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Curve Fitting With Clothoidal Splines

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Clothoids, i.e. curves $Z(s)$ in R^2 whose curvatures $\varkappa(s)$ are linear fitting functions of arclength s, have been used for some time for curve fitting purposes in engineering applications. The first part of the paper deals with some basic interpolation problems for clothoids and studies the existence and uniqueness of their solutions.

The second part discusses curve fitting problems for clothoidal splines, i.e. C^2-curves, which are composed of finitely many clothoids. An iterative method is described for finding a clothoidal spline $Z(s)$ passing through given points $Z_i \varepsilon R^2$. $i = 0,1,..., n+1$, which minimizes the integral $\int_Z \varkappa(s)^2 ds$.

This algorithm is superlinearly convergent and needs only $0(n)$ operations per iteration. A similar algorithm is given for a related problem of smoothing by clothoidal splines.

Key words: Approximation; clothoids; computer-aided design; Cornu-spirals; curvature; curve fitting; Fresnel-integrals; interpolation; splines

Introduction

The characteristic property of curves known as *Cornu-spirals* or *clothoids* is that their curvature $\varkappa(s)$ is a linear function of the arc length, $\varkappa(s) = \varkappa_0 + \lambda s$. Straight lines ($\varkappa_0 = 0, \lambda = 0$) and circles ($\lambda = 0$) may be considered as limiting cases. We are interested in constructing C^2-curves in the plane R^2 which are composed of finitely many Cornu-spirals; that is, C^2-curves whose curvature is a continuous piecewise linear function of their arc lengths. We will call such curves *clothoidal splines*. Typical elementary problems encountered in such an effort are to construct a clothoid joining a given straight line and a given circle, or joining two circles. Composite curves of this type have been used by engineers, for instance, for the construction of highway sections, some of which are specified to be straight lines and circles. A more complex problem is to construct a clothoidal spline Z through a sequence of finitely many points $(x_i, y_i) \varepsilon R^2, i = 0, 1, \ldots, n+1$ such that the integral

$$K = \int_Z \varkappa(s)^2 ds$$

along the curve is minimal. This problem can be considered as an approximation to the "true" problem of curve fitting in R^2, namely that of finding a curve $Z(\cdot)$ minimizing this integral among *all* C^2-curves passing through the given points. The latter problem has been studied by several authors (Lee, Forsythe [7],[1] Mehlum [8]), and its exact solution leads to a multipoint boundary value problem for elliptic functions (Reinsch [14]). Mehlum [8] also proposed to approximate its solution by solving the corresponding multipoint boundary value problem for clothoidal spline functions, however the resulting clothoidal spline does in general not minimize the integral K among all interpolating clothoidal splines (see also Pal and Nutbourne [10] for a related use of clothoidal splines in computer aided geometric design).

There is also the problem of smoothing: for given points (x_i, y_i), $i = 0, 1, \ldots, n+1$, the problem is to find a clothoidal spline Z in such a way that its deviation (in the least squares sense) from the given points is not greater than a prescribed tolerance and the integral K along Z is minimal (compare Reinsch [13] for the related problem for spline functions).

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[1]Figures in brackets indicate literature references at the end of this paper.

Cornu-spirals can be easily computed in terms of Fresnel integrals, though admittedly not as easily as the cubic polynomials generally used for spline functions. In contrast to the latter, however, clothoidal splines are represented in terms of the natural parameter of plane curves; namely, the curvature as function of arc length. Furthermore, we hope that they do not exhibit the drawbacks observed with other schemes for curve fitting which have been observed in practice, namely, a tendency toward oscillations.

In the first section we list some elementary properties of Cornu-spirals and Fresnel integrals, mainly taken from Abramowitz and Stegun [1]. The second section deals with simple interpolation problems for a single Cornu-spriral. Section 3 is devoted to interpolation with clothoidal spirals; section 4 to the problem of smoothing.

1. Elementary properties of Cornu-spirals

By definition, a Cornu-spiral or clothoid is a curve,

$$Z(s) = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}, s \varepsilon R ,$$

whose curvature $\varkappa(s) = \varkappa_0 + \lambda s$ is a linear function of arc length s. If its tangent vector is

$$\dot{Z}(s) = \begin{bmatrix} \cos\phi(s) \\ \sin\phi(s) \end{bmatrix} ,$$

then

$$\varkappa(s) = \dot{\phi}(s) ,$$

so that

$$\phi(s) = \phi_0 + \int_0^s \varkappa(\tau)d\tau = \phi_0 + \varkappa_0 s + \frac{\lambda}{2}s^2 ,$$

$$Z(s) = Z_0 + \int_0^s \begin{bmatrix} \cos\phi(t) \\ \sin\phi(t) \end{bmatrix} dt . \tag{1.1}$$

According to the sign of λ, Z is called positively or negatively oriented. *In the sequel, we restrict ourselves to the case of $\lambda > 0$.* Similar results will hold for $\lambda < 0$.

Using the Fresnel integrals,

$$C(z) := \int_0^z \cos\frac{\pi t^2}{2}\,dt \ , \ S(z) := \int_0^z \sin\frac{\pi t^2}{2}\,dt \ , F(z) := \begin{bmatrix} C(z) \\ S(z) \end{bmatrix} ,$$

$Z(s)$ can be expressed in closed form by [see [1], formulas (7.4.38), (7.4.39)]

$$Z(s) = Z_0 + \sqrt{\pi/\lambda}\, V\left(\phi_0 - \frac{\varkappa_0^2}{2\lambda}\right)\left\{ F\left(\frac{\varkappa_0+\lambda s}{\sqrt{\pi\lambda}}\right) - F\left(\frac{\varkappa_0}{\sqrt{\pi\lambda}}\right)\right\}, \text{if } \lambda > 0, \tag{1.2}$$

where $V(\alpha)$ is the orthogonal matrix,

$$V(\alpha) \ := \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Note that $F(s)$ also describes a Cornu-spiral with arc length s, curvature $\varkappa(s) = \pi s$ and phase angle $\phi(s) = (\pi/2)s^2$. The Fresnel integrals have the following properties [see [1], (7.3.17), (7.3.20)] which we list without proof:

$$F(z) = -F(-z)$$

$$\lim_{z\to+\infty} F(z) = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}, \quad \lim_{z\to-\infty} F(z) = -\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}. \tag{1.3}$$

Moreover, $F(z)$ can be expressed in the following way [see [1], (7.3.9), (7.3.10)]:

$$F(z) = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - V\left(\frac{\pi}{2}z^2\right)h(z) \quad , \tag{1.4}$$

where the component functions $g(z)$ and $f(z)$ of

$$h(z) = \begin{bmatrix} g(z) \\ f(z) \end{bmatrix}$$

satisfy [see [1], (7.3.5), (7.3.6), (7.3.21), (7.3.27)-(7.3.31)],

(a) $h(0) = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}, \ \lim_{z\to+\infty} h(z) = 0$

(b) $g(z)$ and $f(z)$ are strictly monotonically decreasing for $z \in [0, +\infty]$

(c) $f'(z) = -\pi z g(z)\,,\ g'(z) = \pi z f(z) - 1,$ for $z \in R$ (1.5)

(d) For $z > 0$ the following estimates hold for $g(z)$ and $f(z)$:

$$\frac{1}{\pi^2 z^3}\left(1 - \frac{15}{(\pi z^2)^2}\right) < g(z) < \frac{1}{\pi^2 z^3}$$

$$\frac{1}{\pi z}\left(1 - \frac{3}{(\pi z^2)^2}\right) < f(z) < \frac{1}{\pi z}$$

$$\frac{-3}{\pi^3 z^5} < f(z) - \frac{1}{\pi z} < -\frac{3}{\pi^3 z^5}\left(1 - \frac{35}{(\pi z^2)^2}\right)$$

Approximations of $f(z)$, $g(z)$ suitable for the calculation of $F(z)$ are given in [1], (7.3.32), (7.3.33), and in Boersma [2].

