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Acceptance Probabilities for a Sampling Procedure Based on the Mean and an Order Statistic

Mary C. Croarkin* and Grace L. Yang**

National Bureau of Standards, Washington, DC 20234

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A dual acceptance criterion based on the sample mean and an extreme order is used in many inspection procedures. Computation of the acceptance probability for such a dual criterion is investigated. An approximation and a lower bound to the acceptance probability are derived and are applicable to any continuous distribution. In addition, the connection between this dual criterion and hypothesis testing of scale and location parameters is studied. In the case of the exponential distribution the exact evaluation of the acceptance probability yields the power of the test.

Key words: acceptance probability; compliance sampling; dual acceptance criteria; mixed sampling plan; order statistics; statistical methods.

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*Center for Applied Mathematics, National Engineering Laboratory.
**Center for Applied Mathematics, National Engineering Laboratory and the University of Maryland, College Park, MD.

1. Introduction

Suppose that a random sample of size n from a lot is measured with respect to a particular variable and that the acceptance or rejection of the lot depends upon whether or not the measurements satisfy certain criteria. "Lot" can refer to a group of individual items or to a specified amount of material which can be sampled randomly.

There is widespread interest in sampling procedures that specify acceptance criteria involving the sample mean and a proportion of defectives in the sample [1], [4], [5], [9], [11] and [14].[1] Such a sampling procedure might specify that the lot is to be accepted only if the sample mean is greater than a value μ_0, say, and if no more than a specified percentage of the sample is less than a lower limit L. The purpose of a dual acceptance criterion is to ensure, for example, that the lot is at least a stated amount, μ_0, of the specified variable on the average and that the number of so called "defectives" or items that violate the lower limit is controlled. Obviously, depending on the application, the acceptance criteria can be specified in the opposite direction; i.e., the lot is to be accepted only if the sample mean is less than μ_0 and at least a certain percentage of the sample is greater than an upper limit U.

Specifically, let $X_1,\cdots,X_n$ be a random sample of n measurements, and let $X_{(1)} \leqslant \cdots \leqslant X_{(n)}$ be the corresponding order statistics. It is assumed that the random variables $X_1,\cdots,X_n$ are independent and identically distributed (i.i.d.) with a probability density function $f(x)$, and that the X_j have finite mean μ and variance σ^2. Let $\overline{X}$ be the sample mean and N_L be the number of defectives or measurements having values smaller than the specified (lower) limit L.

The sampling procedure to be considered is such that the lot is accepted whenever

$$[\overline{X} \geqslant \mu_0 \text{ and } N_L \leqslant k] \tag{1.1}$$

where μ_0 and k are specified in the sampling plan.

In terms of the order statistics, (1.1) is equivalent to the criterion

$$[\overline{X} \geqslant \mu_0 \text{ and } X_{(k+1)} > L] \tag{1.2}$$

and the probability of accepting the lot is defined to be

$$P_n = P[\overline{X} \geqslant \mu_0, N_L \leqslant k]. \tag{1.3}$$

The sampling procedure discussed above is a mixed variables-attributes acceptance criterion based on one sample. There are various ways of designing a mixed sampling plan. The type studied by Schilling and Dodge [19] is a double sampling procedure involving variables inspection in the first sample. If the variables inspection does not lead to acceptance, a second sample is taken and an attribute inspection is conducted on the combined samples. In their work, Schilling and Dodge assume a normal distribution with unknown mean and known variance.

We concentrate on a single sample plan where both the variables inspection as specified by the sample mean and attributes inspection as specified by k, the number of allowable defectives, are conducted on the same sample. This causes difficulties in the computation of the acceptance probabilities because of the lack of independence of the sample mean and the order statistics.

Investigations, of which we are aware, into the statistical properties of sampling procedures of this type assume a normal distribution with unknown mean and known variance. For instance in a compliance sampling application, Weed [21] simulates a two-stage procedure used in specifications for the thickness of paving material in which both stages involve a variable and an attribute inspection. Elder and Muse [8] develop a large sample approximation for the acceptance probability used in U.S. Department of Agriculture inspection procedures (1.3) and compare the approximation to an exact numerical procedure.

[1] Figures in brackets indicate literature references at the end of this paper.

It is noted that the dual sampling criterion leads to an acceptance region for testing hypotheses concerning the mean μ and the probability of item defectiveness simultaneously. The probability of a defective is defined to be $p = P[X \leqslant L]$. The acceptance region in (1.1) or (1.2) may be used for testing the null hypothesis

$$H_0: \mu = \mu^* \text{ and } p = p^*$$

versus the alternatives (1.4)

$$H_1: \mu < \mu^* \text{ or } p > p^*$$

Through reparametrization, these hypotheses may be formulated in terms of the location and scale parameters. Evidently, this depends on the properties of the distribution under consideration.

In the case of the normal distribution $N(\mu, \sigma^2)$, the probability of a defective is

$$p = \Phi\left(\frac{L - \mu}{\sigma}\right) \tag{1.5}$$

where

$$\Phi\ (z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\{-u^2/2\} du.$$

Thus,

$$\sigma = (L - \mu)/\Phi^{-1}(p). \tag{1.6}$$

Consequently, $\mu = \mu^*$ and $p = p^*$ if and only if

$$\mu = \mu^* \text{ and } \sigma = \sigma^* = (L - \mu^*)/\Phi^{-1}(p^*).$$

Accordingly, the hypothesis testing problem in (1.4) becomes that of testing

$$H_0: \mu = \mu^* \text{ and } \sigma = \sigma^*$$

versus (1.7)

$$H_1: \mu < \mu^* \text{ or } \sigma < \frac{L - \mu}{\Phi^{-1}(p^*)}.$$

Perusal of the literature turned up very few papers that are directly related to a joint test of the location and scale parameters. Eisenberger [7] develops an asymptotic joint test for the mean and variance of a normal distribution based on a quantile. Perng [18] develops a joint test for the location and scale parameters of an exponential distribution based on Fisher's method of combining two test statistics. Anderson [2] discusses the likelihood ratio test for simultaneously testing the mean and variance in multivariate normal distributions; both one-sample and k-sample problems are considered. In a recent paper, Perlman [17] shows that the likelihood ratio test is unbiased. None of these papers discusses the computation of acceptance probabilities under alternative hypotheses. Also, unlike (1.7), the alternatives in the quoted papers are rectangular regions.

2. Scope of the Study

It is our intention to investigate the acceptance probability of a dual sampling procedure from several aspects. The investigations are carried out for the normal distribution because of its importance in acceptance sampling and for the exponential and Weibull distributions because of their application in modeling the life span distribution.

First, in section 3, we derive a large sample approximation P_a for the acceptance probability P_n. This is achieved by deriving the asymptotic joint distribution of $\sqrt{n}(\overline{X}-\mu)/\sigma$ and $(N_L-np)/(np(1-p))^{1/2}$ as the sample size approaches infinity. This approximation method applies to any distribution. We illustrate its use in the normal, Weibull, and exponential distributions. The results as given in sections 3.1, 3.2, and 3.3 are compared with a simulation study.

In section 4 a lower bound $\underline{P}$ is established for P_n that amounts to assuming the independence of the sample mean and the k^{th} order statistic. This lower bound for finite samples provides some information ou the accuracy of the approximation. We attempt to determine under what conditions the approximation P_a is a significant improvement over the lower bound. In this connection one notes that a large sample approximation P_a is derived by normalizing the sample mean as $\sqrt{n}(\overline{X}-\mu)/\sigma$ and the number of defectives in the sample as $(N_L-np)/(np(1-p))^{1/2}$. If, instead, we convert N_L to an order statistic $X_{(k)}$ and consider $X_{(k)}$ (or $X_{(n-k)}$) as an extreme statistic, the normalized sample mean $\sqrt{n}(\overline{X}-\mu)$ and $X_{(k)}$ (or equivalently $X_{(n-k)}$) are asymptotically independent (The proof is given in appendix B). This suggests that $\underline{P}$ serves as a possible approximation to P_n when n is large and k is small.

In other words, when comparing P_a and $\underline{P}$, one should keep in mind the relationship between k and n; namely, the ratio k/n. In the case of P_a we have $N_L/n \to p$ and in the case of an extreme statistic we have $k/n \to 0$ as $n \to \infty$. Clearly, one would expect that the lower bound $\underline{P}$ may be a reasonable approximation when k is relatively small compared with n. This is indeed confirmed in our numerical study in section 4. The numerical studies show that P_a is comparable to $\underline{P}$ for small k/n and superior to $\underline{P}$ for larger values of k/n.

Finally, in section 5 the acceptance probabilities are approximated for the normal and Weibull distributions using a procedure proposed by Pearson and Hartley [16]. The exact acceptance probabilities curves are computed for the exponential distribution.

3. Large Sample Approximation of the Joint Distribution of $\overline{X}$ and N_L.

3.1 Derivation

Let $X_1,\cdots,X_n$ be a random sample from the lot with pdf $f(x)$. Assume that X_j has a finite mean μ and variance σ^2.
Introducing indicator random variables I_j,
where

$$I_j = \begin{cases} 1 & \text{if } X_j \leqslant L \\ 0 & \text{if } X_j > L \end{cases} \tag{3.1.1}$$

and letting the probability that an item violates the lower specification limit L be

$$p = P[X_j \leqslant L], \tag{3.1.2}$$

we can write the number of (unit) lower limit violations N_L in the sample as

$$N_L = \sum_{j=1}^{n} I_j. \tag{3.1.3}$$

Note that N_L has a binominal distribution $B(n,p)$, and the event $[N_L \leqslant k]$ is equivalent to the event $[X_{(k+1)} > L]$. In order to develop an approximation formula for the acceptance probability

$$P_n = P[\overline{X} \geqslant \mu_0, N_L \leqslant k],$$

we consider random variables W_n and Y_n defined as

$$\begin{aligned} W_n &= n^{1/2}(\overline{X}-\mu)/\sigma \\ \text{and} \quad Y_n &= (N_L-np)/(np(1-p))^{1/2}. \end{aligned} \tag{3.1.4}$$

Let (W_n, Y_n) be a row vector. We prove the following result.

THEOREM 3.1. *As* $n \to \infty$, *the random vector* $(W_n, Y_n)'$ *converges in distribution to a bivariate normal distribution with mean* $(0,0)'$ *and covariance matrix*

$$\Sigma = \begin{pmatrix} 1 & \varrho \\ \varrho & 1 \end{pmatrix} \tag{3.1.5}$$

where

$$\varrho = E\ (X_j-\mu)I_j/\sigma(p(1-p))^{1/2}. \tag{3.1.6}$$

PROOF: Let t_1 and t_2 be arbitrarily chosen but fixed real numbers. Form the linear combination of W_n and Y_n, $t_1 W_n + t_2 Y_n$.

