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On the Decomposition of Vertex-Transitive Graphs into Multicycles*

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In this paper, we prove that every vertex-transitive graph can be expressed as the edge-disjoint union of symmetric graphs. We define a multicycle graph and conjecture that every vertex-transitive graph can be expressed as the edge-disjoint union of multicycles. We verify this conjecture for several subclasses of vertex-transitive graphs, including Cayley graphs, multidimensional circulants, and vertex-transitive graphs with a prime or twice a prime number of nodes. We conclude with some open questions of interest.

Key words: Cayley graph; circulant; cycle decomposition; edge-transitive graph; grouplike set; line-symmetric graph; multicycle; multidimensional circulant; point-symmetric graph; starred polygon; symmetric graph; vertex-transitive graph.

1. Introduction

Following the notation of [7,9],[1] we denote the set of nodes of a finite, simple graph X by $V(X)$, the set of edges by $E(X)$, and the automorphism group of X by $G(X)$. Throughout, we regard $G(X)$ as a permutation group on the nodes, and sometimes the edges, of X. In particular, a subgroup J of $G(X)$ is said to be *transitive* if for every pair of nodes $u,v \in V(X)$, J contains an automorphism mapping u to v. If, in addition to being transitive, $o(J)=o(V(X))$, then J is a *regular* subgroup of $G(X)$. It is well known (see Lemma 16.3 of [4], for example) that $G(X)$ contains a regular subgroup if and only if X is a Cayley graph. A *Cayley graph* $X_{G,H}$ is the graph defined by $V(X_{G,H})=\{\alpha \mid \alpha \in G\}$ and $E(X_{G,H})=\{(\alpha,\beta) \mid \alpha\beta^{-1} \in H\}$ where G is an abstract group and H is a subset of $G-\{1\}$ closed under inverses.

We are interested primarily in those graphs with a transitive automorphism group. Such graphs are called *vertex-transitive* or, equivalently, *point-symmetric*. Similarly, a graph is called *edge-transitive* or, equivalently, *line-symmetric* if $G(X)$ is transitive on the *edges* of X. Graphs which are both vertex-transitive and edge-transitive are called *symmetric*. As is pointed out in [4,6,8], not every vertex-transitive graph is edge-transitive nor is every edge-transitive graph vertex-transitive. An area of recent interest in the literature involves the relationship between the class of vertex-transitive graphs and the class of edge-transitive graphs, and the nature of their intersection, the class of symmetric graphs [3–6,8].

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[1] Figures in brackets indicate literature references at the end of this paper.

Of particular interest is the class of circulants [1,2,7,8,10,11] and a generalization thereof, the class of multidimensional circulants [7,8]. A *circulant* or, equivalently, a *starred polygon* is a graph whose nodes can be labeled so that there exists a set $S \subset \mathrm{Z}$, where Z is the set of integers, such that $V(X)=\{v_0, \ldots, v_{n-1}\}$ and $E(X)=\{(v_i, v_j) \mid 0 \leqslant i, j < n$ and $\mathrm{mod}(i-j,n) \in S\}$. (By $\mathrm{mod}(x,y)$, we mean the remainder of x upon division by y.) For such a graph, the pair (n,S) is called a *symbol* for X. In [7], we generalize this concept and define a *multidimensional circulant* to be a graph X whose nodes can be labeled so that there exist an integer k, a k-vector $\mathbf{a}$, and a set $S \subset \mathrm{Z}^k$ such that $V(X)=\{v_{\mathbf{i}} \mid \mathbf{0} \leqslant \mathbf{i} < \mathbf{a}\}$ and $E(X)=\{(v_{\mathbf{i}}, v_{\mathbf{j}}) \mid \mathrm{mod}(\mathbf{i}-\mathbf{j},\mathbf{a}) \in S\}$. The pair $(\mathbf{a},S)$ is called a *k-symbol* of X. (We employ the vector notation introduced in [7] whenever discussing multidimensional circulants. In particular, $\mathbf{i}=(i_1, \ldots, i_k)$, $j=(j_1, \ldots, j_k)$, $\mathbf{a}=(a_1, \ldots, a_k)$, $\mathbf{0}=(0, \ldots, 0)$, $\mathrm{mod}(\mathbf{i}-\mathbf{j},\mathbf{a})=(\mathrm{mod}(i_1-j_1, a_1), \ldots, \mathrm{mod}(i_k-j_k, a_k))$, and $\mathbf{0} \leqslant \mathbf{i} < \mathbf{a}$ if and only if $0 \leqslant i_l < a_l$ for $1 \leqslant l \leqslant k$.)