As a simple consequence of (1.5d) we note the following estimates for the euclidean norms of the vectors $h(z)$ and

$$\bar{h}(z) := \begin{bmatrix} g(z) \\ f(z) - 1/(\pi z) \end{bmatrix}$$

to be used later on:

$$\text{(a)}\quad 1-\frac{15}{(\pi z^2)^2}\leqslant \|h(z)\|\left((1/\pi z)\sqrt{1+\frac{1}{(\pi z^2)^2}}\right)^{-1}\leqslant 1 \text{ for } z>0\,,$$

$$\text{(b)}\quad 1-\frac{35}{(\pi z^2)^2}\leqslant \|\bar{h}(z)\|\left((1/\pi^2 z^3\sqrt{1+\frac{9}{(\pi z^2)^2}}\right)^{-1}\leqslant 1 \text{ for } z>0\,, \tag{1.6}$$

$$\text{(c)}\quad \lim_{z\to+\infty} z\bar{h}(z)=\begin{bmatrix}0\\0\end{bmatrix},\ \lim_{z\to+\infty}\pi z h(z)=\begin{bmatrix}0\\1\end{bmatrix}.$$

By (1.5a), (1.5b), $\|h(z)\|$ decreases strictly monotonically toward 0 for $z \to +\infty$. The same holds for $\bar{h}(z)$:

$$\|\bar{h}(z)\| \textit{ decreases strictly monotonically toward } 0 \textit{ as } z\to+\infty. \tag{1.7}$$

The following is a consequence of (1.6) and (1.5):

$$\frac{1}{2}\,\frac{d}{dz}\,\|\bar{h}(z)\|^2=\frac{1}{\pi z^2}\,(f(z)\ -\frac{1}{\pi z})<0 \quad \text{for } z>0.$$

It follows from (1.3), (1.4) that $F(z)$ has the form shown in figure 1.

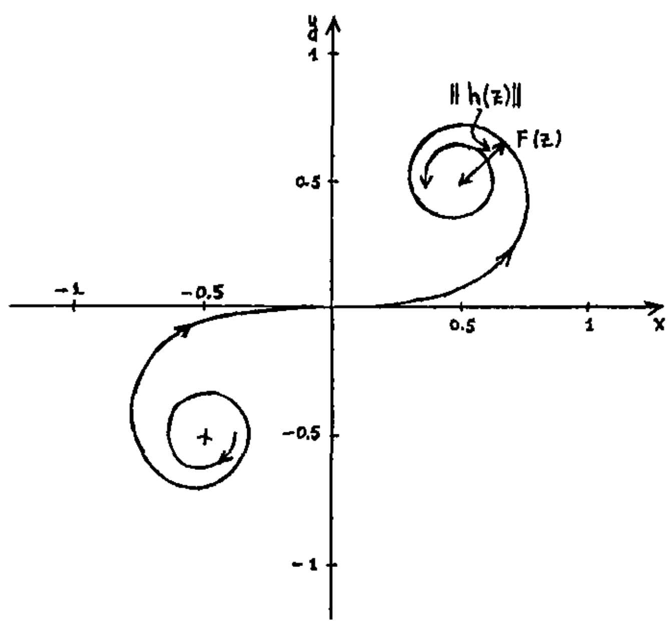


FIGURE 1. Positively Oriented Cornu-Spiral with $Z_o = X_o = o$ and $\lambda = \pi$

For $z \geqslant 0$ the vector

$$h(z)=\|h(z)\|\cdot\begin{bmatrix}\cos o(z)\\ \sin o(z)\end{bmatrix}>0\,,\ \ o(z):=\arctan\,(f(z)/g(z))$$

stays in the interior of the first quadrant of R^2

$$0 < \sigma(z) < \frac{\pi}{2}, \ \sigma(0) = \frac{\pi}{4}, \ \sigma(+\infty) = \pi/2 \ .$$

Moreover, the vector

$$r(z) := F(z) - \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} = -\mathrm{V}\left(\frac{\pi}{2}z^2\right)h(z) =: \|h(z)\| \cdot \begin{bmatrix}\cos\varrho(z)\\ \sin\varrho(z)\end{bmatrix}$$

where

$$\varrho(z) := \sigma(z) + \frac{\pi}{2}z^2 + \pi, \frac{\pi}{2}z^2 + \pi < \varrho(z) < \frac{\pi}{2}z^2 + \frac{3}{2}\pi$$

rotates counterclockwise for $z \geqslant 0$ as z tends to $+\infty$. This follows from (1.5):

$$\dot{\varrho}(z) = \dot{\sigma}(z) + \pi z = \frac{d}{dz}\operatorname{arctant}(f(z)/g(z)) + \pi z = \frac{f(z)}{f(z)^2 + g(z)^2} > 0 \text{ for } z \geqslant 0$$

Therefore, the curve $F(z)$ crosses any fixed ray

$$d_\alpha := \left\{ \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \sigma\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix} \,\middle\|\, \sigma \geqslant 0 \right\}$$

infinitely often at abscissae $0 \leqslant z_1 < z_2 < \ldots$, for which

$$\lim_{i \to \infty} z_i = +\infty$$

$$(z_i)^2 + 4n-1 \leqslant (z_{i+n})^2 \leqslant (z_i)^2 + 4n+1 \quad , \quad i \geqslant 1, n \geqslant 1 \tag{1.8}$$

$$4n-1 \leqslant (z_n)^2 \leqslant 4n+1$$

These estimates easily imply the following bounds

$$\frac{4n-1}{z_i}\left(1 + \sqrt{1 + \frac{4n-1}{(z_i)^2}}\right)^{-1} \leqslant z_{i+n} - z_i \leqslant \frac{4n+1}{z_i}\left(1 + \sqrt{1 + \frac{4n+1}{(z_i)^2}}\right)^{-1}, i,n \geqslant 1$$

$$\sqrt{4n-1} \leqslant z_n \leqslant \sqrt{4n+1} \quad , \tag{1.9}$$

which we note for later reference.