Direct computation and application of the central limit theorem give

$$t_1 W_n + t_2 Y_n \xrightarrow{D} N\ (0,\ t_1{}^2 + t_2{}^2 + t_1 t_2 \varrho) \text{ as } n \to \infty$$

It then follows from application of the Cramer-Wold device that

$$\begin{pmatrix} W_n \\ Y_n \end{pmatrix} \xrightarrow{D} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma\right) \text{ as } \quad n \to \infty$$

where Σ is given in (3.1.5).

Making use of the asymptotic distribution in Theorem 3.1, we note from (3.1.4) that

$$\begin{aligned} \overline{X} &= n^{-1/2}\sigma W_n + \mu \\ &\text{and} \\ N_L &= (np(1-p))^{1/2}\, Y_n + np. \end{aligned}$$

Thus the random vector $(\overline{X}, N_L)'$ has asymptotically a bivariate normal distribution with mean and covariance matrix Γ given by

$$\begin{pmatrix} \mu \\ np \end{pmatrix} \text{ and } \Gamma = \begin{pmatrix} \frac{\sigma^2}{n} & E(X_j-\mu)I_j \\ E(X_j-\mu)I_j & np(1-p) \end{pmatrix} \tag{3.1.7}$$

respectively.

For convenience in computation, write the acceptance probability P_n as

$$\begin{aligned} P_n &= P[\overline{X} \geqslant \mu_0] - P[\overline{X} \geqslant \mu_0, N_L > k] \\ &= P[\overline{X} \geqslant \mu_o] - P[W_n \geqslant \sqrt{n}(\mu_o-\mu)/\sigma,\ Y_n > (np(1-p))^{-1/2}(k-np)]. \end{aligned} \tag{3.1.8}$$

Making use of (3.1.7) and the continuity correction factor 0.5 for the random variable N_L, we see that for sufficiently large n, P_n may be approximated by

$$P_a = \frac{1}{\sqrt{2\pi}} \int_a^\infty \exp(-z^2/2)dz - \int_a^\infty \int_b^\infty g(x,y,\varrho)dxdy \tag{3.1.9}$$

where

$$a = \sqrt{n}(\mu_0 - \mu)/\sigma, \tag{3.1.10}$$

$$b = (np(1-p))^{-1/2}(k + 0.5 - np), \tag{3.1.11}$$

$$g(x,y,\varrho) = (2\pi)^{-1}(1-\varrho^2)^{-1/2}\exp\{-(x^2+y^2-2\varrho xy)/2(1-\varrho^2)\}, \tag{3.1.12}$$

and ϱ is defined in (3.1.6).

In order to compute the $P[\overline{X} \geqslant \mu_0, N_L \leqslant k]$ using the approximation P_a, we need to know the mean μ and the variance σ^2 of the distribution in question, the proportion defective p as defined in (3.1.2) and the correlation coefficient ϱ as defined in (3.1.6). The computation of the bivariate normal term is described in more detail in Appendix A.

3.2 Normal Distribution

Assume that the sample comes from a normal distribution $N(\mu,\sigma^2)$.
The item defective probability from (3.1.2) is

$$p = P[X \leqslant L] = \Phi\{(L-\mu)/\sigma\}, \tag{3.2.1}$$

where $\Phi\{(L-\mu)/\sigma\}$ is the cdf of the $N(0,1)$ given in (1.5).
In order to compute the approximation P_a given in (3.1.9), we need to compute the correlation coefficient given in (3.1.6).
The expectation $E\{(X-\mu)I_{[X\leqslant L]}\}$ is evaluated as

$$E\{(X-\mu)I_{[X\leqslant L]}\} = -\frac{\sigma}{\sqrt{2\pi}} \exp\{-(L-\mu)^2/2\sigma^2\}.$$

Consequently the correlation coefficient is

$$\varrho = -(2\pi p(1-p))^{-1/2}\exp\{-(L-\mu)^2/2\sigma^2\}.$$

In order to compare the approximation P_a in (3.1.9) with an approximation developed by Elder and Muse [8], the lower limit L is chosen under the assumption that $\mu = 0$, $\sigma = 1$, and according to the criterion

$$P[N_L \leqslant k] = 1-\alpha, \tag{3.2.2}$$

where $0 < \alpha < 1$.
Because N_L is $B(n,p)$, the lower limit L is determined from

$$\sum_{j=0}^{k} \binom{n}{j} p^j(1-p)^{n-j} = 1-\alpha, \tag{3.2.3}$$

where $p = \Phi(L)$.
Values of L as tabulated by Elder and Muse for $\alpha = 0.10$, 0.05, and 0.01 are shown in table I. Once

TABLE I. *Lower Limits used in Computation of Acceptance Probabilities for Normal Distribution*

		Lower Limit L		
n	k	$\alpha=0.10$	$\alpha=0.05$	$\alpha=0.01$
5	0	2.036	2.319	2.877
	1	1.215	1.429	1.843
	2	0.685	0.881	1.250
10	0	2.309	2.568	3.089
	1	1.602	1.789	2.157
	2	1.196	1.358	1.670
20	0	2.559	2.799	3.289
	1	1.928	2.095	2.428
	2	1.586	1.726	2.001
30	0	2.696	2.928	3.402
	1	2.100	2.258	2.574
	2	1.783	1.914	2.172

L is determined the correlation coefficient of $\overline{X}$ and N_L can be evaluated as

$$\varrho = -[2\pi p(1-p)]^{-1/2} \exp\{-L^2/2\}. \tag{3.2.4}$$

The Elder-Muse approximation along with their exact results are compared with the corresponding values of P_a in table II where L is chosen such that $\alpha = 0.10$.

The comparison with the exact values derived in [8] shows that even for small sample size P_a provides an excellent approximation to the acceptance probability P_n, and its effectiveness increases as k gets larger. When $k = 0$, the percent error in P_a as compared to the exact results is approximately 3 percent. For $k = 1$, it is about 1 percent and for $k = 2$, it is less than 1 percent. The percentage errors in both P_a and the Elder-Muse approximation when $\mu = 0$ are shown below.

	Percent Error in Approximations					
	$k = 0$		$k = 1$		$k = 2$	
n	P_a	Elder Muse	P_a	Elder Muse	P_a	Elder Muse
5	3.3	1.0	1.0	1.8	0.6	1.2
10	3.1	0.6	1.0	1.0	0.6	1.2
20	3.0	0.2	0.8	0.6	0.6	0.8
30	2.6	0.2	0.8	0.8	0.4	0.6

3.3 Weibull Distribution

Assume that the sample $X_1,\ldots,X_n$ comes from a two parameter Weibull distribution $W(\lambda,\theta)$ with scale parameter λ, shape parameter θ and *pdf*

$$f(x) = (\theta/\lambda)\ (x/\lambda)^{\theta-1} \exp\{-(x/\lambda)^{\theta}\} \quad \text{for } x > 0, \lambda > 0, \theta > 0 \tag{3.3.1}$$

The mean and variance are

$$\mu = \lambda\Gamma(1+1/\theta) \tag{3.3.2}$$

and

$$\sigma^2 = \lambda^2 \{\Gamma(1+2/\theta) - [\Gamma(1+1/\theta)]^2\} \tag{3.3.3}$$

respectively where $\Gamma(\cdot)$ is the gamma function.

For $0 < \theta \leqslant 1$, X has a decreasing failure rate (DFR) distribution; for $\theta \geqslant 1$, X has an increasing failure rate (IFR) distribution. For further information see Johnson and Kotz [13].

In the case of the Weibull distribution, the proportion defective p is defined from (3.1.2) and (3.2.2) as

$$p = [X \leqslant L] = 1 - \exp\{-(L/\lambda)^{\theta}\}. \tag{3.3.4}$$

The expectation

$$\begin{aligned} EXI_{[X \leqslant L]} &= \frac{\theta}{\lambda} \int_0^L x(x/\lambda)^{\theta-1} \exp\{-(x/\lambda)^{\theta}\} dx \\ &= \lambda\, \mathrm{I}\{(L/\lambda)^{\theta}, 1/\theta\} \end{aligned} \tag{3.3.5}$$

and $I(c,d)$ is related to the incomplete Γ-function [12].

Combining (3.1.6), (3.3.4) and (3.3.5), we find that the correlation coefficient is

TABLE II. *Comparison of Approximation* P_a *with Elder – Muse Values for* $P[X \geqslant \mu_o^*, N_L \leqslant k]$ *where* $P(N_L \leqslant k) = 0.90$ *for Normal Distribution* $N(0,1)$

		$k=0$			$k=1$			$k=2$		
n	μ	Exact	P_a	Elder Muse	Exact	P_a	Elder Muse	Exact	P_a	Elder Muse
5	−.8	0.035	0.034	0.032	0.036	0.036	0.034	0.037	0.036	0.037
	−.6	0.087	0.082	0.085	0.089	0.088	0.089	0.089	0.089	0.091
	−.4	0.180	0.168	0.181	0.184	0.181	0.188	0.185	0.184	0.189
	−.2	0.318	0.300	0.323	0.324	0.320	0.332	0.326	0.325	0.333
	.0	0.488	0.472	0.493	0.496	0.491	0.505	0.499	0.496	0.505
	.2	0.659	0.667	0.663	0.669	0.672	0.674	0.671	0.672	0.674
	.4	0.801	0.814	0.802	0.811	0.814	0.811	0.813	0.814	0.812
	.6	0.899	0.910	0.899	0.908	0.910	0.906	0.909	0.910	0.906
	.8	0.956	0.963	0.955	0.962	0.963	0.959	0.963	0.963	0.959
10	−.6	0.027	0.026	0.026	0.028	0.028	0.026	0.028	0.028	0.027
	−.4	0.098	0.091	0.097	0.100	0.098	0.099	0.101	0.101	0.101
	−.2	0.252	0.236	0.253	0.257	0.252	0.261	0.260	0.257	0.264
	.0	0.480	0.465	0.483	0.490	0.485	0.495	0.494	0.491	0.500
	.2	0.713	0.732	0.714	0.725	0.735	0.728	0.731	0.736	0.733
	.4	0.876	0.897	0.876	0.888	0.897	0.887	0.893	0.897	0.891
	.6	0.956	0.971	0.956	0.966	0.971	0.965	0.969	0.971	0.967
	.8	0.956	0.994	0.985	0.992	0.994	0.991	0.993	0.994	0.993
20	−.4	0.034	.0.032	0.034	0.035	0.034	0.034	0.036	0.035	0.034
	−.2	0.174	0.162	0.174	0.178	0.173	0.178	0.180	0.177	0.181
	.0	0.474	0.460	0.475	0.483	0.479	0.486	0.488	0.485	0.492
	.2	0.781	0.811	0.781	0.795	0.814	0.795	0.802	0.814	0.802
	.4	0.937	0.963	0.937	0.950	0.963	0.950	0.956	0.963	0.955
	.6	0.981	0.996	0.981	0.991	0.996	0.991	0.994	0.996	0.993
30	−.4	0.013	0.012	0.013	0.013	0.013	0.013	0.014	0.014	0.013
	−.2	0.127	0.118	0.127	0.130	0.126	0.129	0.131	0.129	0.131
	.0	0.470	0.458	0.471	0.479	0.476	0.480	0.484	0.482	0.487
	.2	0.824	0.861	0.824	0.839	0.863	0.839	0.847	0.863	0.846
	.4	0.958	0.986	0.958	0.972	0.986	0.972	0.978	0.986	0.977
	.6	0.985	0.999	0.985	0.995	0.999	0.995	0.997	0.999	0.997

$^*\mu_0 = 0$

$$\varrho = [\lambda I\{(L/\lambda)^{\theta}, 1/\theta\} - \mu p]/\sigma(p(1-p))^{1/2} \tag{3.3.6}$$

where μ and σ are defined by (3.3.2), and (3.3.3) respectively.