In this paper, we investigate the decomposition of vertex-transitive graphs into edge-disjoint symmetric graphs. In particular, we prove in section 2 that every vertex-transitive graph can be expressed as the edge-disjoint union of symmetric graphs.

In section 3, we define a grouplike set and a multicycle graph and use their properties to extend the result of section 2. We conjecture that every vertex-transitive graph can be expressed as the edge-disjoint union of multicycles. This conjecture is verified for several subclasses of vertex-transitive graphs, including Cayley graphs, multidimensional circulants, and vertex-transitive graphs with a prime or twice a prime number of nodes.

We conclude by mentioning some related problems of interest in section 4. In particular, we show how to construct a multicycle decomposition from the symbol of any multidimensional circulant.

2. Symmetric Graph Decomposition

Let X be any graph, e an edge of X, and G a subgroup of $G(X)$. The *orbit* of e under G is defined as the subgraph $X_{G,e}$ of X which has nodes $V(X)$ and edges $\{\sigma(e) \mid \sigma \in G\}$. The orbits of X possess several well-known and useful properties. We cite three such properties in the following lemmas. The proofs of these lemmas are not difficult and are deferred until section 3, where we prove similar results for a more general subset of $G(X)$.

LEMMA 1: $\mathrm{X}_{\mathrm{G,e}}$ *is edge-transitive.*

LEMMA 2: $\mathrm{G} \subseteq \mathrm{G}(\mathrm{X}_{\mathrm{G,e}})$.

LEMMA 3: X *can be expressed as the edge-disjoint union of the distinct* $\mathrm{X}_{\mathrm{G,e}}$.

With the use of these lemmas, it is not difficult to prove:

THEOREM 1: *Every vertex-transitive graph* X *can be expressed as the edge-disjoint union of one or more symmetric graphs, each with vertex set* V(X).

PROOF: Let X be any vertex-transitive graph and let $G=G(X)$. Consider the graphs $X_{G,e}$. By definition, they each have vertex set $V(X)$. From Lemma 1, we know that each $X_{G,e}$ is edge-transitive. Since X is vertex-transitive, G must be transitive, and, by Lemma 2, we know that each $X_{G,e}$ is vertex-transitive. Thus, each $X_{G,e}$ is symmetric. Finally, we know from Lemma 3 that X can be expressed as the edge-disjoint union of the distinct $X_{G,e}$.□

3. Multicycle Decomposition

Call a graph a *multicycle* if it can be written as the node-disjoint union of equal length cycles. In particular, for any pair of positive integers b and d, define the multicycle $C_{b,d}$ to be the graph consisting of b node-disjoint d-cycles. Several examples are provided in figure 1. (Note that we have adopted the convention that every edge is a 2-cycle and that every node is a 1-cycle.)

It is not difficult to show that every multicycle is symmetric. We state a partial converse of this fact in the following lemma.

LEMMA 4: *If* X *is a vertex-transitive graph and* $\mathrm{o(E(X))} \leqslant \mathrm{o(V(X))}$, *then* X *is a multicycle.*