Upon inserting (1.4) into (1.2), we get the following representation of $Z(s)$ in terms of the vector h:

$$Z(s) = Z(0) - \sqrt{\frac{\pi}{\lambda}}\left(V(\phi(s))h\left(\frac{x(s)}{\sqrt{\pi\lambda}}\right) - V(\phi_0)h\left(\frac{x_0}{\sqrt{\pi\lambda}}\right)\right) \tag{1.10}$$

where (see (1.1))

$$x(s) := x_0 + \lambda s, \qquad \phi(s) := \phi_0 + x_0 s + \frac{\lambda}{2}s^2 \quad .$$

Note that because of $\lambda > 0$ and (1.5) (a), (1.3)

$$\begin{aligned}
&\text{(a)}\ Z(+\infty) = Z(0) + \sqrt{\frac{\pi}{\lambda}}\,V(\phi_0)h\left(\frac{x_0}{\sqrt{\pi\lambda}}\right)\\
&\text{(b)}\ Z(-\infty) = Z(+\infty) - \sqrt{\frac{\pi}{\lambda}}\,V\left(\phi_0 - \frac{x^2{}_0}{2\lambda}\right)\cdot\begin{bmatrix}1\\1\end{bmatrix}\\
&\text{(c)}\ Z(s) - Z(+\infty) = -\sqrt{\frac{\pi}{\lambda}}\,V(\phi(s))h\left(\frac{x(s)}{\sqrt{\pi\lambda}}\right)
\end{aligned} \tag{1.11}$$

The evolute of Z, that is the locus of all centers of curvature $M(s)$ of $Z(s)$ for $s \varepsilon R$, is given by

$$\begin{aligned}
M(s) &= Z(s) + \frac{1}{x(s)}\begin{bmatrix}-\sin\phi(s)\\ \cos\phi(s)\end{bmatrix} = Z(s) + \frac{1}{x(s)}\,V(\phi)s))\begin{bmatrix}0\\1\end{bmatrix}\\
&= Z(0) - \sqrt{\frac{\pi}{\lambda}}\left(V(\phi(s))\bar{h}\left(\frac{x(s)}{\sqrt{\pi\lambda}}\right) - V(\phi_0)h\left(\frac{x_0}{\sqrt{\pi\lambda}}\right)\right)
\end{aligned} \tag{1.12}$$

because $\bar{h}(z) = h(z) - \begin{bmatrix}0\\ \pi z\end{bmatrix}$. Again, the evolute M is a spiral type of curve with the following properties:

$$\begin{aligned}
&\text{(a)}\quad M(+\infty) = Z(+\infty)\\
&\text{(b)}\quad M(-\infty) = Z(-\infty)\\
&\text{(c)}\quad M(s) - M(+\infty) = -\sqrt{\frac{\pi}{\lambda}}\ V(\phi(s))\bar{h}\left(\frac{x(s)}{\sqrt{\pi\lambda}}\right)\\
&\text{(d)}\quad M(s_1) \neq M(s_2) \text{ for } s_1 \neq s_2 \quad .
\end{aligned} \tag{1.13}$$

(a) follows directly from (1.7) and (1.11), (b) and (c) follow from (1.11), and (d) from (1.7], since $V(\phi(s))$ is an orthogonal matrix. Furthermore,

if $x(\bar{s}) > 0$, $\lambda > 0$, then for every $s > \bar{s}$

$$||M(\bar{s}) - M(s)|| < \frac{1}{x(\bar{s})} - \frac{1}{x(s)}\ , \tag{1.14}$$

$$||M(\bar{s}) - Z(s)|| < \frac{1}{x(\bar{s})}\ ,$$

that is, for $s > \bar{s}$ the osculating circle of Z at s and $Z(s)$ are contained in the interior of the osculating circle of Z at $\bar{s}$.

Indeed, according to a well-known result of differential geometry (see, e.g., [15]), the arclength $\sigma(s)$ of the evolute $M(s)$ of any curve $Z(s)$ is given relative to the curvature $x(s)$ of $Z(s)$ by

$$\dot{\sigma}(s) = -\frac{d}{ds}x(s)^{-1}$$

so that in our case for $s > \bar{s}$

$$\sigma(s) - \sigma(\bar{s}) \;=\; \frac{1}{\kappa(\bar{s})} - \frac{1}{\kappa(s)}\ .$$

Since $M(\tau)$, $\tau \in [\bar{s},s]$ is not a straight line, we have the additional inequality

$$\|M(s) - M(\bar{s})\| < \sigma(s) - \sigma(\bar{s}) = \kappa(\bar{s})^{-1} - \kappa(s)^{-1}$$

which proves the first part of (1.14). The second part follows from the first, as

$$\|Z(s) - M(s)\| = \kappa(s)^{-1}$$

2. Interpolation properties of Cornu spirals

In this section we study some simple interpolation problems for Cornu spirals. In stating the results we make use of oriented circles

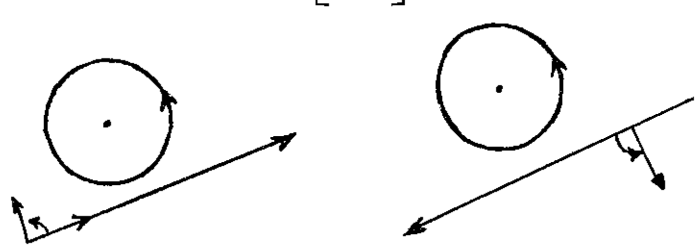
$$K(a,r) \;:=\; \left\{ a + r \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \,\middle|\, 0 \leqslant \phi \leqslant 2\pi \right\}\ ,$$

whose orientation is determined by the sign of the radius $r \neq 0$, and of oriented lines

$$g = g(b,\alpha) \;:=\; \left\{ b + \sigma \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \,\middle|\, \sigma \in R \right\}\ ,$$

whose orientation is deterined by the direction of the vector $(\cos\alpha, \sin\alpha)^T$. We say that the orientations of an oriented line g and of an oriented circle $K(a,r)$ not meeting g are *coherent*, if $K(a,r)$ lies in the same halfplane determined by g which contains the point

$$b + r \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$$



Coherent orientation　　　　Incoherent orientation

FIGURE 2

A first simple result refers to the problem of joining a line to a circle by a Cornu spiral.

(2.1) THEOREM:

1. *For any given oriented circle* K(a,r), r ≠ 0, *not meeting a coherently oriented line* g(b,α) *there exists exactly one oriented Cornu-spiral* Z(s) *which joins* g to K(a,r) *(in this order) such that the resulting composite curve is a* C^2 *curve with a coherent orientation.*
2. *If* g meets K *or the orientation of* g *and* K *are not coherent, then there is no such interpolating Cornu-spiral.*

Of course, a similar result holds for joining an oriented circle K to an oriented line (in this order) by an oriented Cornu-spiral which we do not state explicitly.

PROOF: 1. Without loss of generality we may assume that $r=1/\kappa>0$ and g is the x-axis in R^2 with its usual orientation. Since $K(a,r)$ is coherently oriented with g, the center $a=(x_0,y_0)^T$ of K is such that $\bar{y}:=y_0/r=y_0\kappa>1$.

