Journal of Research of the National Bureau of Standards
Volume 90, Number 5, September-October 1985

Note on Weighings Carried Out on the NBS-2 Balance

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Accepted: August 28, 1985

The NBS-2 balance was designed and built at NBS and transferred to the BIPM in 1972. It is presently used for the comparison of national prototype kilograms with international standards. Excellent environmental conditions at the BIPM have resulted in a long-term standard deviation of 1 microgram (1×10^{-9}) for a comparison of two 1-kilogram standards. With this remarkable precision, one has begun to observe and quantify systematic biases of less than 5 micrograms. The nature of these biases is presented as well as the remedy adopted to eliminate their influence on both the final measurement results and the variance assigned to those results.

Key words: balance; errors; kilogram comparator; precision weighing; single-pan balance; weighing schemes.

The NBS-2 balance [1][1] permits mass intercomparisons of standards of nominal value $m_0=1$ kg by a substitution method. The features of this balance which are important to the present analysis are: 1) the balance has a single pan so that substitution weighing must be used; 2) the on-scale range of the balance is limited to about 40 mg; 3) a small "sensitivity weight" can be added or removed from the balance pan by remote control, thereby provided a means of calibrating the scale of the balance in mass units; 4) six weights can be placed on a table within the balance enclosure. The weight table may be raised, lowered, or rotated by remote control, combinations of these operations permitting any of the six weights to be placed on the balance pan; and 5) during the course of a day's measurements, the balance knives are kept in contact with their corresponding flats at full load. After measurements are completed, the balance is fully arrested (i.e., knife-flat contact is broken) until measurements are recommenced the next day.

In what follows, we use the term *weighing* to denote an ensemble of operations carried out successively and decomposable to a set number of elementary operations called *subweighings* in the course of which one determines the mass difference between two standards.

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[1]Figures in brackets indicate literature references.

1. Subweighing

Let A and B be the standards, having masses m_A and m_B, which are used in a subweighing; let S be the sensitivity weight, of known mass m_S, which is used to determine the sensitivity of the balance.

One usually works with five operations (each yielding one position of balance equilibrium). The five operations consist of the balance being successively loaded with A, B, B and S, A and S, and finally A; one notes the corresponding positions of equilibrium, that is to say the readings λ_1, λ_2, λ_3, λ_4, and λ_5 taken from the balance scale.

Let us introduce the notation λ_0 to designate the reading which one would obtain if the balance were loaded with a standard of mass exactly equal to m_0. Because of continuous variations in ambient conditions, successive readings on a single weight can shift from one balance equilibrium to the next. For the moment, we assume that the balance readings have no random errors associated with them. To conserve *notational symmetry*, we adopt the convention that λ_0 corresponds to the third equilibrium observation and we introduce the following *drifts* from this reading:

d_2 between the first and second observation,
d_1 between the second and third,
d_1' between the third and fourth, and
d_2' between the fourth and fifth.

The readings which one would make with a mass standard exactly equal to m_0 would thus be, in succession,

$$\lambda_0-d_1-d_2,\ \lambda_0-d_1,\ \lambda_0,\ \lambda_0+d_1' \text{ and } \lambda_0+d_1'+d_2' .$$

Let a and b be the difference in readings corresponding to m_A-m_0 and m_B-m_0, respectively; let s be the difference in readings corresponding to m_S.

The five equilibrium positions are described by five equations:

Object on Pan	Equations
A	$\lambda_0-d_1-d_2+a=\lambda_1$
B	$\lambda_0-d_1+b=\lambda_2$
B and S	$\lambda_0+b+s=\lambda_3$
A and S	$\lambda_0+d_1'+a+s=\lambda_4$
A	$\lambda_0+d_1'+d_2'+a=\lambda_5$.

At this point, we recognize that the actual balance readings $\lambda_1, \ldots, \lambda_5$ are subject to random errors. One sees that λ_0 is always associated with either a or b, so that it is preferable to write

$$(\lambda_0+a)-d_1-d_2=\lambda_1$$

$$(\lambda_0+b)-d_1=\lambda_2$$

$$(\lambda_0+b)+s=\lambda_3$$

$$(\lambda_0+a)+s+d_1'=\lambda_4$$

$$(\lambda_0+a)+d_1'+d_2'=\lambda_5 .$$

To solve this system (5 equations, 7 unknowns) it is necessary to reduce the number of unknowns. The only way to accomplish this is to make hypotheses about the drift.

1.1 Classical Hypothesis of Drift: One postulates that the drift is the same during the 1st and the 3rd intervals between balance equilibria, so that $d_2=d_1'$, and that it is also the same for the 2nd and the 4th intervals between equilibria, so that $d_1=d_2'$.