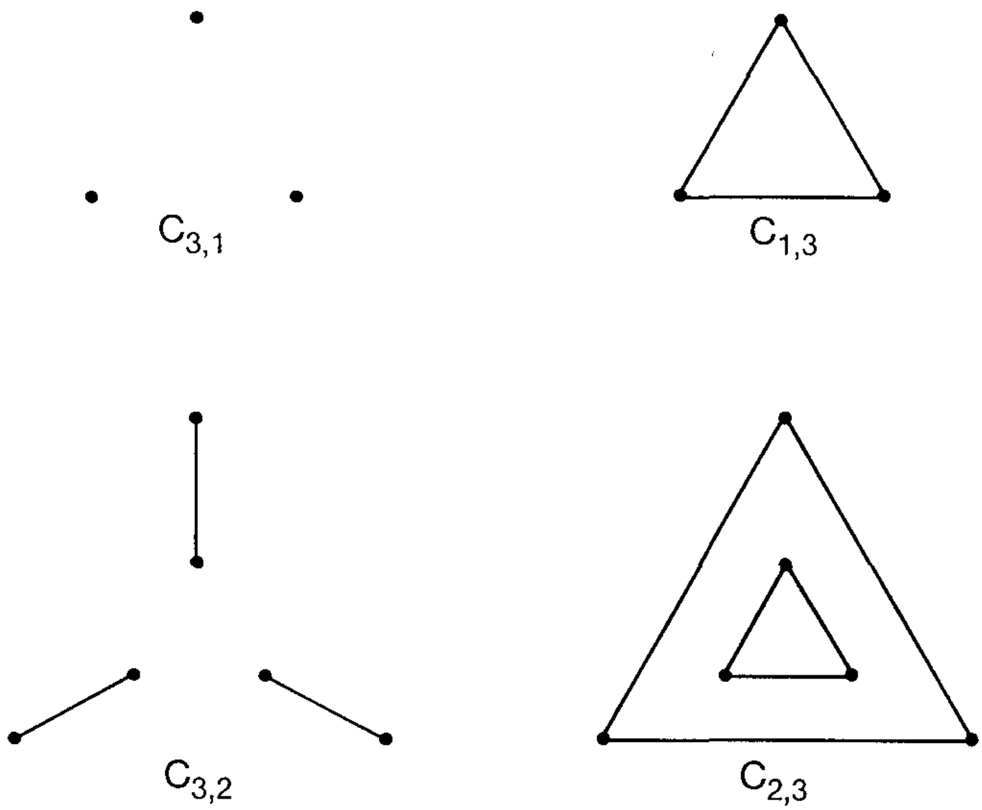

Figure 1

PROOF: Since X is vertex-transitive, all the nodes of X must have the same degree. Since $o(E(X)) \leqslant o(V(X))$, this common degree is 0, 1 or 2. In the first case, $X \simeq C_{n,1}$ where $n = o(V(X))$. In the second case, X consists entirely of node-disjoint edges and $X \simeq C_{n/2,2}$. In the final case, X is the node-disjoint union of cycles and, since X is vertex-transitive, each cycle must have the same length. Thus $X \simeq C_{b,d}$ for some b and d such that $bd = n$.□

We now extend Theorem 1.

THEOREM 2: *Every Cayley graph* X *can be expressed as the edge-disjoint union of multicycles, each with vertex set* V(X).

PROOF: The proof is identical to that of Theorem 1 with an additional observation. Let X be any Cayley graph and let R be a regular subgroup of $G(X)$. By definition, R is transitive and has $o(V(X))$ elements. Thus each $X_{R,e}$ is symmetric and has at most $o(V(X))$ edges. Since $o(E(X_{R,e})) \leqslant o(V(X)) = o(V(X_{R,e}))$ for every $e \in E(X)$, we know by Lemma 4 that each $X_{R,e}$ is a multicycle.□

COROLLARY 1: *Every multidimensional circulant can be expressed as the edge-disjoint union of multicycles. In particular, every vertex-transitive graph with a prime number of nodes can be expressed as the edge-disjoint union of multicycles.*

PROOF: We know from [7] that every vertex-transitive graph with a prime number of nodes is a circulant and that the automorphism group of a multidimensional circulant contains a regular abelian subgroup. Thus, such graphs are Cayley graphs.□

There are some vertex-transitive graphs, however, with automorphism groups which do not contain a regular subgroup. The Petersen graph shown in figure 2 is one such graph.