Any positively oriented Cornu-spiral touching the x-axis at $(0,0)^T$ with $s=0$ (i.e., $\phi_0=0$, $Z(0)=0$) with a curvature $\kappa(0)=\kappa_0=0$ has the form (see (1.2)).

$$Z(s)=\sqrt{\frac{\pi}{\lambda}}F\left(\sqrt{\frac{\lambda}{\pi}}s\right)=:\begin{bmatrix}x(s)\\ y(s)\end{bmatrix}$$

with some $\lambda>0$. In order to solve the problem it suffices to determine $s>0$ and $\lambda>0$ such that Z has at s the curvature κ and $(x_0,y_0)^T$ as center of curvature (see fig. 3).

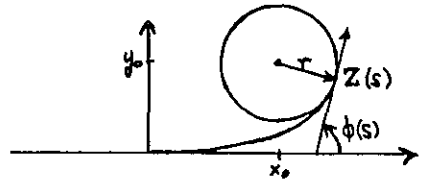


FIGURE 3

This leads to the conditions

$$\begin{aligned}\kappa(s)&=\lambda s=\kappa \quad\rightarrow\quad \lambda=\kappa/s\ ,\\ \phi(s)&=\frac{\lambda}{2}s^2=\kappa s/2\ ,\\ \cos\phi(s)&=(y_0-y(s))\,\kappa=\bar{y}-\kappa\sqrt{\frac{\pi}{\lambda}}\,S\left(\sqrt{\frac{\lambda}{\pi}}\,s\right)\end{aligned}$$

Hence s must satisfy the equation

$$\cos\frac{\kappa s}{2}+\sqrt{\pi s\kappa}\,S\left(\sqrt{\frac{\kappa s}{\pi}}\right)=\bar{y}\quad,$$

or the variable

$$\Psi:=\sqrt{\frac{\kappa s}{2}}\quad,$$

must solve

$$\cos\Psi^2+\Psi\sqrt{2\pi}\,S\left(\sqrt{\frac{2}{\pi}}\,\Psi\right)=\bar{y}\quad.$$

Now the function

$$\begin{aligned}p(\Psi)\ :&=\cos\Psi^2+\Psi\sqrt{2\pi}\,S\left(\sqrt{\frac{2}{\pi}}\,\Psi\right)\\ &=\cos\Psi^2+2\Psi\int_0^{\Psi}\sin t^2dt\end{aligned}$$

is strictly monotonically increasing for $\Psi\geqslant 0$ because

$$p'(\Psi)=2\int_0^{\Psi}\sin t^2dt>0\ \text{ for }\ \Psi>0.$$

Since $\bar{y}>1$, $p(0)=1$ and $\lim_{\tau\to\infty}p(\tau)=+\infty$, there exists therefore a unique solution $\bar{\Psi}>0$ of (2.2), which can be found by Newton's method. In terms of $\overline{\Psi}$, the solution of the problem is

$$s = 2\overline{\Psi}^2/\kappa \quad , \quad \lambda = \kappa/s$$

$$Z(s) = \sqrt{\frac{\pi}{\lambda}}\ F\left(\sqrt{\frac{\lambda}{\pi}}\, s\right) \ , \ x_0 = x(s) - \frac{1}{\kappa} \sin \overline{\Psi}^2$$

The proof of (2) is straightforward.

We now turn to the problem of joining two oriented circles,

$$K_i\,(a_i, 1/\kappa_i)\,, \quad i = 1,2, \quad ,$$

by an oriented Cornu spiral.

We first show an auxiliary result for the family of Cornu spirals $Z_\lambda\,(s)$, $\lambda>0$ with

$$\kappa_0 = 0\,, \quad \phi_0 = 0\,, \quad Z_\lambda\,(0) = 0$$

$$\kappa(s) = \lambda s\,, \quad \phi\,(s) = \frac{\lambda}{2}\, s^2$$

given by [see (1.10), (1.5a)]

$$Z_\lambda\,(s) = -\sqrt{\frac{\pi}{\lambda}} \left(V\left(\frac{\lambda s^2}{2}\right) h \left(\frac{\lambda s}{\pi\lambda}\right) - \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} \right)$$

For their center of curvature $M_\lambda\,(s)$ taken at arclength $s{:}= \overline{\kappa}/\lambda$ for which $\kappa(s) = \overline{\kappa}$, the following holds:

$$M_\lambda\,(\overline{\kappa}/\lambda) = -\sqrt{\frac{\pi}{\lambda}} \left(V\left(\frac{\overline{\kappa}^2}{2\lambda}\right) \overline{h} \left(\frac{\overline{\kappa}}{\sqrt{\pi\lambda}}\right) - \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} \right)$$

so that because of (1.6) (c), (1.11) and (1.13)

$$\begin{aligned}
&\text{(a)} \quad \lim_{\lambda \downarrow 0} M_\lambda\,(\overline{\kappa}/\lambda) - \sqrt{\frac{\pi}{\lambda}} \begin{bmatrix}0.5\\0.5\end{bmatrix} = 0 \quad \text{if } \overline{\kappa} > 0 \\
&\text{(b)} \quad \lim_{\lambda \downarrow 0} M_\lambda\,(\overline{\kappa}/\lambda) + \sqrt{\frac{\pi}{\lambda}} \begin{bmatrix}0.5\\0.5\end{bmatrix} = 0 \quad \text{if } \overline{\kappa} < 0 \\
&\text{(c)} \quad \lim_{\lambda \to +\infty} M_\lambda\,(\overline{\kappa}/\lambda) \quad = \quad \begin{bmatrix}0\\1/\overline{\kappa}\end{bmatrix}.
\end{aligned} \tag{2.3}$$

As an easy consequence, we get

(2.4) THEOREM: *Let* $K_i\ (a_i, 1/\kappa_i)$, $i = 1,2$ *be two oriented circles.*

1. *If* K_i and K_2 *are coherently oriented*, i.e. *if* $\kappa_i \cdot \kappa_2 > 0$, *then there exists an oriented Cornu spiral joining* K_1 *to* K_2 *(in this order) and having both* K_1 *and* K_2 *as osculating circle if and only if their centers* a_i *are different and one of the circles contains the other in its interior.*

2. *If* $\kappa_1 \cdot \kappa_2 < 0$, *then there exists an oriented Cornu spiral joining* $\underline{K}_1$ and $\underline{K}_2$ *(in this order) and having both* K_1 *and* K_2 *as osculating circles if and only if neither circle contains the other, i.e.* $\|a_1 - a_2\| > K_1\|^{-7} + \|K_2\|^{-7}$.

