i, S_{12}^n=j, S_1^n=k, S_2^n=m\}$$

for $i,j,k,m=0,1$. Thus we have the result. □

Corollary 4.1 *Suppose that*

$$P\{(Z_n-b_n)/a_n\le x\}\xrightarrow{w}H(x)$$

for some nondegenerate distribution H. Then, for fixed $r\ge 1$ *and* $x_1>\cdots>x_r$

$$\left| P\{(Z_n-b_n)/a_n\le x_1,\ldots,(Z_{n-r+1}-b_n)/a_n\le x_r\}\right.$$
$$\left. -P\{(Z_n^*-\beta_n)/\alpha_n\le x_1,\ldots,(Z_{n-r+1}^*-\beta_n)/\alpha_n\le x_r\}\right|\to 0,$$
as $n\to\infty$,

where Z_{n-i}^* *is the* $(i+1)$*-th multivariate extreme statistic from the distribution H,* $i=0,\ldots,r-1$, *and* $H^n(\alpha_n x+\beta_n)=H(x)$, $n=1,2,\ldots$

Example 4.1 Let F be the bivariate normal distribution with the correlation coefficient less than one. Then

$$\left| P\{(Z_n-b_n)/a_n\le x_1,\ldots,(Z_{n-r+1}-b_n)/a_n\le x_r\}\right.$$
$$\left. -P\{(Z_n^*-(\log n)\mathbf{1}\le x_1,\ldots,(Z_{n-r+1}^*-(\log n)\mathbf{1}\le x_r\}\right|\to 0,$$
as $n\to\infty$,

where Z_{n-i}^* is the $(i+1)$-th multivariate extreme statistic from the bivariate exponential distribution whose marginals are equal to the standard exponential distribution and they are independent. For the univariate case, it is a well known result (see Weissman [8], Theorem 3).

Acknowledgments

The author thanks Prof. M. Sibuya and Prof. J. Galambos for their advice.

5. References

[1] J. Galambos, The Asymptotic Theory of Extreme Order Statistics. 2nd ed., Krieger, Malabar, FL (1987).
[2] S. I. Resnick, Extreme Values, Regular Variation and Point Processes. Springer-Verlag, Berlin (1987).
[3] M. Sibuya, Bivariate extreme statistics. Ann. Inst. Statist. Math. **11**, 195–210 (1960).
[4] A. W. Marshall and I. Olkin, Domains of attraction of multivariate extreme value distributions. Ann. Probab. **11**, 168–177 (1983).
[5] R. Takahashi, Characterizations of a multivariate extreme value distribution. Adv. Appl. Probab. **20**, 235–236 (1988).
[6] R. Takahashi, Asymptotic independence and perfect dependence of vector components of multivariate extreme statistics. Statist. Probab. Letters **19**, 19–26 (1994).
[7] M. R. Leadbetter, G. Lindgren, and H. Rootzén, Extremes and Related Properties of Random Sequences and Processes. Springer-Verlag, New York (1983).
[8] I. Weissman, Estimation of parameters and large quantiles based on the *k* largest observations. J. Amer. Statist. Assoc. 73, 812–815 (1978).

About the author: *Rinya Takahashi is Associate Professor, Department of Information Systems Engineering, Kobe University of Mercantile Marine, Fukaeminami 5-1-1, Higashinada, Kobe, 658, Japan.*