The justification for this hypothesis could be that in the course of the 1st and 3rd interval there is an exchange of standards A and B, hence rotation of the weight table; whereas, in the course of the 2nd and 4th there is only manipulation of the sensitivity weight, S.

One is thus led to a system having five unknowns:

$$(\lambda_0+a)-d_1-d_1'=\lambda_1$$

$$(\lambda_0+b)-d_1=\lambda_2$$

$$(\lambda_0+b)+s=\lambda_3$$

$$(\lambda_0+a)+s+d_1'=\lambda_4$$

$$(\lambda_0+a)+d_1+d_1'=\lambda_5 .$$

The solution is:

$$(\lambda_0+a)=(\lambda_1+\lambda_5)/2$$

$$(\lambda_0+b)=(\lambda_2+\lambda_3-\lambda_4+\lambda_5)/2$$

$$s=(-\lambda_2+\lambda_3+\lambda_4-\lambda_5)/2$$

$$d_1=(-\lambda_2+\lambda_3-\lambda_4+\lambda_5)/2$$

$$d_1'=(-\lambda_1+\lambda_2-\lambda_3+\lambda_4)/2$$

By postulating that, after accounting for drift, the remaining *variations of balance readings are proportional to differences of mass*, one has

$$\frac{m_A-m_B}{m_S}=\frac{a-b}{s}=\frac{(\lambda_0+a)-(\lambda_0+b)}{s} .$$

Now $(\lambda_0+a)-(\lambda_0+b)=(\lambda_1-\lambda_2-\lambda_3+\lambda_4)/2$

from which

$$\boxed{\frac{m_A-m_B}{m_S}=\frac{\lambda_1-\lambda_2-\lambda_3+\lambda_4}{-\lambda_2+\lambda_3+\lambda_4-\lambda_5}} . \qquad (1)$$

If one assumes that the five independent readings λ_i have the same variance var(λ), then var($a-b$)=var(λ).

Remark: The hypothesis according to which variations of *balance reading are directly proportional to differences of* mass is, perhaps, not strictly verified. Let us introduce, therefore, a non-linearity in the form of a second-order term. In place of the differences in readings a, b, $b+s$ and, $a+s$, proportional to differences of mass m_A-m_0, m_B-m_0, $m_B+m_S-m_0$, and $m_A+m_S-m_0$, we write $a+ka^2$, $b+kb^2$,

$b+s+k(b+s)^2$, and $a+s+k(a+s)^2$. The five equations are thus written:

$$\lambda_0+a+ka^2-d_1-d_1'=\lambda_1$$

$$\lambda_0+b+kb^2-d_1=\lambda_2$$

$$\lambda_0+b+s+k(b+s)^2=\lambda_3$$

$$\lambda_0+a+s+k(a+s)^2+d_1'=\lambda_4$$

$$\lambda_0+a+ka^2+d_1+d_1'=\lambda_5 \ .$$

Then

$$\lambda_1-\lambda_2-\lambda_3+\lambda_4=2(a-b)[1+k(a+b+s)] \ ,$$

$$-\lambda_2+\lambda_3+\lambda_4-\lambda_5=2s[1+k(a+b+s)] \ ,$$

and it is still true that

$$\frac{\lambda_1-\lambda_2-\lambda_3+\lambda_4}{-\lambda_2+\lambda_3+\lambda_4-\lambda_5}=\frac{a-b}{s}=\frac{m_A-m_B}{m_S} \ .$$

Equation (1) therefore remains valid for the new hypothesis.

1.2 Linear Drift as a Function of the Sequence Number of the Equilibrium Observation: One thus has $d_2=d_1=d_1'=d_2'$ (Let us call it d) and the system of five equations and four unknowns is written:

$$(\lambda_0+a)-2d=\lambda_1$$

$$(\lambda_0+b)-d=\lambda_2$$

$$(\lambda_0+b)+s=\lambda_3$$

$$(\lambda_0+a)+s+d=\lambda_4$$

$$(\lambda_0+a)+2d=\lambda_5 \ .$$

One can solve this system by the method of least squares. The solution is:

$$(\lambda_0+a)=(3\lambda_1+\lambda_2-\lambda_3+\lambda_4+3\lambda_5)/7$$

$$(\lambda_0+b)=(-3\lambda_1/4+5\lambda_2+2\lambda_3-2\lambda_4+11\lambda_5/4)/7$$

$$s=(-\lambda_1/4-3\lambda_2+3\lambda_3+4\lambda_4-15\lambda_5/4)/7$$

$$d=(-\lambda_1+\lambda_5)/4 \ .$$

One extracts from this

$$a-b=(15\lambda_1-16\lambda_2-12\lambda_3+12\lambda_4+\lambda_5)/28 \ ,$$

from which

$$\boxed{\frac{m_A-m_B}{m_S}=\frac{a-b}{s}=\frac{15\lambda_1-16\lambda_2-12\lambda_3+12\lambda_4+\lambda_5}{-\lambda_1-12\lambda_2+12\lambda_3+16\lambda_4-15\lambda_5}} \ .$$

With the same hypothesis as above, var$(a-b)$= 55/56 var(λ).