PROOF: (1) Assume $\kappa_2 > \kappa_1 > 0$ without loss of generality and let K_1 contain K_2 in its interior; that is,

$$0 < \|\ a_1 - a_2\ \| < 1/\kappa_1 - 1/\kappa_2 \quad . \tag{2.5}$$

Then by (2.3) (a), (c)

$$\lim_{\lambda \downarrow 0} \| M_\lambda(x_1/\lambda) - M_\lambda(x_2/\lambda) \| = 0$$

$$\lim_{\lambda \to +\infty} \| M_\lambda(x_1/\lambda) - M_\lambda(x_2/\lambda) \| = 1/x_1 - 1/x_2 \quad .$$

Since $M_\lambda(x/\lambda)$ depends continuously on $\lambda > 0$, there is a $\lambda' > 0$ such that

$$\| M_{\lambda'}(x_1/\lambda') - M_{\lambda'}(x_2/\lambda') \| = \| a_1 - a_2 \| ,$$

that is the Cornu spiral $Z_{\lambda'}$ has two osculating circles of radii $1/x_1$ and $1/x_2$ respectively, whose centers $M_{\lambda'}(x_i/\lambda')$, $i = 1,2$ have the desired distance. This proves the "if" part of (1). To prove the "only if" part, note that by (1.13)(d), the centers of curvature of any Cornu spiral are different for different arclengths, so that $a_1 \neq a_2$ is a necessary condition for the existence of a Cornu spiral joining two different circles K_1, K_2. The rest follows from (1.14).

(2.) Assume $x_1 > 0 > x_2$ and $\| a_1 - a_2 \| > 1/x_1 - 1/x_2$. Then, because of (2.3)

$$\lim_{\lambda \to +\infty} \| M_\lambda(x_1/\lambda) - M_\lambda(x_2/\lambda) \| = 1/x_1 - 1/x_2$$

$$\lim_{\lambda \downarrow 0} \| M_\lambda(x_1/\lambda) - M_\lambda(x_2/\lambda) \| = +\infty$$

Hence by a continuity argument there exists $\lambda' > 0$ such that

$$\| M_{\lambda'}(x_1/\lambda') - M_{\lambda'}(x_2/\lambda') \| = \| a_1 - a_2 \|$$

which proves the "if" part of (2). The "only if" part is trivial. We next turn to the following problems:

(2.6) PROBLEM: *For a given oriented circle* K *and two points* $P_0 \varepsilon K$ and $P_1 \varepsilon K$ *find an oriented Cornu-spiral connecting* P_0 to P_1 *(in this order) which has* K *as osculating circle at* P_0 *(see figs. 4 (A), (B)).*

(2.6) is equivalent to the problem of connecting a point $P_1 \varepsilon K$ to a point $P_0 \varepsilon K$ (in this order) on an oriented circle K by an oriented Cornu spiral which has K as osculating circle at P_0. Using suitable reflections and changes of orientation [compare fig. 4 (B), (C)], (2.6) is seen to be equivalent to the following, which involves only positive orientations:

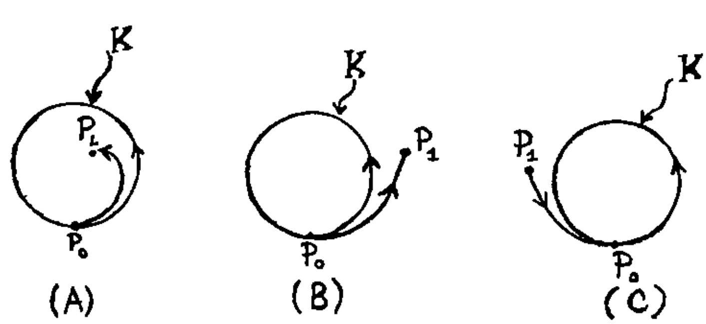


FIGURE 4

(2.6′) PROBLEM: *For a given positively oriented circle* $K = K(M_0, 1/\kappa)$, $\kappa > 0$, *and two points* $P_0 \varepsilon K$ *and* $P_1 \varepsilon K$ *find a positively oriented Cornu-spiral with* K *as osculating circle at* P_0, *which leads from* P_0 *to* P_1, *if* P_1 *is inside,* K *and leads from* P_1 *to* P_0 *if* P_1, *if* P_1 *is outside* K.

Clearly, (2.6′) depends only on κ and the relative positions of P_0 and P_1 so that we may assume without loss of generality

$$M_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \underline{P}_0 = \begin{bmatrix} 0 \\ -1/\kappa \end{bmatrix}, \quad \underline{P}_1 = \quad r \begin{bmatrix} \sin\alpha \\ -\cos\alpha \end{bmatrix} \quad, r \geqslant 0 \quad ,$$

(see Fig. 5).

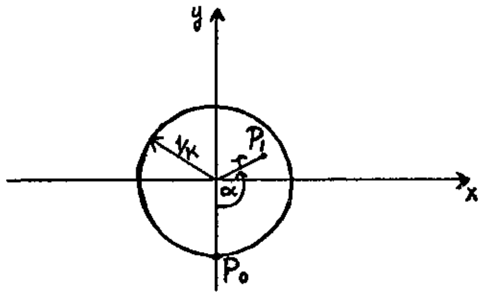


FIGURE 5

By (1.10) the class of positively oriented Cornu spirals Z with

$$Z(0) = P_0 = \begin{bmatrix} 0 \\ -1/\kappa \end{bmatrix}, \quad \kappa(0) = \kappa \quad , \quad \phi(0) = 0$$

is given by

$$\begin{aligned} C_\lambda(s) &= \begin{bmatrix} 0 \\ -1/\kappa \end{bmatrix} + \sqrt{\frac{\pi}{\lambda}}\,\underline{h}\left(\frac{\kappa}{\sqrt{\pi\lambda}}\right) - \sqrt{\frac{\pi}{\lambda}}\; V(\phi_\lambda(s))\,h(\kappa_\lambda(s)/\sqrt{\pi\lambda}\,) \\ &= \sqrt{\frac{\pi}{\lambda}}(\bar{h}\;(\kappa/\sqrt{\pi\lambda}) \quad - \quad V(\phi_\lambda(s))\,h(\kappa_\lambda(s)/\sqrt{\pi\lambda})) \end{aligned}$$

where

$$\kappa_\lambda(s) \;:=\; \kappa + \lambda s \quad , \quad \phi_\lambda(s) \;:=\; \kappa s + (\lambda/2)s^2.$$

Essentially, we will show [Theorem (2.25)] that for $r \neq 1/\kappa$, i.e. $P_1 \varepsilon K$, there are countably many numbers $\lambda_1 > \lambda_2 > \ldots > 0$ and arclengths s_i, $i \geqslant 1$, such that $\underline{C}_{\lambda_i}(s_i) = P_i$ for all $i \geqslant 1$. To prove this, we need some auxiliary results. From (1.5) (a) and (1.14) follow

$$C_\lambda(+\infty) \quad = \quad \sqrt{\frac{\pi}{\lambda}}\,\bar{h}\;(\kappa/\sqrt{\pi\lambda}) \quad , \quad \|\; C_\lambda(+\infty) \;\| < 1/\kappa \quad . \tag{2.8}$$

We show next:

(2.9) *For any fixed bounded interval* $I = [s_1, s_2]$ *such that for all* $\lambda > 0$ *and all* $s \,\varepsilon\, I$, $\kappa(s) = \kappa + \lambda s > 0$ *there holds*

$$\limsup_{\lambda \downarrow 0} \sup_{s \varepsilon I} \;\|\; C_\lambda(s) \;\| = 1/\kappa \;. \tag{2.9}$$