The limits of integration for the approximation (3.1.9) are

$$a = \frac{n^{1/2}[\mu_0 - \lambda\Gamma(1+1/\theta)]}{\lambda\{\Gamma(1+2/\theta) - [\Gamma(1+1/\theta)]^2\}^{1/2}} \tag{3.3.7}$$

and b as defined in (3.1.11).

As is the case in the normal distribution, the lower limit L is determined according to (3.2.3) and (3.3.4) for specified values of k and α.

Explicitly

$$L = \lambda\,[-\log_e(1-p)]^{1/\theta}. \tag{3.3.8}$$

The proportion defective p is tabulated in table III for $\alpha = 0.10$, 0.05 and 0.01, $n = 5, 10, 20, 30$ and $k = 0, 1, 2, 3$. Corresponding lower limits L where $\lambda = 1$ are shown in table IV.

TABLE III. *Proportion Defectives p used in Computation of Acceptance Probabilities*

		Proportion Defective p		
n	k	α=0.10	α=0.05	α=0.01
5	0	0.0208	0.0102	0.00200
	1	.112	.0765	.0330
	2	.247	.1890	.106
	3	.416	.3425	.222
10	0	0.0105	0.00511	0.00100
	1	.0545	.0365	.0155
	2	.1155	.0870	.0475
	3	.1875	.1500	.0930
20	0	0.00525	0.00256	0.000500
	1	.0269	.0180	.00759
	2	.0564	.0422	.0227
	3	.0902	.0713	.0435
30	0	0.00350	0.00171	0.000335
	1	.0178	.0120	.00500
	2	.0373	.0278	.0149
	3	.0594	.0468	.0285

The approximation P_a is compared to a simulation study where the acceptance probability was computed from 5,000 random samples. Simulation for the Weibull distribution was done by generating independent uniform random deviates U_i using a congruential random number generator and making the transformation

$$X_i = \lambda(-\log_e U_i)^{1/\theta}$$

The X_i are independent $W(\lambda,\theta)$ r.v.s with pdf as shown in (3.3.1).

Values of P_a and simulated acceptance probabilities are tabulated in table V for Weibull distribution $W(1,\theta)$ for $\theta = 1, 2, 3.5$.

The accuracy of the approximation P_a as gauged by the simulation results is dependent on several factors; i.e., namely, the value of the shape parameter θ; α, the probability that the sample will contain more than the allowable number of defectives; n, the size of the sample; and k, the number of allowable defectives or number of measurements less than the lower limit L.

The worst accuracy is for a Weibull distribution with $\theta = 1$ where α is small, $\alpha = 0.01$, and n is small, $n = 5$. *The error is 9 percent for this case but drops to 2 percent when the sample size is in-*

TABLE IV. *Lower Limits Used in Computation of Acceptance Probabilities for Weibull Distribution*

	n	k	Lower Limit L $\alpha=0.10$	$\alpha=0.05$	$\alpha=0.01$
$\theta=1$	5	0	0.0210	0.0103	0.0020
	5	1	.1188	.0796	.0336
	5	2	.2837	.2095	.1120
	5	3	.5379	.4193	.2510
	10	0	.0106	.0051	.0010
	10	1	.0560	.0372	.0156
	10	2	.1227	.0910	.0487
	10	3	.2076	.1625	.0976
	20	0	.0053	.0026	.0005
	20	1	.0273	.0182	.0076
	20	2	.0581	.0431	.0230
	20	3	.0945	.0740	.0445
	30	0	.0035	.0017	.0003
	30	1	.0180	.0121	.0050
	30	2	.0380	.0282	.0150
	30	3	.0612	.0479	.0289
$\theta=2$	5	0	.1450	.1013	.0447
	5	1	.3446	.2821	.1832
	5	2	.5326	.4577	.3347
	5	3	.7334	.6475	.5010
	10	0	.1027	.0716	.0316
	10	1	.2367	.1928	.1250
	10	2	.3503	.3017	.2206
	10	3	.4557	.4031	.3124
	20	0	.0726	.0506	.0224
	20	1	.1651	.1348	.0873
	20	2	.2409	.2076	.1515
	20	3	.3075	.2720	.2109
	30	0	.0592	.0414	.0183
	30	1	.1340	.1099	.0708
	30	2	.1950	.1679	.1225
	30	3	.2475	.2189	.1700
$\theta=3.5$	5	0	.3317	.2702	.1694
	5	1	.5441	.4852	.3791
	5	2	.6977	.6398	.5351
	5	3	.8376	.7801	.6737
	10	0	.2724	.2216	.1390
	10	1	.4390	.3904	.3047
	10	2	.5492	.5042	.4216
	10	3	.6382	.5950	.5144
	20	0	.2233	.1818	.1140
	20	1	.3573	.3181	.2482
	20	2	.4434	.4073	.3402
	20	3	.5097	.4752	.4109
	30	0	.1988	.1620	.1017
	30	1	.3171	.2831	.2202
	30	2	.3929	.3607	.3013
	30	3	.4502	.4198	.3633

TABLE V. *Comparison of Approximation P_a with Simulation for $P(\bar{X} \geq \mu_0^*.\ N_L \leq k)$ where $P(N_L \leq k) = 1-\alpha$ for Weibull Distribution $W(1.\theta)$.*

	n	k	Probability of Acceptance					
			$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.01$	
			P_a	Simul	P_a	Simul	P_a	Simul
$\theta=1$	5	0	0.645	0.634	0.698	0.663	0.712	0.672
	5	1	.668	.652	.698	.673	.712	.672
	5	2	.678	.664	.699	.680	.711	.674
	5	3	.688	.674	.702	.684	.711	.675
	10	0	.705	.707	.768	.755	.785	.773
	10	1	.726	.724	.768	.763	.785	.776
	10	2	.733	.734	.766	.768	.785	.778
	10	3	.738	.743	.767	.772	.784	.779
	20	0	.775	.795	.848	.828	.868	.875
	20	1	.795	.801	.846	.836	.868	.877
	20	2	.798	.807	.844	.840	.867	.879
	20	3	.801	.810	.843	.841	.867	.878
	30	0	.815	.826	.893	.872	.914	.914
	30	1	.835	.837	.890	.879	.914	.918
	30	2	.837	.838	.887	.878	.914	.915
	30	3	.838	.841	.886	.881	.913	.917
$\theta=2$	5	0	.681	.703	.720	.731	.744	.738
	5	1	.709	.723	.729	.734	.744	.739
	5	2	.721	.736	.733	.737	.744	.740
	5	3	.729	.740	.736	.739	.744	.740
	10	0	.743	.764	.788	.807	.824	.822
	10	1	.768	.780	.803	.808	.824	.825
	10	2	.777	.794	.808	.808	.823	.826
	10	3	.784	.800	.812	.810	.823	.827
	20	0	.810	.827	.865	.885	.906	.903
	20	1	.832	.839	.876	.884	.906	.905
	20	2	.837	.848	.884	.882	.905	.907
	20	3	.840	.851	.884	.882	.905	.907
	30	0	.844	.860	.897	.924	.946	.947
	30	1	.865	.866	.903	.922	.946	.950
	30	2	.867	.867	.909	.919	.946	.949
	30	3	.869	.872	.914	.918	.945	.952
$\theta=3.5$	5	0	.801	.829	.864	.859	.880	.872
	5	1	.830	.844	.865	.873	.880	.875
	5	2	.839	.855	.868	.875	.880	.876
	5	3	.843	.853	.869	.878	.879	.877
	10	0	.853	.864	.931	.911	.951	.942
	10	1	.877	.868	.930	.915	.951	.947
	10	2	.882	.884	.928	.922	.951	.946
	10	3	.885	.889	.928	.925	.951	.950
	20	0	.882	.890	.968	.938	.990	.983
	20	1	.902	.894	.964	.946	.990	.984
	20	2	.903	.899	.960	.946	.990	.987
	20	3	.903	.894	.958	.947	.989	.986
	30	0	.887	.893	.974	.938	.998	.984
	30	1	.908	.896	.970	.947	.998	.989
	30	2	.907	.894	.966	.948	.997	.990
	30	3	.907	.898	.964	.956	.996	.990

$^*\mu_0 = 0.75$

creased to $n = 10$. For other Weibull distributions and combinations of α and n, the worst accuracies occur when $k = 0$, and in this case the errors are as large as 6 percent for $n = 5$ and 4 percent for $n = 30$. However, the approximation P_a works very well when $k > 0$. The disagreement between P_a and the simulation is less than 1 percent for a large proportion of the points when $k > 0$.

3.4 Exponential Distribution

Assume that the sample $X_1,...,X_n$ comes from an exponential distribution $E(\lambda,\beta)$ with location parameter β and scale parameter λ and pdf

$$f(x) = (1/\lambda) \exp\{-(x-\beta)/\lambda\} \qquad x > \beta, \lambda > 0 \tag{3.4.1}$$

The mean and variance of X are given by $\mu = \lambda + \beta$ and $\sigma^2 = \lambda^2$ respectively.