1.3 Quadratic Drift as a Function of the Sequence Number of the Equilibrium Observation: It is sufficient to add to the total drift with respect to the third equilibrium position a second-order term: δ for the second and fourth equlibria and 4δ for the first and fifth.

The new system of five equations and five unknowns is:

$$(\lambda_0+a) \qquad -2d+4\delta=\lambda_1$$

$$(\lambda_0+b) \qquad -d+\delta=\lambda_2$$

$$(\lambda_0+b)+s \qquad =\lambda_3$$

$$(\lambda_0+a) \qquad +s+d+\delta=\lambda_4$$

$$(\lambda_0+a) \qquad +2d+4\delta=\lambda_5 \ .$$

The solution is:

$$(\lambda_0+a)=(-\lambda_1+4\lambda_2-4\lambda_3+4\lambda_4-\lambda_5)/2$$

$$(\lambda_0+b)=(-\lambda_1+3\lambda_2-\lambda_3+\lambda_4)/2$$

$$s=(\lambda_1-3\lambda_2+3\lambda_3-\lambda_4)/2$$

$$d=(-\lambda_1+\lambda_5)/4$$

$$\delta=(\lambda_1-2\lambda_2+2\lambda_3-2\lambda_4+\lambda_5)/4 \ ,$$

from which one extracts

$$a-b=(\lambda_2-3\lambda_3+3\lambda_4-\lambda_5)/2 \ ,$$

so that

$$\boxed{\frac{m_A-m_B}{m_S}=\frac{a-b}{s}=\frac{\lambda_2-3\lambda_3+3\lambda_4-\lambda_5}{\lambda_1-3\lambda_2+3\lambda_3-\lambda_4}} \ .$$

With the same hypothesis as above, var$(a-b)$=5var(λ).

Remark: At present, it is the first hypothesis [viz. I.1] which seems the best verified. In particular, the hypothesis of a quadratic drift is not verified.

2. Weighings

Classical weighing involves four standards, the mass differences of which are determined from pair-wise intercomparisons in all possible combinations. Let A, B, C, and D be standards having mass m_A, m_B, m_C, and m_D, respectively. In the *classic design* one determines successively:

$$m_A - m_B = m_1$$

$$m_A - m_C = m_2$$

$$m_A - m_D = m_3$$

$$m_B - m_C = m_4$$

$$m_B - m_D = m_5$$

$$m_C - m_D = m_6 \ ,$$

where each m_i $(i=1,\ldots,6)$ is the result of a subweighing. One considers m_A as known. Thus the differences $m_B - m_A$, $m_C - m_A$, and $m_D - m_A$, which we will designate m'_B, m'_C, and m'_D, constitute the three actual unknowns. We should also point out that each m_i is subject to random errors associated with the subweighing measurements. The six conditional equations are thus written:

$$-m'_B = m_1$$

$$-m'_C = m_2$$

$$-m'_D = m_3$$

$$m'_B - m'_C = m_4$$

$$m'_B - m'_D = m_5$$

$$m'_C - m'_D = m_6 \ .$$

One can solve this system by the method of least squares. The normal equations are:

$$3m'_B - m'_C - m'_D = -m_1 + m_4 + m_5$$

$$-m'_B + 3m'_C - m'_D = -m_2 - m_4 + m_6$$

$$-m'_B - m'_C + 3m'_D = -m_3 - m_5 - m_6 \ .$$

By addition, one obtains

$$m'_B + m'_C + m'_D = -(m_1 + m_2 + m_3) \qquad (2)$$

from which

$$m'_B = (-2m_1 - m_2 - m_3 + m_4 + m_5)/4$$

$$m'_C = (-m_1 - 2m_2 - m_3 - m_4 + m_6)/4$$

$$m'_D = (-m_1 - m_2 - 2m_3 - m_5 - m_6)/4$$

By supposing the determinations of $m_1, m_2, \ldots, m_6$ to be independent and $\text{var}(m_1) = \text{var}(m_2) = \ldots = \text{var}(m_6) = \text{var}(m)$, one has immediately