The automorphism group of this graph does contain a 10-element, transitive subset, however, which is very similar to a subgroup in structure. This subset is $M = \{\alpha^i \gamma^j \mid 0 \leqslant i < 2 \text{ and } 0 \leqslant j < 5\}$ where $\alpha = (1\ 6)\ (2\ 8\ 5\ 9)\ (3\ 10\ 4\ 7)$ and $\gamma = (1\ 2\ 3\ 4\ 5)\ (6\ 7\ 8\ 9\ 10)$. Note that M is not a subgroup as $\alpha^2 = (1)\ (6)\ (2\ 5)\ (3\ 4)\ (7\ 10)\ (8\ 9) \notin M$. Define $X_{M,e}$ to be the subgraph of the

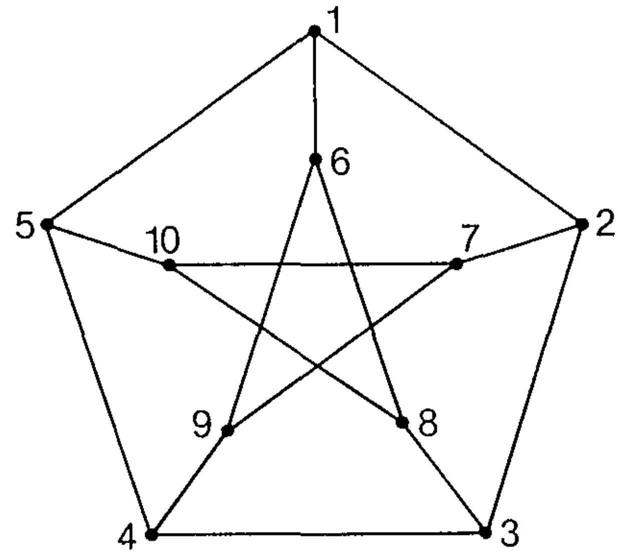

Figure 2

Petersen graph with ten nodes and edge set $\{\sigma(e)|\sigma\in M\}$ for any edge e. The subgraph $X_{M,e}$ is quite similar in structure to an orbit subgraph. In fact, it is not difficult to show that the $X_{M,e}$ satisfy the conditions stated in Lemmas 1-3. Since M has $o(V(X))=10$ elements, we may then apply the arguments of Theorem 2 to conclude that the Petersen graph may be expressed as the edge-disjoint union of multicycles. We now generalize this result.

DEFINITION: Given a graph X, a subset M of $G(X)$ is *grouplike* if for every edge $e\in E(X)$, the following three conditions are met:

GL1) $\forall\sigma_1\in M, \exists\sigma_2\in M$ such that $\sigma_2(e)=\sigma_1^{-1}(e)$,

GL2) $\forall\sigma_1,\sigma_2\in M, \exists\sigma_3\in M$ such that $\sigma_3(e)=\sigma_1\sigma_2(e)$, and

GL3) $\exists\sigma\in$ M such that $\sigma(e)=e$.

Note that the definition of grouplike is very similar to that of a subgroup. The only difference is that we have reversed the order of the $\forall e$ and $\exists\sigma$ quantifiers in forming the definition of grouplike. Thus any subgroup of $G(X)$ is grouplike but not conversely. As an example, it is easily checked that $M=\{\alpha^i\gamma^j|0\leqslant i<2 \text{ and } 0\leqslant j<5\}$ is a grouplike subset, but not subgroup, of the automorphism group of the Petersen graph.

Let X be a graph, M a grouplike subset of $G(X)$ and e an edge in X. Define $X_{M,e}$ to be the subgraph of X with nodes $V(X)$ and edges $\{\sigma(e)|\sigma\in M\}$. The following are generalizations of Lemmas 1-3.

LEMMA 5: $X_{M,e}$ *is edge-transitive.*

PROOF: Given any $e_1,e_2\in E(X_{M,e})$, we know from the definition that $\exists\sigma_1,\sigma_2\in M$ such that $\sigma_1(e)=e_1$ and $\sigma_2(e)=e_2$. By GL1, we know that $\exists\sigma_3\in M$ such that $\sigma_3(e_1)=\sigma_1^{-1}(e_1)=e$. By GL2, we have that $\exists\sigma_4\in M$ such that $\sigma_4(e_1)=\sigma_2\sigma_3(e_1)=\sigma_2(e)=e_2$. Thus $X_{M,e}$ is edge-transitive.□

LEMMA 6: $M\subseteq G(X_{M,e})$.