We have

$$p = 1-\exp(-(L-\beta)/\lambda) \tag{3.4.2}$$

and

$$EXI_{[X \leqslant L]} = \lambda p - (1-p)(L-\beta) + \beta p. \tag{3.4.3}$$

Combining (3.4.2) and (3.4.3), we get

$$\varrho = -(1-p)^{1/2}(L-\beta)/\lambda p^{1/2}. \tag{3.4.4}$$

Using values for the proportion defective p that are given in table III, the corresponding limits L as determined by

$$L = \beta - \lambda \log(1-p) \tag{3.4.5}$$

are found in table VI for $\beta = 0$ and $\lambda = 0.5, 1, 2$.

The values a and b appearing in the approximation P_a (3.1.9) are given by

$$\begin{aligned} a &= n^{1/2}\lambda^{-1}(\mu_0-\lambda-\beta) \\ b &= (np(1-p))^{-1/2}(k + 0.5 - np) \end{aligned} \tag{3.4.6}$$

and ϱ is defined by (3.4.4.)

Values of P_a and simulated acceptance probabilities are tabulated in table VII for the exponential distribution $E(\lambda,0)$ for $\lambda = 0.5,1,2$.

The accuracy of the approximation P_a is more dependent on n, the sample size and less dependent on k, the number of allowable defectives for the exponential distribution than for Weibull distributions. The worst accuracy is for an exponential distribution with $\lambda = 1$, where $k = 0$ and $n = 5$. The disagreement with the simulation in this case is 7 percent, dropping to 1 percent when the sample size is increased to $n = 10$. In general, the accuracies are not dependent upon the parameter λ but are somewhat dependent upon the way in which the lower limit L is chosen, and the accuracies tend to worsen as the probability of the sample containing more than the allowable number of defectives increases. Accuracies of about 2 percent are characteristic of the results over all values of k.

4. A Lower Bound for the Acceptance Probability

A lower bound for the acceptance probability is provided by the following lemma.

LEMMA 4.1: *Let* $X_1,...,X_n$ *be i.i.d random variables from a continuous distribution. Let* $\overline{X}$ *be the* *sample mean and* $X_{(r)}$ *be the* r^{th} *smallest order statistic of* $X_1,\ldots,X_n$. *Then for arbitrarily fixed real numbers* a, b *and positive integer* r, $1 \leqslant r \leqslant n$,

$$P[\overline{X} \geqslant a, X_{(r)} \geqslant b] \geqslant P[\overline{X} \geqslant a]\, P[X_{(r)} \geqslant b] \tag{4.1}$$

$$P[\overline{X} < a, X_{(r)} < b] \geqslant P[\overline{X} < a]\, P[X_{(r)} < b]. \tag{4.2}$$

The lemma is an easy eonsequenee of a general theorem (Esary, Proschan, and Walkup [10]). For easy reference, we quote the theorem below, as well as the definition of "associatedness." Random

TABLE VI. *Lower Limits used in Computation of Acceptance Probabilities for Exponential Distribution*

			Lower Limit L		
	n	k	α=0.10	α=0.05	α=0.01
λ=0.5	5	0	0.0105	0.0051	0.0010
	5	1	.0594	.0398	.0168
	5	2	.1418	.1047	.0560
	5	3	.2689	.2097	.1255
	10	0	.0053	.0026	.0005
	10	1	.0280	.0186	.0078
	10	2	.0614	.0455	.0243
	10	3	.1038	.0813	.0488
	20	0	.0026	.0013	.0003
	20	1	.0136	.0091	.0038
	20	2	.0290	.0216	.0115
	20	3	.0473	.0370	.0222
	30	0	.0018	.0009	.0002
	30	1	.0090	.0060	.0025
	30	2	.0190	.0141	.0075
	30	3	.0306	.0240	.0145
λ=1.0	5	0	.0210	.0103	.0020
	5	1	.1188	.0796	.0336
	5	2	.2837	.2095	.1120
	5	3	.5379	.4193	.2510
	10	0	.0106	.0051	.0010
	10	1	.0560	.0372	.0156
	10	2	.1227	.0910	.0487
	10	3	.2076	.1625	.0976
	20	0	.0053	.0026	.0005
	20	1	.0273	.0182	.0076
	20	2	.0581	.0431	.0230
	20	3	.0945	.0740	.0445
	30	0	.0035	.0017	.0003
	30	1	.0180	.0121	.0050
	30	2	.0380	.0282	.0150
	30	3	.0612	.0479	.0289
λ=2	5	0	.0420	.0205	.0040
	5	1	.2376	.1592	.0671
	5	2	.5674	.4190	.2241
	5	3	1.0757	.8386	.5021
	10	0	.0211	.0102	.0020
	10	1	.1121	.0744	.0312
	10	2	.2455	.1820	.0973
	10	3	.4153	.3250	.1952
	20	0	.0105	.0051	.0010
	20	1	.0545	.0363	.0152
	20	2	.1161	.0862	.0459
	20	3	.1891	.1479	.0889
	30	0	.0070	.0034	.0007
	30	1	.0359	.0241	.0100
	30	2	.0760	.0564	.0300
	30	3	.1225	.0959	.0578

TABLE VII. *Comparison of Approximation P_a with Simulation for $P[\bar{X} \geq \mu_0, N_L \leq k]$ where $P[N_L \leq k] = 1-\alpha$ for Exponential Distribution $E(\lambda, 0)$*

		Probability of Acceptance					
		$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.01$	
n	k	P_a	Simul	P_a	Simul	P_a	Simul
$\lambda=0.5, \mu_0=0.25$							
5	0	0.781	0.825	0.850	0.853	0.868	0.893
5	1	.804	.846	.847	.871	.868	.896
5	2	.811	.857	.846	.874	.867	.897
5	3	.819	.869	.847	.885	.866	.899
10	0	.842	.875	.922	.929	.943	.957
10	1	.862	.889	.919	.927	.943	.958
10	2	.865	.893	.915	.933	.942	.958
10	3	.867	.897	.913	.935	.941	.956
20	0	.878	.902	.964	.948	.987	.985
20	1	.898	.905	.960	.954	.987	.984
20	2	.898	.907	.956	.955	.986	.984
20	3	.898	.899	.953	.948	.985	.984
30	0	.886	.901	.973	.952	.997	.987
30	1	.906	.904	.969	.951	.997	.990
30	2	.906	.903	.965	.948	.996	.990
30	3	.906	.903	.962	.950	.995	.989
$\lambda=1.0, \mu_0=0.75$							
5	0	.645	.632	.698	.651	.712	.686
5	1	.668	.657	.698	.660	.712	.689
5	2	.673	.665	.699	.667	.711	.690
5	3	.688	.676	.702	.674	.711	.691
10	0	.705	.712	.768	.754	.785	.776
10	1	.726	.726	.768	.761	.785	.777
10	2	.733	.734	.767	.766	.785	.777
10	3	.738	.745	.767	.769	.784	.778
20	0	.775	.789	.848	.846	.868	.873
20	1	.795	.800	.846	.849	.868	.876
20	2	.798	.804	.844	.849	.867	.875
20	3	.801	.806	.843	.851	.867	.873
30	0	.815	.839	.893	.890	.915	.920
30	1	.835	.846	.890	.887	.914	.921
30	2	.837	.842	.887	.884	.914	.922
30	3	.838	.846	.886	.891	.913	.921
$\lambda=2.0, \mu_0=0.75$							
5	0	.825	.872	.899	.915	.919	.950
5	1	.846	.883	.895	.927	.919	.954
5	2	.851	.891	.893	.929	.918	.953
5	3	.856	.894	.892	.935	.916	.955
10	0	.870	.894	.954	.944	.976	.987
10	1	.889	.889	.950	.944	.976	.987
10	2	.891	.898	.945	.948	.975	.986
10	3	.891	.894	.942	.945	.974	.985
20	0	.887	.902	.974	.952	.997	.991
20	1	.906	.903	.970	.956	.997	.989
20	2	.906	.905	.965	.956	.996	.991
20	3	.906	.904	.962	.955	.995	.992
30	0	.889	.907	.976	.956	1.000	.990
30	1	.909	.906	.972	.949	.999	.991
30	2	.908	.902	.968	.948	.999	.991
30	3	.908	.901	.965	.952	.998	.990

variables $X_1,\ldots,X_n$ are said to be associated if

$$\mathrm{Cov}(f(T), g(T)) \geqslant 0$$

for all non-decreasing functions f and g in each X_j for which $Ef(T)$, $Eg(T)$, $Ef(T)g(T)$ exist and T denotes $\{X_1,\ldots,X_n\}$.

THEOREM 4.1. *Let* $\mathrm{T} = \{\mathrm{X}_1,\ldots,\mathrm{X}_n\}$ *be associated,* $\mathrm{S}_i = \mathrm{f}_i(\mathrm{T})$ *and* f_i *be nondecreasing for* $i=1,\ldots,k$. *Then*

$$\mathrm{P}[\mathrm{S}_1 \leqslant s_1,\ldots,\mathrm{S}_k \leqslant s_k] \geqslant \prod_{i=1}^{k} \mathrm{P}[\mathrm{S}_i \leqslant s_i] \tag{4.3}$$

$$\mathrm{P}[\mathrm{S}_1 > s_1,\ldots,\mathrm{S}_k > s_k] \geqslant \prod_{i=1}^{k} \mathrm{P}[S_i > s_i] \tag{4.4}$$

for all $s_1,\ldots,s_k$.

PROOF OF LEMMA 4.1: In our case the X_i's are statistically independent and hence associated. Let $S_1 = \overline{X}$ and $S_2 = X_{(r)}$. Clearly, S_1 and S_2 are non-decreasing functions in each of the X_i's; hence (4.1) and (4.2) hold. Moreover, $\mathrm{Cov}(S_1, S_2) = \mathrm{Cov}\,(\overline{X}, X_{(r)}) \geqslant 0$. This completes the proof.

From Lemma 4.1, we have a lower bound $\underline{P}$ to the acceptance probability

$$\underline{P} = P[\overline{X} \geqslant a]\, P[X_{(k+1)} > L] \leqslant P_n = P[\overline{X} \geqslant a, X_{(k+1)} > L], \tag{4.5}$$

where $k + 1$ corresponds to r.

The r.v. $X_{(k+1)}$ can be transformed to a r.v. Z with Beta distribution with parameters $n-k$ and $k+1$. Thus

$$P[X_{(k+1)} > L] = P[Z < 1-F_X(L)] = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k)} \int_0^{1-p} z^{n-k-1}(1-z)^k dz. \tag{4.6}$$

The lower bound $\underline{P}$ in (4.5) can be computed using the marginal distribution of the sample mean and the Beta distribution.