$$\text{var}\,(m'_B) = \text{var}(m'_C) = \text{var}(m'_D) = (1/2)\ \text{var}(m)$$

$$\text{covar}\,(m'_B, m'_C) = \text{covar}(m'_C, m'_D)$$

$$= \text{covar}(m'_D, m'_B) = (1/4)\ \text{var}(m) \ .$$

The residual deviations are:

$$g_1 = m_1 + m'_B \qquad = (2m_1 - m_2 - m_3 + m_4 + m_5)/4$$

$$g_2 = m_2 + m'_C \qquad = (-m_1 + 2m_2 - m_3 - m_4 + m_6)/4$$

$$g_3 = m_3 + m'_D \qquad = (-m_1 - m_2 + 2m_3 - m_5 - m_6)/4$$

$$g_4 = m_4 - m'_B + m'_C = (m_1 - m_2 + 2m_4 - m_5 + m_6)/4$$

$$g_5 = m_5 - m'_B + m'_D = (m_1 - m_3 - m_4 + 2m_5 - m_6)/4$$

$$g_6 = m_6 - m'_C + m'_D = (m_2 - m_3 + m_4 - m_5 + 2m_6)/4 \ .$$

3. Errors

Following numerous weighings, G. Girard [of BIPM] observed that the mass value found for a standard varies in a rather reproducible fashion with the position taken by this standard in the sequence of the four standards used during the weighing. The reference standard (of supposedly known mass) was, of course, always the same artifact. Another indication of a problem arose in the comparison of values obtained by a global treatment of the subweighings made in the course of the following six weighings involving six standards: (ABCD), (ABCE), (ABCF), (ABDE), (ABDF), and (ABEF) compared with those which one obtains from the following weighings: (ABCD), (BCDE), (CDEF), (DEFA), (EFAB), and (FABC). Each set of six weighings supplies 36 observations. Mass values of B, C, D, E, and F can be obtained for each set by the method of least squares assuming that the mass of A is known. The values obtained for the masses of B, C, D, E, and F using the two different ts of six weighings shown above differ significantly. (For future reference, we refer to the first set above as Set I and the second set as Set II).

The subweighings are therefore tainted by errors, which are to some extent reproducible. These errors are compensated in the case of Set II (where all the standards play identical roles), but not in the case of Set I, nor in the case of a single weighing.

It was first supposed that all the subweighings were tainted by the same error, which could be caused by a non-linearity of the drift in equilibrium position of the balance at constant load. The global treatment of a number of subweighings has not confirmed this hypothesis, a result consistent with the remark made in 1.3.

One then supposed that the error depends on the rank of the subweighing—that is to say that the result of a subweighing of rank i ($i=1,2,\ldots,6$) gives a result tainted by an error ϵ_i.

Thus each m_i, in addition to random error, is biased by a systematic error ϵ_i. To notate this explicitly, consider the measured mass difference m_1. The quantity m_1, we now must admit, has two components which we separate in the following way:

$$m_1=m_1'+\epsilon_1$$

where m_1' is an estimate of m_A-m_B subject only to random errors having a variance var(m), and ϵ_1 is the bias in the measurement. (As a first approximation, we suppose that the ϵ_i's do not themselves have a random component.) Thus, it is incorrect to treat m_1 as an unbiased estimate of m_A-m_B. We now assume that m_1 is in reality an unbiased estimate of $m_A-m_B+\epsilon_1$, i.e.

$$m_A-m_B+\epsilon_1=m_1.$$

The correct conditional equations corresponding to the classic design of the weighing (ABCD) are thus

$$m_A-m_B+\epsilon_1=m_1$$

$$m_A-m_C+\epsilon_2=m_2$$

$$m_A-m_D+\epsilon_3=m_3$$

$$m_B-m_C+\epsilon_4=m_4$$

$$m_B-m_D+\epsilon_5=m_5$$

$$m_C-m_D+\epsilon_6=m_6$$

If one now assumes that each ϵ_i keeps a constant value for weighings made on different days, but according to a schedule and a procedure as invariant as possible, one may make a global solution of a number of weighings and avail oneself of sufficient conditional equations to find the six unknown ϵ_i's thus introduced. (Note that the ϵ_i's cannot be uniquely determined from Set I but can be uniquely determined from Set II.)