PROOF. It follows from (2.7):

$$C_\lambda(s) = \sqrt{\frac{\pi}{\lambda}}\left(\bar{h}\ (x/\ \sqrt{\pi\lambda}) - V(\phi_\lambda(s))\ \bar{h}\left(\frac{x_\lambda(s)}{\pi\lambda}\right)\right) - V(\phi_\lambda(s)\begin{bmatrix} 0 \\ 1/x_\lambda(s)\end{bmatrix} .$$

By (1.6)(c), the first two terms tend to 0 uniformly in $s \varepsilon I$ as $\lambda \downarrow 0$. Hence,

$$\limsup_{\lambda\downarrow 0\ \ s\varepsilon I} \|C_\lambda(s)\| = \limsup_{\lambda\downarrow 0\ \ s\varepsilon I} 1/x_\lambda(s) = 1/x \text{ , QED.}$$

With the abbreviations

$$\bar{h}_\lambda := \sqrt{\frac{\pi}{\lambda}} = \bar{h}\ (x/\sqrt{\pi\lambda}) \ ,\ h_\lambda(s) := \sqrt{\frac{\pi}{\lambda}}\ h(x_\lambda(s)/\ \pi\lambda)$$

$$\bar{r}_\lambda := \|\ \bar{h}_\lambda\ \| \ ,\ r_\lambda(s) := \|\ h_\lambda(s)\ \| \ ,$$

we have from (2.7)

$$C_\lambda(s) = \bar{h} - V(\phi_\lambda(s))\, h_\lambda(s) \tag{2.10}$$

and from (1.6), (2.8), the estimates

$$\frac{\lambda}{x^3}\left(1 - \frac{35\lambda^2}{x^4}\right) \leqslant \bar{r}_\lambda \ / \sqrt{1 + \frac{9_\lambda{}^2}{x^4}} \leqslant \lambda/x^3 \ ,\ \bar{r}_\lambda < 1/x$$
$$\frac{1}{x_\lambda(s)}\left(1 - \frac{15\lambda^2}{x_\lambda(s)^4}\right) \leqslant r_\lambda(s) \ / \sqrt{1 + \frac{\lambda^2}{x_\lambda(s)^4}} \leqslant \frac{1}{x_\lambda(s)} \tag{2.11}$$

for all s with $x_\lambda(s) = x + \lambda s > 0$.

Two cases are possible with respect to the location of the target point

$$P_1 = r\begin{bmatrix}\sin \alpha \\ -\cos \alpha\end{bmatrix}$$

which will be treated somewhat differently.

Case (1): $0 \leqslant r < 1/x$, P_1 lies in the interior of K
Case (2): $r > 1/x$, P_1 lies outside of K.

In Case (1) there is a sufficiently small $\bar{\lambda} > 0$ such that

$$C_\lambda(+\infty) = \bar{h}_\lambda \neq P_1 \text{ for all } 0 < \lambda \leqslant \bar{\lambda} \tag{2.12}$$

Note this is exactly true if $r = 0, P_1 = 0$, for then by (2.8),

$$C_\lambda(+\infty) = \bar{h}_\lambda \neq 0 \text{ for all } \lambda > 0 \ .$$

If $r > 0$, a suitable $\bar{\lambda} > 0$ can be found because of (2.11). With $\bar{\lambda} > 0$ satisfying (2.12), consider the rays

$$d_\lambda := \{\bar{h}_\lambda + \sigma(P_1 - \bar{h}_\lambda) \mid \sigma \geqslant 0\} \quad , \quad 0 < \lambda \leqslant \bar{\lambda}$$

extending from $\bar{h}_\lambda$ towards P_1 (see fig. 6).

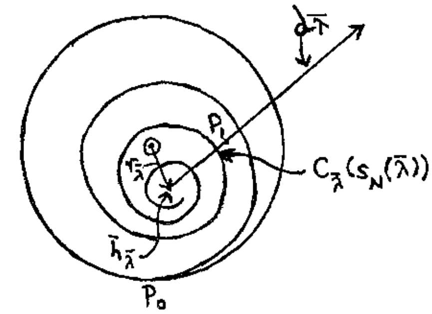


FIGURE 6

Because of (2.10) and using the same reasoning as with (1.8), every Cornu spiral $C_\lambda(s)$, $0 < \lambda \leqslant \bar\lambda$ cuts d_λ infinitely often at abscissae $0 \leqslant s_1(\lambda) < s_2(\lambda) < \ldots$, which satisfy estimates of the form [cf. (1.8)].

$$\begin{aligned} \phi_\lambda(s_n(\lambda)) + 3\pi/2 \quad &\leqslant \quad \phi_\lambda(s_{n+1}(\lambda)) \\ 2(n-1)\pi \;\leqslant\; 2(n - \frac{1}{4})\pi \;\leqslant\; \phi_\lambda(s_n(\lambda)) &\leqslant\; 2(n+\frac{1}{4})\pi \;\leqslant\; 2(n+1)\pi \end{aligned} \tag{2.14}$$

for $n \geqslant 1$, so that

$$\begin{aligned} s_{n+1}(\lambda) - s_n(\lambda) \;&\geqslant\; \frac{3\pi}{\varkappa+\lambda s_n(\lambda)}\left(1 \;+\; 1 \;+\; \sqrt{\frac{3\pi\lambda}{[\varkappa+\lambda s_n(\lambda)]^2}}\right) \\ \bar s_{n-1}(\lambda) \;\leqslant\; s_n(\lambda) \;&\leqslant\; \bar s_{n+1}(\lambda) \, . \end{aligned} \tag{2.15}$$

where

$$\bar s_m(\lambda) := \frac{4m\pi}{\varkappa}\left(1 \;+\; \sqrt{1+\frac{4m\pi\lambda}{\varkappa^2}}\;\right)$$

is the solution of the quadratic equation,

$$\phi_\lambda(s) \equiv \varkappa s + \frac{\lambda}{2}\, s^2 = 2m\pi \quad .$$

As a consequence,

$$\lim_{n\to\infty} s_n(\lambda) \;=\; +\infty, \lim_{n\to\infty} C_{\bar\lambda}(s_n(\bar\lambda)) \;=\; \bar h_{\bar\lambda} \quad ,$$

and therefore there exists an N such that for all $n \geqslant N$ (see fig. 6),

$$C_{\bar\lambda}(s_n(\bar\lambda)) \;\;\varepsilon\;\; [\bar h_{\bar\lambda}, P_1] := \{\bar h_{\bar\lambda} + \sigma(P_1 - \bar h_{\bar\lambda}) \mid \leqslant \sigma \leqslant 1\} \;,$$

that is, $C_{\bar\lambda}$ intersects $d_{\bar\lambda}$ between $\bar h_{\bar\lambda}$ and P_1 at the abscissae $s_n(\bar\lambda)$, $n \leqslant N$. Consider any fixed $n \leqslant N$. By (2.14), $s_n(\lambda)$ is bounded