Because the computation of the lower bound $\underline{P}$ is much easier than the computation of the acceptance probability P_n it would be an immense simplification if the lower bound could serve as an approximation for P_n.

Therefore, it is of practical importance to determine the sample size n and values of k that are necessary in order that the lower bound be an acceptable approximation for P_n. In other words, it is of interest to know the smallest value of n and the range of k values which makes the independence of $\overline{X}$ and $X_{(k+1)}$ acceptable.

5. Comparison of the Exact Probability of Acceptance with the Approximation and the Lower Bound

5.1 Acceptance Probability Curves

The acceptance probabilities computed using either simulation or numerical integration along with the corresponding lower bound $\underline{P}$ and the approximation P_a are plotted as a function of one parameter of the distribution in question. This provides a comparison of the relative accuracy of P_a to $\underline{P}$ as a technique for approximating P_n. The curves are varied over n and k in order to examine the effect of sample size and number of allowable defectives k on P_n, P_a and $\underline{P}$.

5.2 Normal Distribution

Assuming that $X_1,\ldots, X_n$ are i.i.d. $N(\mu,1)$, the acceptance probability

$$P_n = P_\mu[\overline{X} \geqslant \mu_0, X_{(k+1)} > L]$$

for L chosen according to (3.2.3) and $\mu_0 = 0$ was computed using a technique for simulating random normal deviates due to Box and Muller [3]. The resulting acceptance probabilities as a function of μ are shown as the solid line in figures 1-4.

The corresponding lower bound $\underline{P}$ was computed from (4.5) and the approximation $\bar{P}_a$ was computed for (3.1.9).

The relationships among the probability of acceptance P_n, its approximation P_a, and its lower bound $\underline{P}$ as a function of sample size n and allowable number of defectives k is depicted in figures 1-4 for samples of size $n = 10$ and $n = 30$. The following convention is used for all figures; namely, P_n is shown as a solid line; P_a is shown as a heavy dashed line; and $\underline{P}$ is shown as a lighter dotted line.

From figure 1 it is obvious that when $k = 0$ and n is small, P_a is a better approximation to the acceptance probability than the lower bound as long as $\mu < 0.25$. As n increases the superiority of P_a to $\underline{P}$ increases as k is allowed to become larger. For example, when $k = 3$ as in figure 4, the lower bound

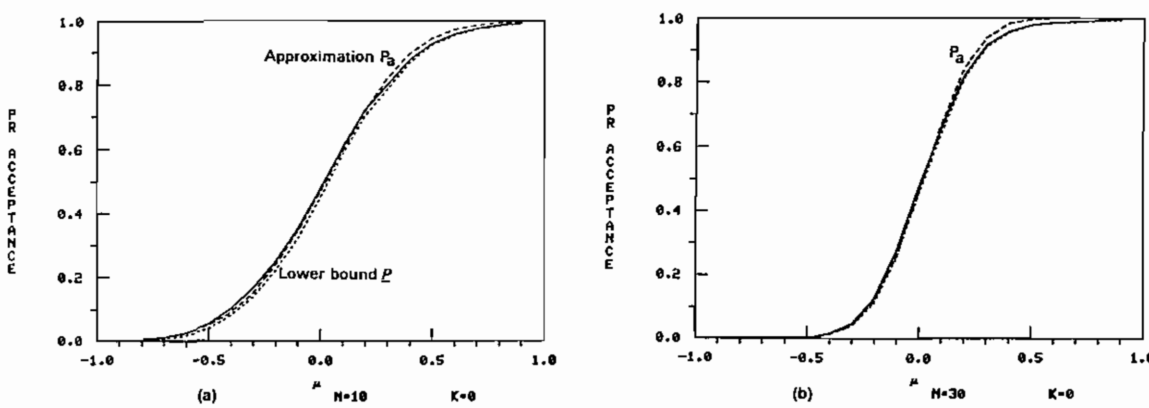

FIGURE 1. Acceptance probabilities when the number of allowable defectives $k = 0$ and n observations are drawn from the normal distribution $N(\mu, 1)$.

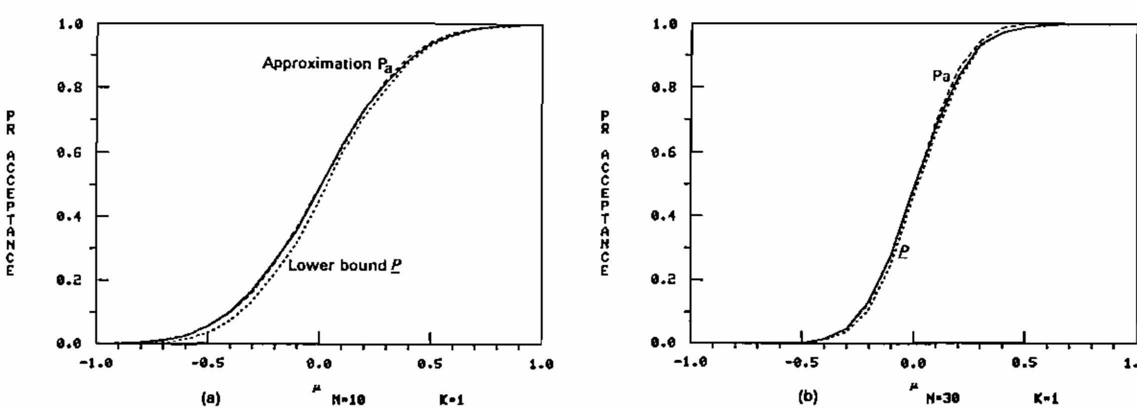

FIGURE 2. Acceptance probabilities when the number of allowable defectives $k = 1$ and n observations are drawn from the normal distribution $N(\mu, 1)$.

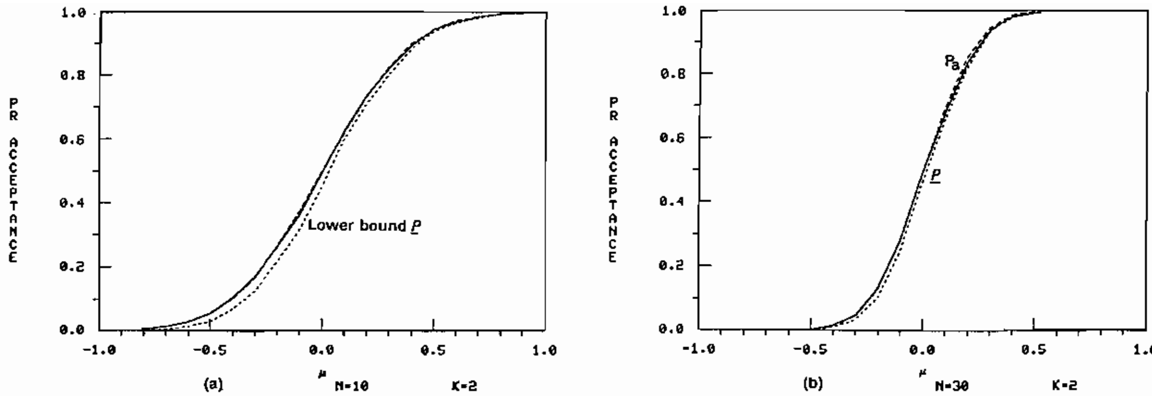

FIGURE 3. Acceptance probabilities when the number of allowable defectives $k = 2$ and n observations are drawn from the normal distribution $N(\mu, 1)$.

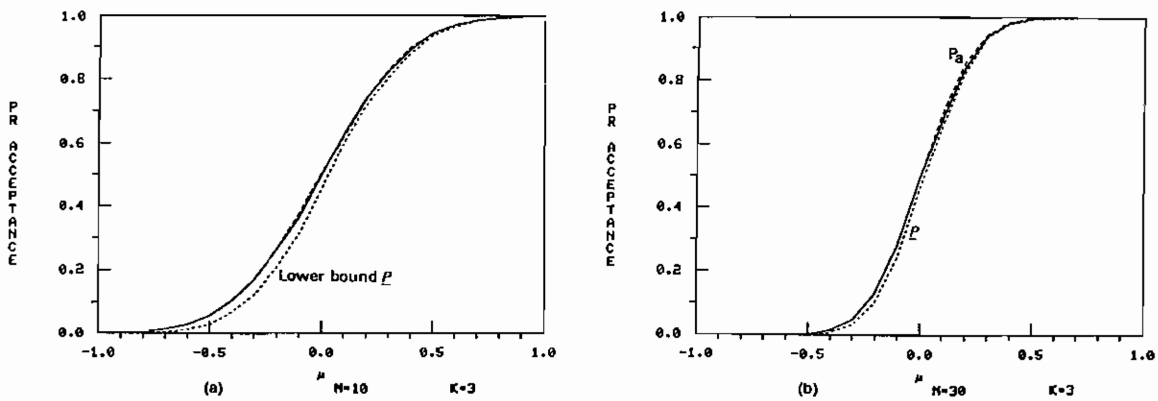

FIGURE 4. Acceptance probabilities when the number of allowable defectives $k = 3$ and n observations are drawn from the normal distribution $N(\mu, 1)$.

does not give a satisfactory approximation for the smaller sample size, and P_a is clearly preferable. Even for $n = 30$, P_a is at least as accurate as $\underline{P}$ over the entire range of μ.

5.3 Weibull Distribution

Assuming that $X_1,...,X_n$ are i.i.d. $W(1,\theta)$, and that $\mu_0 = 0.75$ and that L is chosen according to (3.3.8) with $\theta = 1$, the acceptance probability was computed by simulation and is shown as the solid line in figures 5-8. The corresponding lower bound $\underline{P}$ was also computed using simulation and is

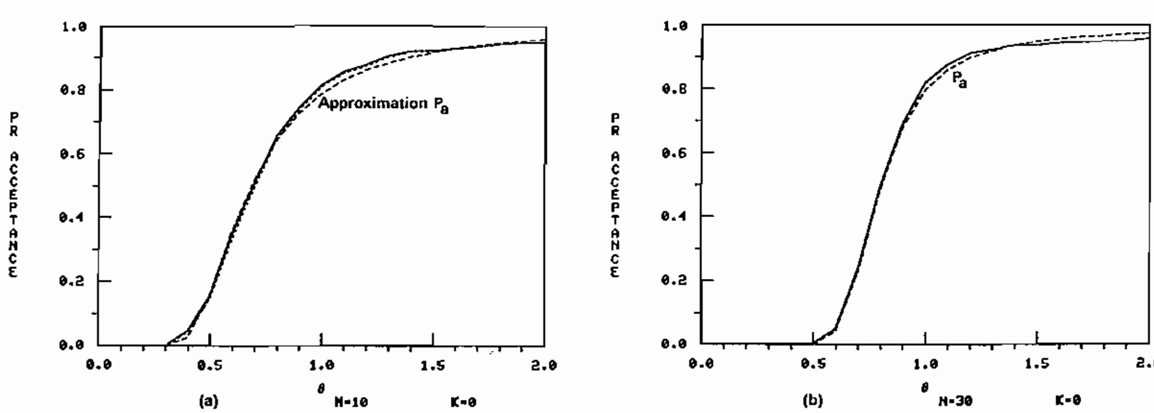

FIGURE 5. Acceptance probabilities when the number of allowable defectives $k = 0$ and n observations are drawn from a Weibull distribution $W(1, \theta)$.