The first such estimate gave[2]

$$\epsilon_2\approx\epsilon_3\approx\epsilon_4\approx\epsilon_5\approx\epsilon_6\approx 2.8\ \mu\text{g, denoted by } \epsilon;$$

$$\epsilon_1\approx-1.4\ \mu\text{g}\approx-\epsilon/2.$$

Introduction of the ϵ_i's into the treatment of results from Set II has, in addition, led to a notable reduction of the residual deviations. Typical data are presented in tables 1 and 2.

Let us take up again the conditional equations used for the classical weighing design (ABCD) in the form

$$m_A-m_B=m_1'+\epsilon_1$$

$$m_A-m_C=m_2'+\epsilon_2$$

$$m_A-m_D=m_3'+\epsilon_3$$

$$m_B-m_C=m_4'+\epsilon_4$$

$$m_B-m_D=m_5'+\epsilon_5$$

$$m_C-m_D=m_6'+\epsilon_6$$

Looking at the conditional equations in this form, we can estimate how the ϵ_i's would effect results calculated in ignorance of the bias which, in fact, exists.

It is very clear that the results obtained in section 2 are immediately applicable, by replacing m_i by ϵ_i, in order to find both the errors arising from the ϵ_i's as well as the contribution of the ϵ_i's to the residual deviations.

One obtains the following expressions and, taking account of the estimates given above for the ϵ_i's, the following numerical values:

$$\Delta m_B'=(-2\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4+\epsilon_5)/4\approx+\epsilon/4\approx+0.7\ \mu\text{g}$$

$$\Delta m_C'=(-\epsilon_1-2\epsilon_2-\epsilon_3-\epsilon_4+\epsilon_6)/4\approx-5\epsilon/8\approx-1.8\ \mu\text{g}$$

$$\Delta m_D'=(-\epsilon_1-\epsilon_2-2\epsilon_3-\epsilon_5-\epsilon_6)/4\approx-9\epsilon/8\approx-3.2\ \mu\text{g}$$

where $\Delta m_B'$, for instance, is the error in the calculated mass value of B which is incurred by ignoring the existence of bias.

[2]The results for the primary kilogram comparator used at NBS suggest that $\{\epsilon_i\}\approx\epsilon\approx 2.6\ \mu$g.[2]).

Table I. Results from a modified Set II design (only five weights were used). The column labeled "w" lists deviations to the least squares solution of the 11×30 design matrix shown. The column labeled "w/o" shows deviations to the least squares solution of the 5×30 design matrix in which the ϵ_i's are ignored.

A	B	C	D	E	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6			
Constraint													
+											*0.1030 mg*		
Observations													
												w	w/o
+	−				+						−0.1792	−1.4 μg	−2.5 μg
+		−				+					−0.0562	−1.5	1.0
+			−				+				−0.2378	0.1	3.3
	+	−						+			0.1226	1.2	3.1
	+		−						+		−0.0589	2.7	5.5
		+	−							+	−0.1790	1.7	4.9
	+	−			+						0.1188	0.6	−0.7
	+		−			+					−0.0612	0.7	3.2
	+			−			+				0.1059	−0.8	2.2
		+	−					+			−0.1822	−0.4	1.7
		+		−					+		−0.0139	−0.9	1.9
			+	−						+	0.1702	−0.9	2.1
		+	−		+						−0.1861	−1.2	−2.2
		+		−		+					−0.0161	−2.8	−0.3
−		+					+				0.0613	1.0	4.1
			+	−				+			0.1703	0.3	2.2
−			+						+		0.2434	−0.4	2.3
−				+						+	0.0755	−0.6	2.5
			+	−	+						0.1698	2.9	1.7
−			+			+					0.2446	1.0	3.5
	−		+				+				0.0683	0.8	3.9
−				+				+			0.0735	−1.6	0.5
	−			+					+		−0.1018	−1.0	1.9
+	−									+	−0.1723	1.3	4.4
−				+	+						0.0711	−0.8	−1.9
	−			+		+					−0.0984	2.6	5.3
		−		+			+				0.0179	−1.0	2.1
+	−							+			−0.1741	0.5	2.6
+		−							+		−0.0548	−0.3	2.4
	+	−								+	0.1210	−1.5	1.5
				Σg_i^2:								60 μg^2	260 μg^2
				Variance of the Fit:								3 μg^2	10 μg^2
				Std. Dev. of the Fit:								1.7 μg	3.1 μg
				Degrees of Freedom:								20	26

The following set of Δg_i are the residual deviations to the least squares fit due only to the ϵ_i's.