PROOF: Given any $e\in E(X)$, $\sigma_1\in M$, and $e'\in E(X_{M,e})$, it suffices to show that $\sigma_1(e')\in E(X_{M,e})$. For then it will be clear that σ_1 preserves the edge structure of $X_{M,e}$ and thus that $\sigma_1\in G(X_{M,e})$ and that $M\subseteq G(X_{M,e})$. By definition, $\exists\sigma_2\in M$ such that $\sigma_2(e)=e'$. By GL2, $\exists\sigma_3\in M$ such that $\sigma_3(e)=\sigma_1\sigma_2(e)=\sigma_1(e')$. Thus $\sigma_1(e')\in E(X_{M,e})$ as desired.□

LEMMA 7: X *can be expressed as the edge-disjoint union of the distinct* $X_{M,e}$.

PROOF: We first show that for any $e_1,e_2\in E(X)$, either $E(X_{M,e_1})=E(X_{M,e_2})$ or $E(X_{M,e_1})\cap E(X_{M,e_2})=\emptyset$. In particular, choose $e\in E(X_{M,e_1})\cap E(X_{M,e_2})$. From the definition, we know that $\exists\sigma_1\in M$ such that $\sigma_1(e_1)=e$. By the transitivity of X_{M,e_2}, we know that given any $e_3\in E(X_{M,e_2})$, $\exists\sigma_2\in M$ such that $\sigma_2(e)=e_3$. Again applying GL2, this means that $\exists\sigma_3\in M$ such that $\sigma_3(e_1)=$

$\sigma_2\sigma_1(e_1)=\sigma_2(e)=e_3$. Thus $E(X_{M,e_2})\subseteq E(X_{M,e_1})$. By reversing e_1 and e_2 in the above argument, it is equally simple to show that $E(X_{M,e_1})\subseteq E(X_{M,e_2})$. Thus either $E(X_{M,e_1})\cap E(X_{M,e_2})=\emptyset$ or $E(X_{M,e_1})=E(X_{M,e_2})$.

The argument is completed by observing that every edge of X is included in some $X_{M,e}$ by GL3.□

We now state the corresponding generalization of Theorem 2.

THEOREM 3: *If the automorphism group of a vertex-transitive graph* X *contains an* o(V(X))*-element transitive grouplike subset, then* X *can be expressed as the edge-disjoint union of multicycles.*

PROOF: The proof is nearly identical to that of Theorem 2 and follows trivially from Lemmas 5–7.□

As we have been unable to find a vertex-transitive graph with an automorphism group which does not contain an $o(V(X))$-element transitive grouplike subset, we make the following conjecture.

CONJECTURE: *Every vertex-transitive graph can be expressed as the edge-disjoint union of multicycles.*

In Theorem 2, we verified the conjecture for all Cayley graphs, multidimensional circulants, and, in particular, for all vertex-transitive graphs with a prime number of nodes. Using a different approach, we now verify the conjecture in another case, one which has received attention recently [2,8,10].

THEOREM 4: *Every vertex-transitive graph with twice a prime number of nodes can be expressed as the edge-disjoint union of multicycles.*

PROOF: Let X be a vertex-transitive graph with $2p$ nodes where p is a prime. Since X is vertex-transitive, the subgroup of automorphisms of X which fix a given node has index $2p$ in $G(X)$. Thus $2p\,|\,o(G(X))$ and, by Sylow's Theorem, $G(X)$ contains an element γ of order p. Since $o(\gamma)=p$, γ is either the composition of two p-cycles or the composition of one p-cycle and p fixed elements. Label the nodes of X so that $V(X)=\{v_{i,j}\,|\,0\leqslant i<2 \text{ and } 0\leqslant j<p\}$ and $\gamma=(v_{0,0}\ldots v_{0,p-1})(v_{1,0}\ldots v_{1,p-1})$ or $\gamma=(v_{0,0}\ldots v_{0,p-1})(v_{1,0})\ldots(v_{1,p-1})$, depending on the structure of γ. We consider the two cases separately.

CASE 1: $\gamma=(v_{0,0}\ldots v_{0,p-1})(v_{1,0})\ldots(v_{1,p-1})$.