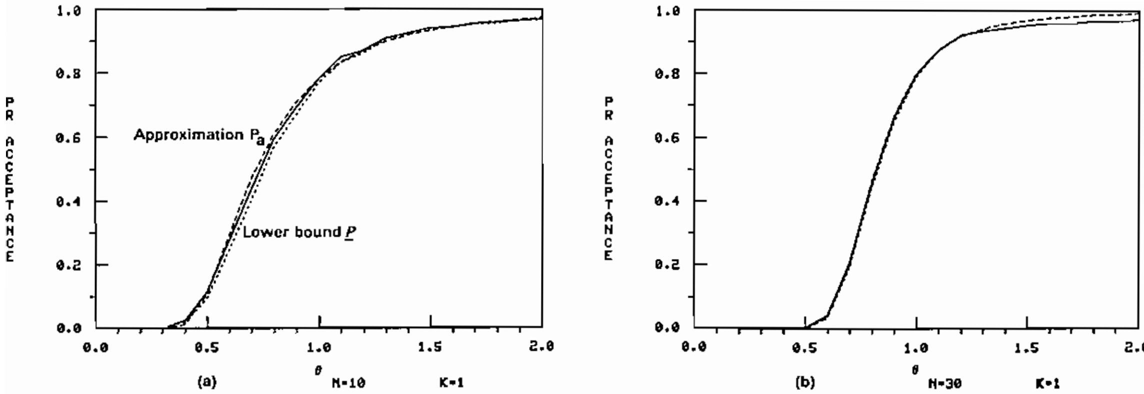

FIGURE 6. Acceptance probabilities when the number of allowable defectives $k = 1$ and n observations are drawn from a Weibull distribution $W(1, \theta)$.

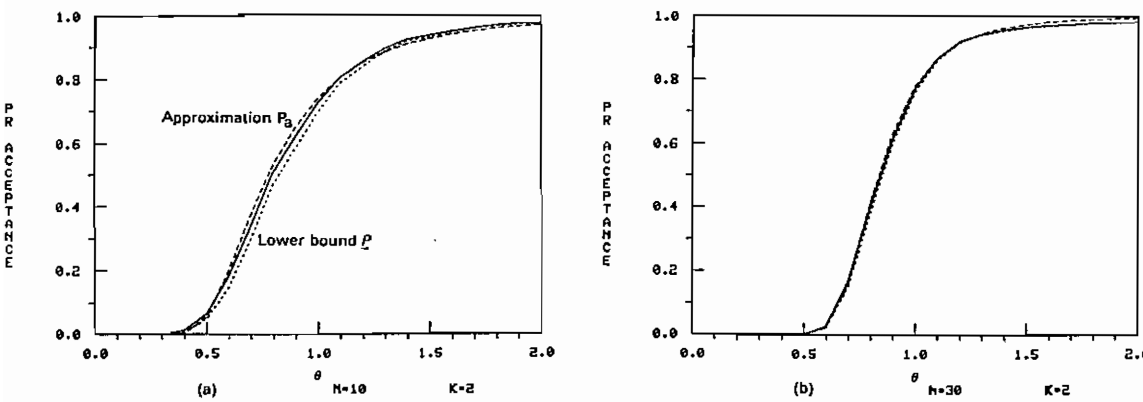

FIGURE 7. Acceptance probabilities when the number of allowable defectives $k = 2$ and n observations are drawn from a Weibull distribution $W(1, \theta)$.

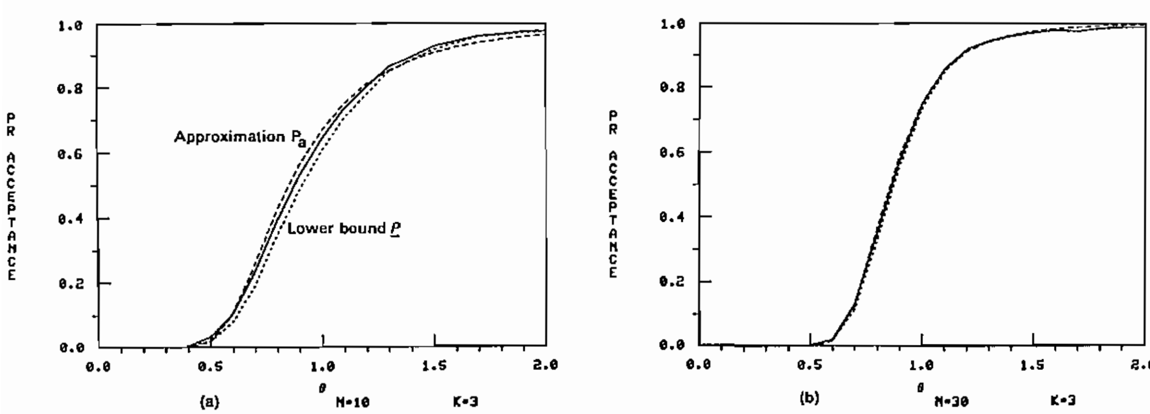

FIGURE 8. Acceptance probabilities when the number of allowable defectives $k = 3$ and n observations are drawn from a Weibull distribution $W(1, \theta)$.

represented by the dotted line in the same figures. The approximation P_a is shown by the heavy dashed line in the figures.

The figures show that P_a is not a particularly good approximation to P_n when $k = 0$, and one would do much better using the lower bound $\underline{P}$. However, P_a shows the same characteristic for the Weibull distribution as for the normal distribution; namely, that as k/n increases the accuracy of the approximation increases. For $n = 10$ and $k = 3$, P_a is superior to $\underline{P}$; for $n = 30$, $\underline{P}$ is indistinguishable from the simulated acceptance probability.

5.4 Exponential Distribution

5.4.1 Comparison with a UMP Test

As discussed in section 1, we may view the problem of finding an optimal sampling procedure as a hypothesis testing problem formulated in (1.4). In general there exists no uniformly most powerful (UMP) test for (1.4). However, it is interesting to note that in the exponential distribution the dual acceptance criterion for $k = 0$ corresponds to a test which is UMP for a subset of alternatives specified in (1.4). Specifically, suppose the sample comes from the exponential pdf given in (3.4.1).

The UMP acceptance region for testing

$$H_0\colon \lambda = \lambda^* \text{ and } \beta = \beta^*$$

versus

$$H_1\colon 0 < \lambda < \lambda^* \text{ and } 0 < \beta < \beta^*$$

is given by

$$[\overline{X} \geqslant \mu_0, X_{(1)} \geqslant \beta^*]. \tag{5.4.1}$$

This testing problem is equivalent to testing

$$H_0\colon \lambda = \lambda^* \text{ and } p = p^*$$

versus

$$H_1\colon 0<\lambda<\lambda^* \text{ and } p > 1 - (1-p^*)^{\lambda^*/\lambda}$$

where

$$p^* = 1-\exp\{-(L-\beta^*)/\lambda^*\}$$

or

$$\beta^* = L + \lambda^* \log (1-p^*).$$

Under H_0, $P_{\lambda^*,\beta^*}[X_{(1)} \geqslant \beta^*] = 1$, and μ_0 is determined by the equation

$$P_{\lambda^*,\beta^*}[\overline{X} \geqslant \mu_0] = 1-\alpha, \tag{5.4.2}$$

where α is a predetermined level of significance (Lehmann [15]).

If we set $\mathrm{L} = \beta^*$ and $\mathrm{k} = 0$, the test specified by (5.4.1) clearly is the same test specified by (1.3), and the acceptance probability

$$P_n = P_{\lambda,\beta}\{\overline{X} \geqslant \mu_0, X_{(1)} \geqslant \beta^*\} \tag{5.4.3}$$

can be computed either by the approximation shown in section 3.4 or by numerical integration using an exact formula for the distribution of $\overline{X}$ and N_L as shown in the next section.

5.4.2 Exact Distribution of $\overline{X}$ and N_L

The joint distribution of $\overline{X}$ and N_L can be obtained from the order statistics.

Let

$$Z_1 = nX_{(1)}$$

$$Z_i = (n-i+1)(X_{(i)}-X_{(i-1)}).$$

We have the pdf of $Z_{(1)}$,

$$g_1(z_1) = \lambda^{-1} \exp\{-(z_1-n\beta)/\lambda\}, z_1>\beta \tag{5.4.4}$$

and for $i \geqslant 2$, Z_i has a pdf

$$g_i(z_i) = \lambda^{-1}\exp(-z_i/\lambda), z_i \geqslant 0.$$

To compute the acceptance probability P_n for an arbitrary k, we make use of the fact that the Z's are independent r.v.'s, and that

$\sum_{j=1}^{n} X_j = \sum_{j=1}^{n} Z_j = \sum_{j=1}^{n} X_{(j)}$ and proceed as follows:

$$P_n = P_{\lambda,\beta}[\bar{X} \geqslant \mu_0, N_L \leqslant k]$$

$$= \int_A \cdots \int P[\sum_1^n Z_i \geqslant n\mu_0, \sum_1^{k+1} Z_i/(n-i+1) > L \,|\, z_1,\cdots,z_{k+1}] \left\{\prod_1^{k+1} g_i(z_i)\right\} dz_1\cdots dz_{k+1}$$

$$= \int_A \cdots \int P[\sum_{k+2}^{n} Z_i \geqslant n\mu_0 - \sum_1^{k+1} z_i] \left\{\prod_1^{k+1} g_i(z_i)\right\} dz_1\cdots dz_{k+1} \tag{5.4.5}$$

where $A = \{(z_1,\cdots,z_{k+1}): \sum_1^{k+1} z_i/(n-i+1) > L$ and $n\mu_0 - \sum_1^{k+1} z_i \geqslant 0\}$.