$$m_n \leqslant s_n(\lambda) \leqslant M_n \quad \text{for all } 0 < \lambda \leqslant \bar\lambda \tag{2.16}$$

by some positive constants m_n, M_n. Hence by (2.15), also the differences

$$s_{n+1}(\lambda) - s_n(\lambda) \geqslant \overline{m}_n > 0 \text{ for all } 0 < \lambda \leqslant \overline{\lambda} \tag{2.17}$$

are bounded below by a positive $\overline{m}_n > 0$. Moreover, for each $n \geqslant N$, $s_n(\lambda)$ is a continuous function of λ, hence also $C_\lambda(s_n(\lambda))$, for $0 < \lambda \leqslant \overline{\lambda}$. Since $s_n(\lambda)$ is bounded above (2.16), (2.9) gives for every fixed n

$$\lim_{\lambda \downarrow 0} \; || C_\lambda(s_n(\lambda)) || \; = 1/x$$

that is, the points $P_{\lambda,n} := C_\lambda(s_n(\lambda)) \; \varepsilon \; d_\lambda$ tend to the boundary of the circle K as λ tends to 0. Therefore, by the continuity of $P_{\lambda,n}$ and because of

$$\underline{P_{\overline{\lambda}}}, \underline{n} \; \varepsilon \; [\underline{\overline{h}_{\overline{\lambda}}}, \underline{P}_1]$$

there is a λ_n, $0 < \lambda_n \leqslant \overline{\lambda}$ such that $P_{\lambda_n,n} = P_1$.

Because of (2.17),

$$|| \; C_{\lambda_n}(s_{n+1}(\lambda_n)) - \overline{h}_{\lambda_n} \; || \; = \; r_{\lambda_n}[s_{n+1}(\lambda_n)] \; < \; r_{\lambda_n}[s_n(\lambda_n)] \; = \; || \; P_1 - \overline{h}_{\lambda_n} || \quad ,$$

so that

$$P_1 \neq C_{\lambda_n}(s_{n+1}(\lambda_n)) \quad \varepsilon \quad [\overline{h}_{\lambda_n}, P_1] \quad ,$$

and therefore $\lambda_{n+1} < \lambda_n$.

This proves that in case (1) there are indeed countably many positively oriented different Cornu-spirals C_{λ_n}, $n \geqslant 1$, and abscissae s_n', namely

$$s_n' := s_n(\lambda_n),$$

having K as osculating circle at $s = 0$ and passing through P_1,

$$C_{\lambda_n}(s_n') = P_1 , \quad \text{for all } n \geqslant 1.$$

In case (2), $r > 1/x$, a similar reasoning applies: Here we consider the Cornu-spiral C_λ (s) for $0 \geqslant s > -x/\lambda$, that is for all $s \leqslant 0$ for which

$$x_\lambda(s) = x + \lambda s > 0$$

is still positive. We will show that:

(2.18) *To every integer* n $\geqslant 1$ *there exists a* $\overline{\lambda} > 0$ *and an integer* N $\geqslant 1$ *such that for every* $0 < \lambda \leqslant \overline{\lambda}$ *the Cornu-spiral* $C_\lambda(s)$, $0 \geqslant s \geqslant -x/\lambda$, *cuts* d_λ *at abscissae* $0 \geqslant s_{-1}(\lambda) > s_{-2}(\lambda) \ldots > s_{-N-n}(\lambda)$ *such that*

(a) $s_{-N}(\lambda) > -x/\overline{\lambda} > -x/\lambda$,

(b) $s_{-i}(\lambda) - s_{-i-1}(\lambda) \geqslant m_i > 0$ *for* $i = 1,2,\ldots,N+n-I$, $0 < \lambda \leqslant \overline{\lambda}$, (2.19)

(c) $r_{\overline{\lambda}}[s_{-N-1}(\overline{\lambda})] \geqslant r + \dfrac{1}{x}$.

(c) means that for $\lambda = \overline{\lambda}$, C_λ (s) has at least n cutting points, namely

$$C_{\overline{\lambda}}\ |s_{-N-i}(\overline{\lambda})|\ \ \varepsilon\ \{\overline{h}_{\overline{\lambda}} + \sigma(\mathrm{P}_1 - \overline{h}_{\overline{\lambda}})|\ \ \sigma \geqslant 1\}\,,\ \ i = 1, 2, \ldots, n$$

with $d_{\overline{\lambda}}$ which lie beyond P_1.

Once (2.18) is proved, then as in case (1), a simple limiting argument $\lambda \downarrow 0$ gives the existence of n values $\lambda_i, \overline{\lambda} \geqslant \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$ such that

$$C_{\lambda_i}(s_{-N-i}(\lambda_i)) \quad = \quad P_1\,,$$

since for $\lambda \downarrow 0$ by (2.9) each $C_{\lambda_i}(s_{-N-i}(\lambda))$, $i \geqslant 1$, tends to the circle K and so, by the continuity of $s_{-N\text{-}i}(\lambda)$ has to pass the point P_1 for a certain parameter value λ_i.

Since by (2.18) n is arbitrary, this gives the existence of countably many Cornu-spirals satisfying the interpolation requirement.

For the proof of (2.18) let γ be defined by $\gamma/x := r + 1/x$, so that $\gamma > 2$. Let $n \geqslant 1$ be an arbitrary positive integer. Choose any numbers α and β such that

$$\begin{gathered} 0 < \alpha < 1\ \ ,\ \ \sqrt{1-\alpha} \leqslant 1/(2\gamma) \\ \alpha\beta < 1\ \ ,\ \ \beta > 1\,. \end{gathered} \tag{2.20}$$

Choose a natural number N so large that

$$\begin{gathered} N + n + 1 \leqslant \beta N \\ \frac{\alpha^2}{N^2\pi^2(1-\alpha)^2} \leqslant \frac{1}{2} \end{gathered} \tag{2.21}$$

and set

$$\overline{\lambda}\ := \frac{\alpha x^2}{4N\pi}\quad .$$

Consider the solution $\overline{s}_{-\mathrm{m}}$, $\mathrm{N} \leqslant \mathrm{m} \leqslant \beta N$ of the quadratic equation

$$\phi_{\overline{\lambda}}\,(s) \equiv xs + \frac{\overline{\lambda}}{2}s^2 = -2m\pi$$

given by

$$\overline{s}_{-m} = \frac{-4m\pi}{x}\left(1 + \sqrt{1 - \frac{4m\pi\overline{\lambda}}{x^2}}\right)^{-1}\quad .$$

Since by (2.20)

$$0 < \alpha \leqslant \frac{4m\pi\overline{\lambda}}{x^2} = \alpha\,\frac{m}{N} \leqslant \alpha\beta < 1$$

every such $\overline{s}_{-m}$ is real. Moreover,

$$\overline{\lambda}\,\overline{s}_{-N} \ = -\alpha x/(1 + \quad 1-\alpha) = -x \cdot (1 - \quad 1-\alpha)$$

$$\overline{\lambda}\,\overline{s}_{-\beta N} \ = -x\,(1 - \quad 1-\beta\alpha)$$

so that by (2.20)

$$x_{\bar\lambda}(\bar{s}_{-N}) = x + \bar\lambda\bar{s}_{-N} = x\sqrt{1-\alpha} > x_{\bar\lambda}(\bar{s}_{-\beta N}) = x\sqrt{1-\beta\alpha} > 0 \tag{2.23}$$