Without loss of generality, we can assume that $(v_{0,0},v_{1,0})\in E(X)$. Otherwise, X is disconnected and consists of two isomorphic, node-disjoint vertex-transitive graphs on p nodes each. Thus X is a multidimensional circulant and, by Corollary 1, is the edge-disjoint union of multicycles.

Since $\gamma\in G(X)$ and $(v_{0,0},v_{1,0})\in E(X)$, we know that $\gamma^j(v_{0,0},v_{1,0})=(v_{0,j},v_{1,0})\in E(X)$, for $0\leqslant j<p$ and thus $\deg(v_{1,0})\geqslant p$. Since X is vertex-transitive, it must be a regular graph and we know that $\deg(v_{1,j})\geqslant p$ for $0\leqslant j<p$. Thus for any j such that $0\leqslant j<p$, there exists an i such that $(v_{0,i},v_{1,j})\in E(X)$. Again applying the knowledge that $\gamma\in G(X)$, we find that $(v_{0,i},v_{1,j})\in E(X)$ for $0\leqslant i,j<p$. Thus X^c, the complement graph of X, is disconnected and, therefore, a multidimensional circulant. In [7], we show that X is a multidimensional circulant if and only if $G(X)$ contains a regular abelian subgroup. Since $G(X)=G(X^c)$, we conclude that X is also a multidimensional circulant.

CASE 2: $\gamma=(v_{0,0}\ldots v_{0,p-1})(v_{1,0}\ldots v_{1,p-1})$.

Define $V_i=\{v_{i,j}\,|\,0\leqslant j<p\}$ for $0\leqslant i<2$. Let E' be the set of edges of X with one endpoint in V_0 and one endpoint in V_1. Partition E' according to the congruence relation $e_1\sim e_2$ if $e_1=\gamma^j(e_2)$ for some j. Since γ is cyclic over V_0 and V_1, each block of the partition corresponds to p node-disjoint edges (i.e., a $C_{p,2}$ multicycle).

Now consider the edges E_i with both endpoints in V_i for $0\leqslant i<2$. In a similar fashion, partition E_0 and E_1. Each block of this partition corresponds to a p-cycle. Since X is vertex-transitive, it is regular. We already know that each vertex of X is incident to the same number of edges from E'. Thus $o(E_0)=o(E_1)$ and we can pair up p-cycles in E_0 with p-cycles in E_1 to form $C_{2,p}$ multicycles.

Summarizing, if X has at least one edge, then X is decomposable into $C_{p,2}$ multicycles, $C_{2,p}$ multicycles, or both. If $E(X)=\emptyset$, then $X\simeq C_{2p,1}$.□

4. Related Problems

We now consider the question of what multicycle decompositions a vertex-transitive graph can have. The graph in figure 3, for example, has three different multicycle decompositions.