The expression in (5.4.5) is the exact probability of acceptance, P_n.

When $k = 0$, the computation of P_n reduces to

$$P_n = \int_{nL}^{n\mu_0} P[\sum_2^n Z_i \geqslant n\mu_0 - z_1]\, g_1(z_1)dz_1 + \int_{n\mu_0}^{\infty} g_1(z_1)dz_1. \tag{5.4.6}$$

Note that the sum $Y = \sum_2^n Z_i$ has a gamma density.

$$f(y) = \frac{(1/\lambda)^{n-1}}{\Gamma(n-1)}\, y^{n-2} \exp(-y/\lambda). \tag{5.4.7}$$

Substituting (5.4.4) and (5.4.7) in (5.4.6) we obtain

$$P_n = \frac{1}{\Gamma(n-1)} \int_a^b \int_c^\infty e^{-v}\, v^{n-2} \exp\{-(z_1-n\beta)/\lambda\} dv dz_1 + \exp\{-n(\mu_0-\beta)/\lambda\} \tag{5.4.8}$$

where

$$a = nL$$
$$b = n\mu_0$$
$$c = (n\mu_0-z_1)/\lambda$$

The lower bound for P_n is

$$\underline{P} = \int_{2n(\mu_0-\beta)/\lambda}^{\infty} f(x)dx \int_{2n(L-\beta)/\lambda}^{\infty} g(x)dx \tag{5.4.9}$$

where $f(x)$ is the pdf of the $\chi^2(2n)$ and $g(x)$ is the pdf of the $\chi^2(2)$.

5.4.3 Acceptance probabilities

If we assume that $X_1, \cdots, X_n$ are i.i.d. $E(\lambda,0)$, the acceptance probability for $k = 0$

$$P_n = P_\lambda[\bar{X} \geqslant \mu_0, N_L \leqslant 0] = P[\bar{X} \geqslant \mu_0, X_{(1)} > L]$$

is computed from (5.4.8) using a numerical integration technique that takes advantage of the fact that the inner integral is an incomplete Γ-function. Note that μ_0 is determined from $\chi^2(2)$ according to (5.4.2), and L is determined according to (3.4.5). The acceptance probability P_n is shown as the solid line in figure 9.

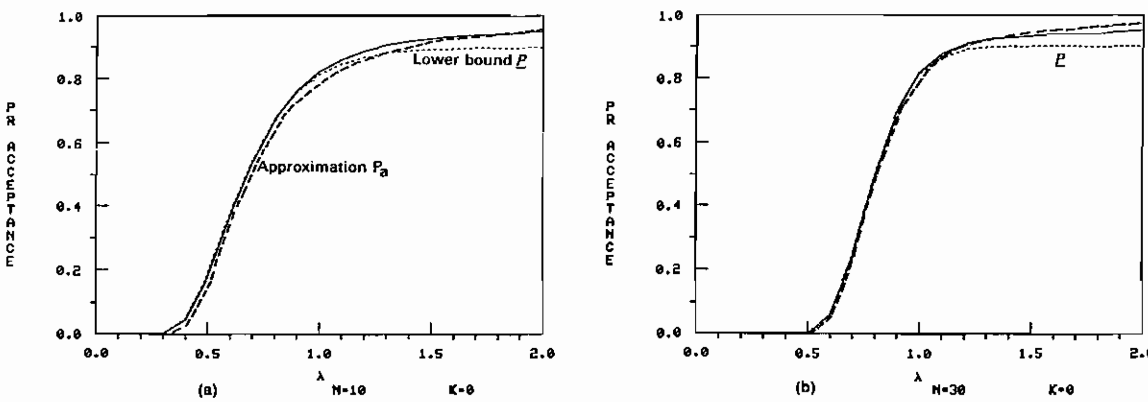

FIGURE 9. Acceptance probabilities when the number of allowable defectives $k = 0$ and n observations are drawn from an exponential distribution $E(\lambda, 0)$.

The acceptance probabilities for $k = 1,2,3$, for $\mu_0 = 0.75$ and L chosen according to (3.4.5) were computed by simulation as were the corresponding values of $\underline{P}$. The approximation P_a was computed from (3.4.6). Results are shown in figures 10-12.

The graphs show that P_a is a better approximation to P_n than the lower bound $\underline{P}$ for small sample size where the superiority of P_a over $\underline{P}$ increases as k increases. For large sample size, say $n = 30$, the two methods give almost identical approximations to P_n.

Values of μ_0 used in Computation of Acceptance Probabilities for UMP Test for Exponential Distribution

	Values of μ_0		
n	α=0.10	α=0.05	α=0.01
5	0.48652	0.39403	0.25582
10	0.62213	0.64254	0.41302
20	0.77626	0.66273	0.55411
30	0.77431	0.71998	0.62475

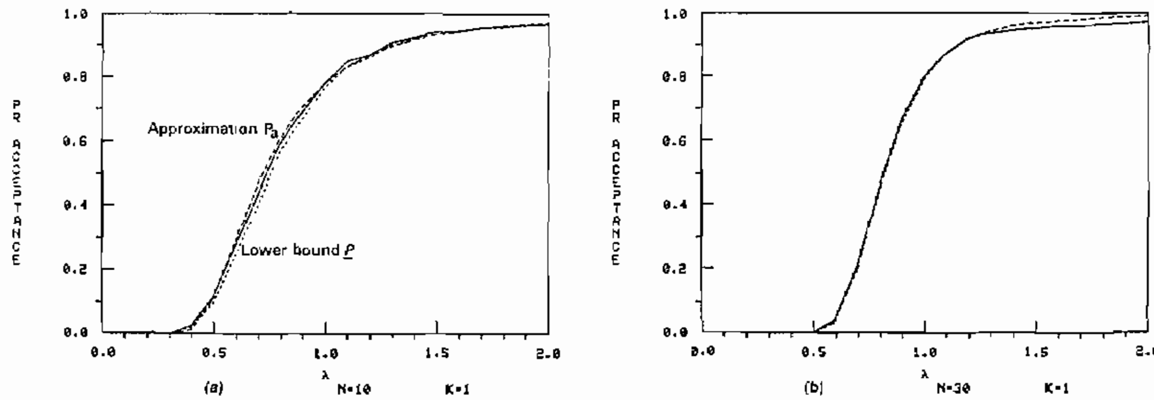

FIGURE 10. Acceptance probabilities when the number of allowable defectives $k = 1$ and n observations are drawn from an exponential distribution $E(\lambda, 0)$.

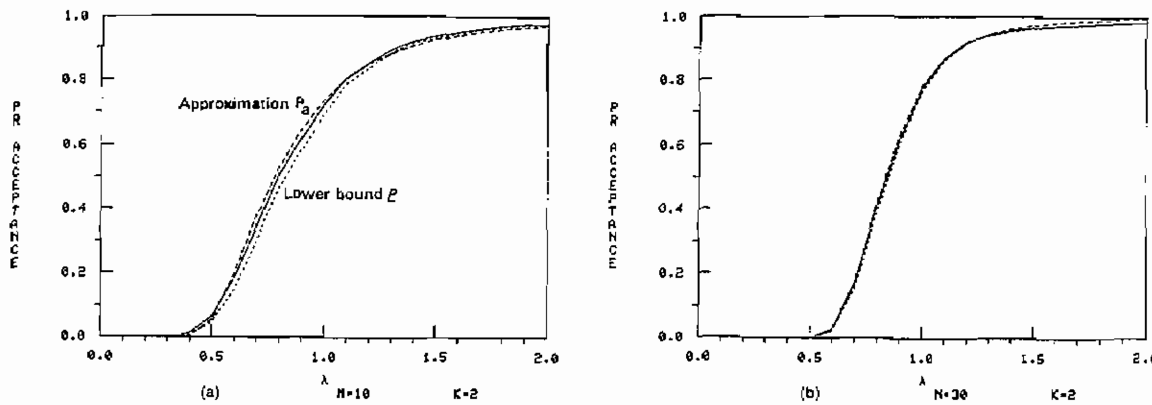

FIGURE 11. Acceptance probabilities when the number of allowable defectives $k = 2$ and n observations are drawn from an exponential distribution $E(\lambda, 0)$.

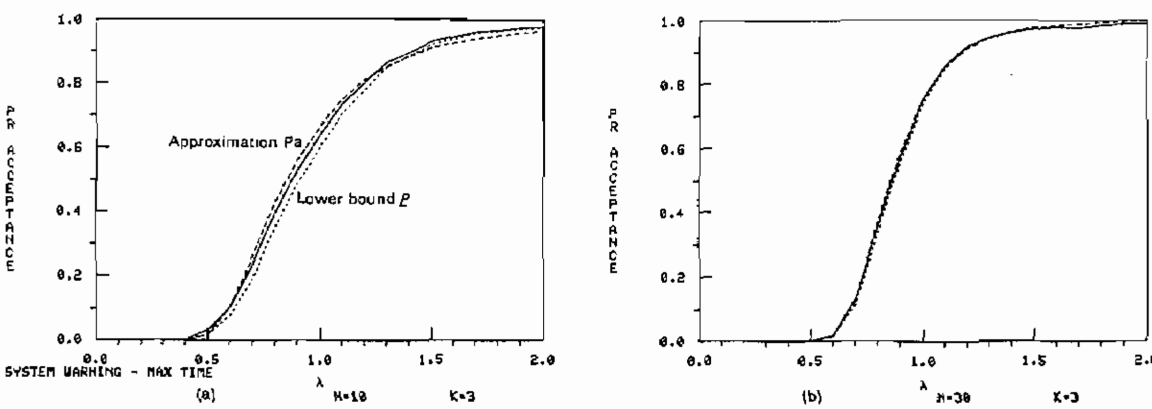

FIGURE 12. Acceptance probabilities when the number of allowable defectives $k = 3$ and n observations are drawn from an exponential distribution $E(\lambda, 0)$.

6. Synopsis

The problem of computing the acceptance probability P_n has been addressed by an approximation P_a that relies on the asymptotic joint distribution of the sample mean and number of defectives in the sample. P_a has the advantage that it is applicable to any continuous distribution. It is computed using a $N(0,1)$ cdf and a bivariate normal cdf which in turn can be reduced to a single variable integration.

The approximation P_a compares very favorably with another published approximation for the normal distribution and with a lower bound $\underline{P}$. Graphs of the acceptance probability as a function of one parameter of the distribution are used to compare the relative accuracies of P_a and $\underline{P}$. The graphs show that for the normal distribution P_a and $\underline{P}$ have comparable accuracies with $k = 0$. As k/n increases, P_a quickly becomes superior to $\underline{P}$, and even for large n and $k > 0$ P_a is superior. In other words, the best results for the normal distribution are obtained with $\underline{P}$ when $k = 0$ and with P_a for all other values of k.