$$\Delta g_1=(2\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4+\epsilon_5)/4 \quad \approx -\epsilon/4 \quad \approx -0.7\ \mu g$$

$$\Delta g_2=(-\epsilon_1+2\epsilon_2-\epsilon_3-\epsilon_4+\epsilon_6)/4 \approx +3\epsilon/8 \approx +1.0\ \mu g$$

$$\Delta g_3=(-\epsilon_1-\epsilon_2+2\epsilon_3-\epsilon_5-\epsilon_6)/4 \approx -\epsilon/8 \quad \approx -0.4\ \mu g$$

$$\Delta g_4=(\epsilon_1-\epsilon_2+2\epsilon_4-\epsilon_5+\epsilon_6)/4 \quad \approx +\epsilon/8 \quad \approx +0.4\ \mu g$$

$$\Delta g_5=(\epsilon_1-\epsilon_3-\epsilon_4+2\epsilon_5-\epsilon_6)/4 \quad \approx -3\epsilon/8 \approx -1.0\ \mu g$$

$$\Delta g_6=(\epsilon_2-\epsilon_3+\epsilon_4-\epsilon_5+2\epsilon_6)/4 \quad \approx +\epsilon/2 \quad \approx +1.4\ \mu g$$

$$\Sigma_i(\Delta g_i)^2 \approx 5\epsilon^2/8 \quad \approx 4.9\ \mu g^2.$$

Note that this result ignores cross terms which would be present if the random error is non-negligible. The cross terms, however, may be either positive or negative so that considerable cancellation occurs in their summation. Thus semiquantitative conclusions may still be drawn even though the cross terms are ignored.

Now, $\Sigma_i g_i^2$ is typically about 12 μg^2. For the classical weighing design with 6 observations and 3 independent unknowns the variance of an observation is $s^2=(1/3)\ \Sigma_i g_i^2=4$ μg^2, and the variance of a single result of the weighing is $s^2/2=2$ μg^2.

One sees that the ϵ_i's contribute in a modest (and, what is more, not directly detectable) way to the residual deviations, while they impose significant errors on the results of the weighing.

Let us pursue this analysis by now considering groups of weighings.

Table 2. The variance/convariance matrix for the 11×30 design shown in table 1 as well as least squares solutions to A, B, C, D, E, and the ϵ_i's. The mass values calculated for A-E would have been the same even if the ϵ_i's had not been included in the model. Note that, assuming cancellation of cross terms, we would estimate Σg_i^2 (w/o)$=\Sigma g_i^2$(w)$+\Sigma(\Delta g_i)^2$, where $\Sigma(\Delta g_i)^2=5(\Sigma\epsilon_i^2)=195$ μg^2. The data of table 1 verify this assumption.

Variance/Covariance Matrix

A	B	C	D	E	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6
0	0	0	0	0	0	0	0	0	0	0
0	2	1	1	1	0	0	0	0	0	0
0	1	2	1	1	0	0	0	0	0	0
0	1	1	2	1	0	0	0	0	0	0
0	1	1	1	2	0	0	0	0	0	0
0	0	0	0	0	3	0	0	0	0	0
0	0	0	0	0	0	3	0	0	0	0
0	0	0	0	0	0	0	3	0	0	0
0	0	0	0	0	0	0	0	3	0	0
0	0	0	0	0	0	0	0	0	3	0
0	0	0	0	0	0	0	0	0	0	3

Divisor=15

	Least Squares Results	**Std. Dev.**
A	0.1030 mg	0.0 μg
B	0.2797	0.6
C	0.1602	0.6
D	0.3441	0.6
E	0.1760	0.6
ϵ_1	−1.1 μg	0.7 μg
ϵ_2	2.6	0.7
ϵ_3	3.1	0.7
ϵ_4	2.0	0.7
ϵ_5	2.8	0.7
ϵ_6	3.1	0.7

For the Set II weighings described above (36 observations, 5 unknown weights), an examination of the least squares solutions with and without the ϵ_i's treated explicitly shows: the ϵ_i's do not impose any error on the results, which would be the same either for a treatment derived from starting equations such as $m_A-m_B=m_1$ or from equations such as $m_A-m_B+\epsilon_1=m_1$. The failure to consider the ϵ_i's explicitly, however, adds a residual deviation of ϵ_i to each subweighing of rank i. The sum of the squares of these contributions is

$$6\Sigma_i\epsilon_i^2\approx6(\epsilon^2/4+5\epsilon^2)=63\epsilon^2/2\ ,$$

a value which contributes to the variance of an observation $(63\epsilon^2/2)/(36-5)=1.02\epsilon^2\approx8$ μg^2. Once again, this is only an estimate because cross terms involving random error components have been neglected. Similar conclusions can be drawn from the Set II weighings shown in tables 1 and 2 (30 observations, 4 unknown weights).