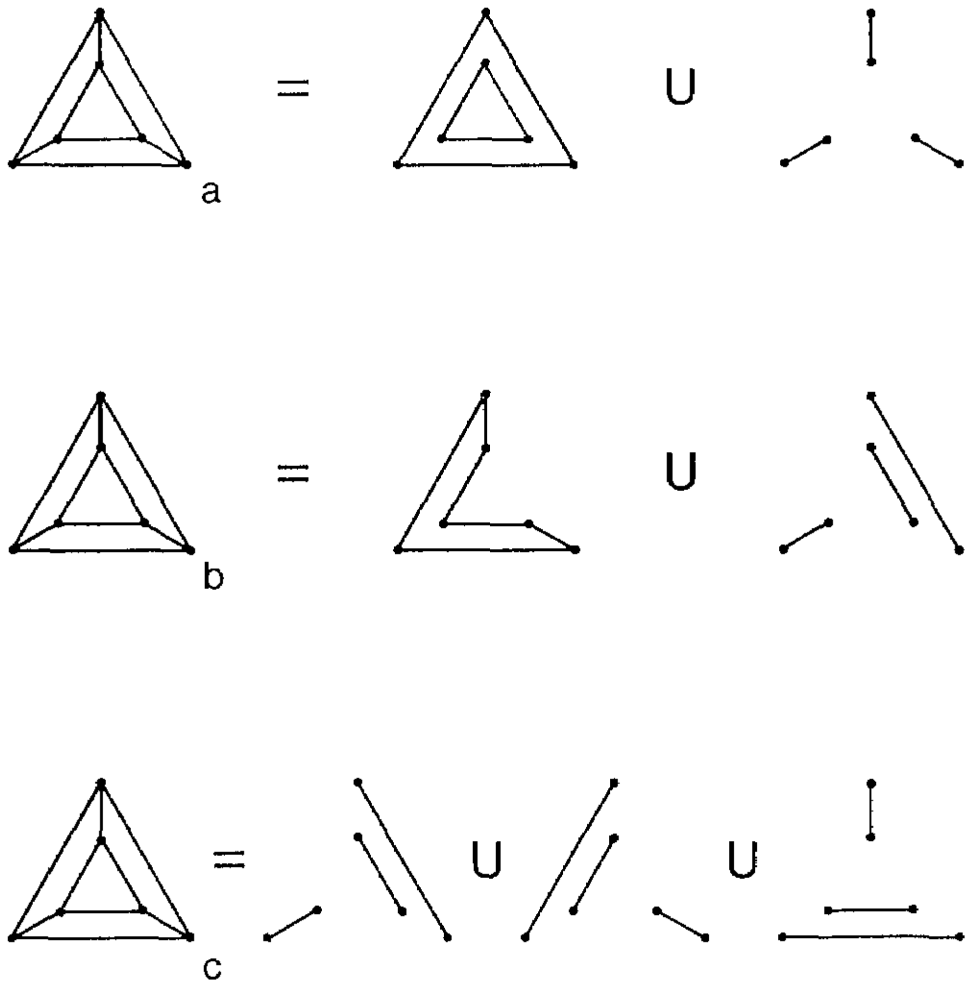

Figure 3

Define a *decomposition vector* $\mathbf{d}$ of a graph to be the vector $(d_1, \ldots, d_n)$ where $d_1, \ldots, d_n$ are such that X can be expressed as the edge-disjoint union of d_1 $C_{n,1}$ multicycles, d_2 $C_{n/2,2}$ multicycles, ..., and d_n $C_{1,n}$ multicycles. By convention, we set $d_1=0$ unless X has no edges, in which case $d_1=1$. We further require that $d_i=0$ for any i which does not integrally divide $n=o(V(X))$. As an example, we observe that the graph displayed in figure 3 has decomposition vectors (0,1,1,0,0,0), (0,1,0,0,0,1), and (0,3,0,0,0,0).

In the new terminology, the problem is to determine in some general way which decomposition vectors a given vertex-transitive graph can have. We now provide a partial solution to this problem.

THEOREM 5: *Let* X *be a multidimensional circulant with k-symbol* $(\mathbf{a},\mathrm{S})$. *For each* $\mathbf{s}=(s_1, \ldots, s_k)\in S$, *define* $\rho(\mathbf{s})=\mathrm{lcm}(\frac{a_1}{\gcd(a_1,s_1)}, \ldots, \frac{a_k}{\gcd(a_k,s_k)})$. *Let* d_i *be the number of distinct unordered pairs* $\{\mathbf{s}, \mathrm{mod}(-\mathbf{s},\mathbf{a})\}$ *such that* $\mathbf{s}\in S$ *and* $\rho(\mathbf{s})=i$ *for* $1\leqslant i\leqslant n$. *If* X *has no edges, then define* $d_1=1$. *Then* $\mathbf{d}=(d_1, \ldots, d_n)$ *is a decomposition vector for* X.

PROOF: Let X be a multidimensional circulant with k-symbol $(\mathbf{a},S)$. We know from the definition of a multidimensional circulant that $E(X)=\{(v_i,v_j)\,|\,\mathrm{mod}(\mathbf{i}-\mathbf{j},\mathbf{a})\in S\}$. For each $\mathbf{s}\in S$, define the subgraph X_s of X to be the graph with nodes $V(X)$ and edges $\{(v_i,v_j)\,|\,\mathrm{mod}(\mathbf{i}-\mathbf{j},\mathbf{a})\in\{\mathbf{s},\mathrm{mod}(-\mathbf{s},\mathbf{a})\}\}$. It is clear that $X_s=X_t$ if and only if $\{\mathbf{s},\mathrm{mod}(-\mathbf{s},\mathbf{a})\}=\{\mathbf{t},\mathrm{mod}(-\mathbf{t},\mathbf{a})\}$. If $X_s\neq X_t$, then it is also clear that $E(X_s)\cap E(X_t)=\emptyset$. Thus X can be expressed as the edge-disjoint union of the X_s.