In the case of Weibull distribution $\underline{P}$ is superior for $k = 0$. As k/n increases, P_a gains in accuracy, and for large n, P continues to have an edge over P_a. The difficulty in computing P for the Weibull distribution may make it desirable to use P_a for all applications.

In the case of exponential distribution, the exact joint distribution of the sample mean and number of defectives in the sample has been derived for $k = 0$. The computation of the acceptance probability P_n in this case involves a two-variable integration. Graphs of the acceptance probabilities show that the lower limit $\underline{P}$ gives a consistently good approximation to the acceptance probability. The approximation P_a and the lower limit $\underline{P}$ have also been computed for the exponential distribution for $1 \leqslant k \leqslant 3$. The graphs for these tests show that $\underline{P}$ is comparable or superior to P_a for large $n (n \geqslant 30)$ with P_a being somewhat superior when n is small, say $n \leqslant 10$.

The numerical integrations for this study were performed using the NBS software package DATAPLOT developed by Dr. J.J. Filliben, and the graphs were prepared using the same package. The authors wish to acknowledge the helpful suggestions for changes in the manuscript made by Dr. P. Smith and Mrs. M. Natrella.

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8. Appendix A

The approximation P_a given in (3.1.9) involves the computation of $L(a,b,\varrho)$ defined as

$$L(a,b,\varrho) = \int_a^\infty \int_b^\infty g(z,y,\varrho) dy dz.$$

The computation of $L(a,b,\varrho)$ can be reduced to a single variable integration. When a and b are both positive [18],

$$L(a,b,\varrho) = \frac{1}{2\pi} \int_{\arccos \varrho}^{\pi} \exp \{- \tfrac{1}{2}(a^2+b^2-2ab \cos w) \operatorname{cosec}^2 w\} dw$$

The following recursion relations hold:

$$\begin{aligned}
L(-a,b,\varrho) &= -L(a,b,-\varrho) + \tfrac{1}{2}[1-h(b)] \\
L(a,-b,\varrho) &= -L(a,b,-\varrho) + \tfrac{1}{2}[1-h(a)] \\
L(-a,-b,\varrho) &= L(a,b,\varrho) + \tfrac{1}{2}[h(a)+h(b)] \\
\text{where } h(x) &= \int_{-x}^{x} \exp(-t^2/2) dt.
\end{aligned}$$

The approximation P_a can be computed for all values of a,b and ϱ using the foregoing equations.

$$\begin{aligned}
P_a &= \Phi(-a) - L(a,b,\varrho), a > 0, b > 0 \\
P_a &= \Phi(-a) - \Phi(-b) + L(-a,b,-\varrho), a < 0, b > 0 \\
P_a &= L(a,-b,-\varrho), a > 0, b < 0 \\
P_a &= \Phi(b) - L(-a,-b,\varrho), a < 0, b < 0
\end{aligned}$$

where $\Phi(x) = \int_{-\infty}^{x} \exp(-t^2/2) dt$.

9. Appendix B

Asymptotic independence of the sample mean and the $(n-k)^{\text{th}}$ extreme statistic.

Let $X_1, \cdots, X_n$ be i.i.d with a p.d.f. $f(x)$. Denote the c.d.f. of the X's by $F(x)$. Assume that X's have a finite mean μ and finite variance σ^2. Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics.

The conditional density of $X_{(1)}, \cdots, X_{(n)}$ given that $X_{(n-k)} = x_{(n-k)}$ is given by

$$L_{x_{(n-k)}} = \frac{(n-k-1)!\prod_{1}^{n-k-1} f(x_{(i)})}{\{F(x_{(n-k)})\}^{n-k-1}} \cdot \frac{k!\prod_{n-k+1}^{n} f(x_{(i)})}{\{1-F(x_{(n-k)})\}^{k}} \tag{1}$$

Clearly, given that $X_{(n-k)} = x_{(n-k)}$, the joint conditional density may be regarded as the joint density of two dependent samples $\{Y_1,\cdots, Y_{n-k-1}\}$ and $\{W_1,\cdots, W_k\}$, where the Y-sample has a p.d.f.

$$\begin{aligned} h(x) &= \frac{f(x)}{F(x_{(n-k)})}, \quad \text{if } x < x_{(n-k)} \\ &= 0 \quad\quad\quad\quad , \quad \text{if } x > x_{(n-k)} \end{aligned} \tag{2}$$

and the W-sample has a p.d.f.

$$\begin{aligned} g(x) &= \frac{f(x)}{1-F(x_{(n-k)})}, \quad \text{if } x > x_{(n-k)} \\ &= 0 \quad\quad\quad\quad , \quad \text{if } x < x_{(n-k)} \end{aligned} \tag{3}$$

THEOREM. *For every fixed k, $\sqrt{n}(\overline{X}-\mu)$ is asymptotically independent of $X_{(n-k)}$ as $n \to \infty$.*

PROOF: Rewrite $\overline{X}$ in terms of the Y's and the W's. We obtain

$$\frac{\sqrt{n}(\overline{X}-\mu)}{\sigma} = \sqrt{n-k-1}\,\frac{(\overline{Y}-\mu)}{\sigma}\,\frac{\sqrt{n-k-1}}{\sqrt{n}} + \frac{(\overline{W}-\mu)\,k}{\sigma\sqrt{n}} + \frac{X_{(n-k)}-\mu}{\sigma\sqrt{n}} \tag{4}$$

From (2) we have

$$EY_i-\mu = \frac{\int_0^{x_{(n-k)}} x\,dF(x) \int_{x_{(n-k)}}^{\infty} dF(x)}{F(x_{(n-k)})} - \int_{x_{(n-k)}}^{\infty} x\,dF(x) \tag{5}$$

Making use of (4) and (5), and letting A be the value of EY_i with $X_{(n-k)}$ replaced by $X_{(n-k)}$, we get

$$\frac{\sqrt{n}(\overline{X}-\mu)}{\sigma} = \sqrt{n-k-1}\;\frac{(\overline{Y}-EY_i)}{\sigma}\;\frac{\sqrt{n-k-1}}{\sqrt{n}}$$

$$+\frac{(n-k-1)\;(A-\mu)}{\sqrt{n}\,\sigma} + \frac{(\overline{W}-\mu)}{\sigma}\;\frac{k}{\sqrt{n}} + \frac{X_{(n-k)}-\mu}{\sigma\sqrt{n}}$$

Since k is fixed, clearly $(\overline{W}-\mu)\;k/\sigma\sqrt{n}\to 0$ in probability as $n \to \infty$. To prove the theorem we need the following two lemmas in which we show that the second and the fourth terms tend to zero in probability. Then the theorem follows from the fact that the first term converges in distribution to $N(0,1)$ which is the "unconditional" limiting distribution of $\sqrt{n}(\overline{X}-\mu)$.

LEMMA 1. *As* $n \to \infty$,

$$\frac{X_{(n-k)}}{\sqrt{n}} \to 0 \text{ in P.}$$

PROOF: For every $\varepsilon > 0$ and for a fixed k, it follows from the Chebyehev inequality that

$$P\left\{\frac{|X_{(n-k)}|}{\sqrt{n}} > \varepsilon\right\} \leqslant \frac{E(X_{(n-k)})^2}{n\varepsilon^2} \leqslant \frac{1}{n\varepsilon} \quad E(\max_{1 \leqslant j \leqslant n} X_j^2)$$

Let $Y_j = X_j^2$ and $H(y) = P[Y_j \geqslant y]$.

Following a proof in Chung (1960),

$$P[\max_{1 \leqslant j \leqslant n} Y_j \geqslant y] = 1 - [H(y)]^n \geqslant n[1-H(y)]$$

and

$$\frac{1}{n} E\, |\max_{1 \leqslant j \leqslant n} X_j^2| = \frac{1}{n} \int_0^\infty \{1-[H(y)]^n\}\, dy \geqslant \int_0^\infty [1-H(y)]dy < \infty$$

On the other hand,

$$\frac{1}{n}\; E\, |\max_{1 \leqslant j \leqslant n} X_j^2| = \int_0^\infty \int_{H(y)}^1 u^{n-1}\, du\, dy.$$

Sinee the expectation is finite, we can take the limit as $n \to \infty$ under the integral sign. As a result

$$\lim_{n \to \infty} \frac{1}{n} \; E\, [max_{1 \leqslant j \leqslant n} X_j^2] = \int_0^\infty \int_{H(y)}^1 \lim_{n \to \infty} u^{n-1}\, du\, dy = 0$$

LEMMA 2. For a fixed k, $0 \leqslant k \leqslant n-1$,

$$\sqrt{n-k-1} \int_{X_{(n-k)}}^\infty x\, dF(x) \to 0 \text{ in P as } n \to \infty.$$

PROOF: Since

$$\sqrt{n-k-1} \int_{X_{(n-k)}}^\infty x\, dF(x) = \sqrt{n-k-1}\, X_{(n-k)}\, [1-F(X_{(n-k)})]$$

$$+ \sqrt{n-k-1} \int_{X_{(n-k)}}^\infty [1-F(x)]\, dx, \tag{8}$$

we will show that each term on the right side of (8) converges in probability to zero.

Set

$$z = 1 - \frac{x}{n-k-1} \tag{9}$$

Then

$$P\{(n-k-1)[1-F(X_{(n-k)})] > x\} = \sum_{i=n-k}^{n} \binom{n}{i} z^i (1-z)^{n-i} \to e^{-x} \sum_{i=0}^{k} \frac{x^i}{i!} \tag{10}$$

We see that $X_{(n-k)}/\sqrt{n-k-1} \to 0$ in P as shown in Lemma 1 and $(n-k-1)\,[1-F(X_{(n-k)})]$ converges in distribution as shown in (10). Thus, the first term on the right side of (8) tends to zero in $\underline{P}$.

Finally, to show that the last term in (8) tends to zero in P, write this term as

$$\sqrt{n-k-1} \int_{X_{(n-k)}}^{\infty} [1-F(X)]\,dx = \sqrt{(n-k-1)\,(1-F(X_{(n-k)})} \left\{ \int_{X_{(n-k)}}^{\infty} ([1-F(x)]dx) \,/\sqrt{1-F(X_{(n-k)})} \right\}$$

Clearly, the part in braekets tends to zero in P ean be seen by the application of the L'Hospitals's rule to it.