Introduction of the six supplementary unknowns ϵ_i into the treatment appreciably reduces the variance of an observation. For the example shown in table 1, which is typical, the variance went from 9.6 μg^2 to 2.9 μg^2.

For the Set I weighings described above, numerical calculation gives

$$\Delta m_B'\approx+0.25\epsilon\approx+0.7\ \mu g$$

$$\Delta m_C'\approx-0.58\epsilon\approx-1.6\ \mu g$$

$$\Delta m_D'\approx-0.78\epsilon\approx-2.2\ \mu g$$

$$\Delta m_E'\approx-0.98\epsilon\approx-2.7\ \mu g$$

$$\Delta m_F'\approx-1.18\epsilon\approx-3.3\ \mu g$$

and, for the variance of an observation, $0.15\epsilon^2\approx1.2$ μg^2. Once again, these are approximate errors which one would suffer through ignorance of the existence of a bias.

One thus sees how imprudent it would have been to choose the design which gives the smaller variance since, for this design, the ϵ_i's impose significant errors on the result of the weighings while, for the design which gives the greater variance, they impose no errors.

4. Remedies

It would not be judicious to retain the classic design for weighings, choosing a set of such designs which ensures that the contribution of the ϵ_i's to the final error of the results is zero. For such a case, the contribution of the ϵ_i's to the variance of an observation would lead to a serious over-estimation of the variance. One could, of course, include the ϵ_i's explicitly in the analysis. However, this approach would greatly increase the number of necessary observations over what had been previously required, and it would be based on the assumption that the ϵ_i's are constant throughout the many days required for a set of measurements.

It goes without saying that the solution lies in discovering the physical cause for the existence of the ϵ_i's and in eliminating it. But, in the meanwhile, one must carry out weighings and, since it is impossible to eliminate the cause, it is necessary to eliminate the effect; that is to say to ensure that the ϵ_i's compensate themselves as exactly as possible, or, put another way, to find an *unbiased* observation which will estimate m_A-m_B, for example.

The reader may have been struck by the fact that we have adopted for all the ϵ_i $(i=2,3...,6)$'s the same value ϵ, while $\epsilon_1=-\epsilon/2$. Without doubt, he or she has good reason to think that it would have been simple to obtain $\epsilon_1=\epsilon$ operationally; for example, by the addition of a preliminary subweighing.

The expressions derived above would be modified in this case to be:

$$\Delta m'_B = -\epsilon/2 = -1.4\ \mu g$$

$$\Delta m'_C = -\epsilon = -2.8\ \mu g$$

$$\Delta m'_D = -3\epsilon/2 = -4.2\ \mu g$$

$$\Delta g_1 = +\epsilon/2 = +1.4\ \mu g$$

$$\Delta g_2 = 0$$

$$\Delta g_3 = -\epsilon/2 = -1.4\ \mu g$$

$$\Delta g_4 = +\epsilon/2 = +1.4\ \mu g$$

$$\Delta g_5 = 0$$

$$\Delta g_6 = +\epsilon/2 = +1.4\ \mu g$$

$$\Sigma_i (\Delta g_i)^2 = \epsilon^2 = 7.8\ \mu g^2 .$$

One would thus obtain both a worsening of the errors attached to the results and an increase in the residual deviations.

For Set II weighings the error attached to the results would, of course, remain zero but the contribution of the ϵ_i's to the variance would be of order $6\times 6\epsilon^2/31 = 1.16\epsilon^2 \approx 9\ \mu g^2$.

For Set I, a numerical calculation indicates that one would then have

$$\Delta m'_B \approx -0.5\epsilon \approx -1.4\ \mu g$$

$$\Delta m'_C \approx -0.9\epsilon \approx -2.7\ \mu g$$

$$\Delta m'_D \approx -1.15\epsilon \approx -3.2\ \mu g$$

$$\Delta m'_E \approx -1.35\epsilon \approx -3.8\ \mu g$$

$$\Delta m'_F \approx -1.55\epsilon \approx -4.3\ \mu g$$

and, for the variance of an observation, $0.23\epsilon^2 \approx 1.8\ \mu g^2$.

Contrary to what one might intuitively think, the act of making the ϵ_i's equal would not automatically eliminate errors in the results.

At this juncture, a remark must be made. In the course of the subweighing of rank i, if one intercompares standards Y and X instead of X and Y, one replaces the equation

$$m_X - m_Y + \epsilon_i = m_{i,1}$$

with

$$m_Y - m_X + \epsilon_i = m_{i,2}$$

which is equivalent to $m_X - m_Y - \epsilon_i = -m_{i,2}$. The sign of ϵ_i is thus reversed.