We now complete the proof by showing that $X_s\simeq C_{n/\rho(s),\rho(s)}$ for all $\mathbf{s}\in S$ where $n=o(V(X))$. By definition, X_s is a vertex-transitive graph with k-symbol $(\mathbf{a},\{\mathbf{s},\mathrm{mod}(-\mathbf{s},\mathbf{a})\})$. Since $o(\{\mathbf{s},\mathrm{mod}(-\mathbf{s},\mathbf{a})\})\leqslant 2$, every node of X_s is adjacent to at most two other nodes. By Lemma 4, we know that X_s is a multicycle. From the definition of $E(X_s)$, we know that the length of each cycle in X_s is the smallest positive integer r such that $\mathrm{mod}(r\mathbf{s},\mathbf{a})=\mathbf{0}$. Note that $\mathrm{mod}(r\mathbf{s},\mathbf{a})=\mathbf{0}$ if and only if $a_i\,|\,rs_i$ for $1\leqslant i\leqslant k$ and thus if and only if $\frac{a_i}{gcd(a_i,s_i)}\,|\,r$ for $1\leqslant i\leqslant k$. Thus $\rho(s)=\mathrm{lcm}\left(\frac{a_1}{gcd(a_1,s_1)},\ldots,\frac{a_k}{gcd(a_k,s_k)}\right)$ is the smallest positive integer r such that $\mathrm{mod}(r\mathbf{s},\mathbf{a})=0$ and $X_s\simeq C_{n/\rho(s),\rho(s)}$.$\square$

Note, however, that some multidimensional circulants may have multicycle decompositions not of the form specified in Theorem 5. For example, the decompositions in figures 3b and 3c do not correspond in any obvious way to the grouplike subsets of the automorphism group of the graph. Thus the complete determination of which decomposition vectors a vertex-transitive graph can have may well be a difficult problem.

Also of interest is the problem of how multicycles can be composed to form a vertex-transitive graph. Not every graph which can be expressed as the edge-disjoint union of multicycles is vertex-transitive. For instance, consider the graph shown in figure 4. This graph can be expressed as the edge-disjoint union of two 7-cycles, yet is not vertex-transitive. This fact is easily seen by observing that the complement of the graph is the node disjoint union of a 4-cycle and a 3-cycle and thus is not vertex-transitive.

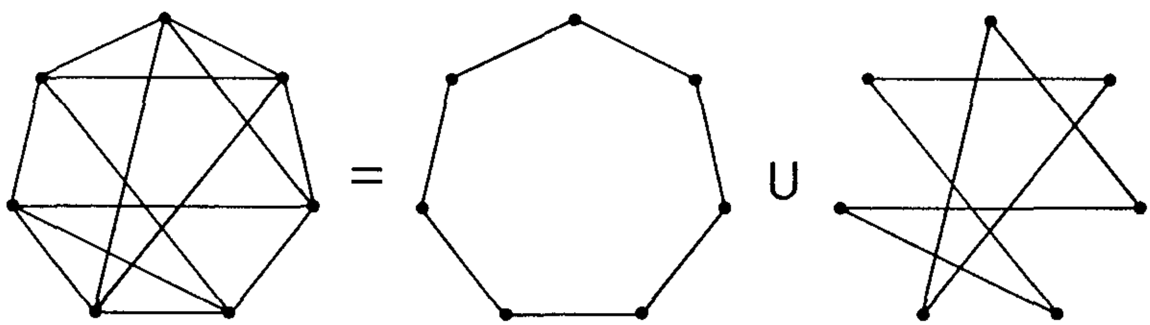

Figure 4

Thus the manner in which one can combine multicycles to form a vertex-transitive graph is not clear. A solution to the problem might well prove useful in settling the conjecture and in the development of a combinatorial characterization of the class of all vertex-transitive graphs.

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