One could hope that by changing the sign of certain, judiciously chosen, ϵ_i's one might eliminate, for each weighing, errors in $\Delta m'_B$, $\Delta m'_C$, $\Delta m'_D$. Unfortunately, there is nothing to be gained from this approach; in essence, one would then have (see eq (2)) $\Delta m'_B + \Delta m'_C + \Delta m'_D = \pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3$ an expression which cannot vanish as long as $\epsilon_1 = \epsilon_2 = \epsilon_3 \neq 0$.

But rather than replacing the intercomparison of X and Y by that of Y and X, it is quite clear that one must make both comparisons, which we will call "opposed comparisons," under as identical conditions as possible and, in particular, in subweighings of the same rank. The two equations written above give, assuming that ϵ_i keeps the same value during the two subweighings,

$$\epsilon_i = (m_{i,1} + m_{i,2})/2$$

and

$$m_X - m_Y = (m_{i,1} - m_{i,2})/2.$$

It is this last value which ought to be introduced into the conditional equations. This approach requires that the ϵ_i's remain essentially constant for only two successive days.

The problem is, therefore, to define a weighing design to which one may make a corresponding "opposed design." *A priori*, if one sticks to the principle of the classic weighing in which four standards are involved for which one determines six paired differences, one can devise $6!2^6 = 46080$ different designs (since one can imagine all permutations of the 6 difference determinations and each difference determination can be realized by one or the other of two opposed comparisons). The ensemble of the designs is composed of 23040 pairs such that, in each of them, subweighings of the same rank correspond to opposed comparisons.

We now note that, in the course of the classic subweighing involving the intercomparison of X and Y (which we will write as (X,Y)), the first and fifth operations involve X. If the following comparison is (X,Z), with of course Z≠Y, the first operation again involves X and there is no rotation of the weight table between these two comparisons. On the other hand, the two opposed comparisons (Y,X) and (Z,X) are separated by a rotation of the weight table. One might worry that the value of ϵ_i is a function of whether the subweighing of rank i is or is not preceded by a rotation of the weight table. It is therefore preferable that absences and presences of these rotations correspond in the opposed design, but we have just seen that an absence of rotation always corresponds to a rotation in the opposed designs. We must therefore exclude all designs for which a subweighing is not preceded by a rotation—either in the first design or in the opposed design. This can be simply expressed through the following conditions: for the two consecutive comparisons (X,Y) and (Z,T) it is necessary that Z≠X and T≠Y. With these restrictions, there still remain 1776 pairs of possible designs.[3]

One of the paired designs is, for example,

(A,B)	(B,A)
(B,C)	(C,B)
(C,D)	(D,C)
(D,A)	(A,D)
(A,C)	(C,A)
(B,D)	(D,B) .

The first column (i.e., the direct weighing) corresponds to the following conditional equations

$$m_A - m_B + \epsilon_1 = m_1$$

$$m_B - m_C + \epsilon_2 = m_2$$

$$m_C - m_D + \epsilon_3 = m_3$$

$$m_D - m_A + \epsilon_4 = m_4$$

$$m_A - m_C + \epsilon_5 = m_5$$

$$m_B - m_D + \epsilon_6 = m_6 \ .$$

This design, used by itself, would not have a marked advantage over the classic design. It can be shown that it would lead, nevertheless, to smaller errors but also to a significant overestimation of the variance of an observation. It is understood that if one chooses it from among designs which satisfy the conditions we have imposed (and which seem equivalent), one must also use the opposed design and take as conditional equations

$$m_A - m_B = (m_{1,1} - m_{1,2})/2$$

$$m_B - m_C = (m_{2,1} - m_{2,2})/2$$

$$m_C - m_D = (m_{3,1} - m_{3,2})/2$$

$$m_D - m_A = (m_{4,1} - m_{4,2})/2$$

$$m_A - m_C = (m_{5,1} - m_{5,2})/2$$

$$m_B - m_D = (m_{6,1} - m_{6,2})/2$$

in which the differences derived for a subweighing of rank i ($i = 1,2,\ldots 6$) are denoted by $m_{i,1}$ for the direct weighing and $m_{i,2}$ for the opposed weighing.

[3]P. Carré has rigorously derived this number. We have chosen, for the sake of brevity, to omit the derivation here.

References

[1] Almer, H.E., J. Res. Nat. Bur. Stand. (U.S.) **76C** (1 and 2): 1-10 (January-June, 1972).

[2] Davis, R.S., J. Res. Nat. Bur. Stand. (U.S.) **90-4** 263-283 (July-August, 1985).