Since by (2.21)

$$\frac{15\bar\lambda^2}{x_{\bar\lambda}(\bar{s}_{-N})^4} = \frac{15\alpha^2}{16N^2\pi^2(1-\alpha)^2} \leqslant \frac{1}{2}$$

we get from (2.11) and (2.20) the estimate

$$r_{\bar\lambda}(\bar{s}_{-N}) \geqslant \frac{0.5}{x_{\bar\lambda}(\bar{s}_{-N})} = \frac{0.5}{x\sqrt{1-\alpha}} \geqslant \frac{\gamma}{x} = r + \frac{1}{x}. \tag{2.24}$$

Since by (2.21)

$$\phi_{\bar\lambda}(\bar{s}_{-\beta N}) = -2\beta N\pi < -2(N+n+1)\pi \quad ,$$

$C_{\bar\lambda}(\bar{s})$ cuts $d_{\bar\lambda}$ at least $N+n$ times within the interval $[\bar{s}_{-\beta N}, 0]$ at abscissae

$$0 > s_{-1}(\bar\lambda) > s_{-2}(\bar\lambda) > \ldots > s_{-N-n}(\bar\lambda) \quad ,$$

satisfying the estimates

$$-2(i-1)\pi \geqslant \phi_{\bar\lambda}(s_{-i}(\bar\lambda)) \geqslant -2(i+1)\pi \quad \text{for } i = 1, 2, \ldots, N+n$$

so that

$$\bar{s}_{-i+1} \geqslant s_{-i}(\bar\lambda) \geqslant \bar{s}_{-i-1} \quad .$$

In particular, we have $0 \geqslant \bar{s}_{-N} \geqslant s_{-N-1}(\bar\lambda)$, so that because of (2.24) and the monotonicity of $r_{\bar\lambda}(s)$, we get (2.19)(c). (2.19)(a) follows from $s_{-N-n}(\bar\lambda) \geqslant \bar{s}_{-N-n-1}$, (2.21) implying $\bar{s}_{-N-n-1} \geqslant \bar{s}_{-\beta N}$ and (2.23). (2.19) (b) is proved as in case (1). All in all, we have shown the following:

(2.25) THEOREM: *For all oriented circles* K *and two points* $P_0 \,\varepsilon\, K$ *and* $P_1 \,\varepsilon\, K$ *there are countably many different Cornu-spirals connecting* P_0 *to* P_1 *(in this order) and all have* K *as osculating circle at* P_0.

3. Interpolation by Clothoidal Splines

A clothoidal spline is a C^2-curve in R^2 whose curvature $x(s)$ is a continuous piecewise linear function of arclength s. More precisely, such a curve $Z(s)$ is given by a finite collection of parameters

$$0 = s_0 < s_1 < \ldots < s_{n+1}$$

$$(Z_i, \phi_i, x_i, \lambda_i)\,,\ \ Z_i \,\varepsilon\, R^2\,,\ \ i = 0,\ 1, \ldots, n$$

such that for each $i = 0, 1, \ldots, n$, $Z^i(s) := Z(s)|[s_i, s_{i+1}]$ is a Cornu-spiral with curvature $x^i(s)$ and phase $\phi^i(s)$ given by

$$
\begin{aligned}
\varkappa^i(s) &:= \varkappa_i + \lambda_i(s-s_i) \\
\phi^i(s) &:= \phi_i + \varkappa_i(s-s_i) + \frac{\lambda_i}{2}(s-s_i)^2 \\
Z^i(s) &:= Z_i + \int_{s_i}^{s} \begin{bmatrix} \cos \\ \sin \end{bmatrix} (\phi^i(t))\, dt
\end{aligned}
\tag{3.1}
$$

so that $Z(s)$ is a c^2-curve; that is, the $Z^i(.)$, $\phi^i(.)$, $\varkappa^i(.)$ satisfy the following continuity conditions for all $i = 0, 1, \ldots, n-1$:

$$
\begin{aligned}
Z^i(s_{i+1}) - Z_i+1 &\equiv Z_i + \int_0^{\tau_i} \begin{bmatrix} \cos \\ \sin \end{bmatrix} (\phi^i(s_i+\tau))\, d\tau - Z_{i+1} = 0 \\
\phi^i(s_{i+1}) - \phi_{i+1} &\equiv \phi_i + \varkappa_i + \varkappa_i \tau_i + \frac{\lambda_i}{2}\tau_i^2 - \phi_{i+1} = 0 \\
\varkappa^i(s_{i+1}) - \varkappa_{i+1} &\equiv \varkappa_i + \lambda_i \tau_i - \varkappa_{i+1} = 0
\end{aligned}
\tag{3.2}
$$

with $\tau_i := s_{i+1} - s_i$. Of course, the parameters s_i are determined by the τ_i, $s_{i+1} = \tau_0 + \tau_1 + \ldots + \tau_i$ so that instead of the s_i, we may take the $\tau_i > 0$ as parameters. Note that we do not require $\lambda_i \neq 0$, so that $Z(s)$ may contain linear or circular segments.

In this section we study the interpolation problem of finding a clothoidal spline passing through a finite number of given points. In this form, the problem is not very meaningful, since by Theorem (2.25) it has arbitrarily many different solutions. More interesting is the problem of finding an interpolating clothoidal spline with minimal $\int \varkappa(s)^2 ds$, in analogy to cubic spline interpolation.

(3.3) PROBLEM: *For a given family* $\{Z_i\}_{i=0,1,\ldots,n+1}$ *of different points* $Z_i \in R^2$ *find parameters* $P_i^T = (\phi_i, \varkappa_i, \lambda_i, \tau_i)$, $i = 0, 1, \ldots, n$ *with* $\tau_i > 0$ *such that these parameters together with the* Z_i *determine a clothoidal spline* $Z(s)$ *by (3.1) satisfying (3.2) and* $Z(s_{n+1}) = Z_{n+1}$ *so that*

$$
\int_0^{s_{n+1}} \varkappa(s)^2 ds = \sum_{i=0}^{n} \int_{s^i}^{s_{i+1}} \varkappa^i(s)^2 ds
$$

is minimal.

With the notation

$$
a_i^T := (\phi_i, \varkappa_i)\, , \, b_i^T := (\lambda_i, \tau_i)
$$

$$
P^T := (P_0^T, P_1^T, \ldots, P_n^T)\, , \, P_i^T := (\varkappa_i, \varkappa_i, \lambda_i, \tau_i)\, , \, i = 0,1, \ldots, n \tag{3.4}
$$

the objective function to be minimized is the function

$$
F(P) := \sum_{i=0}^{n} \int_0^{\tau_i} (\varkappa_i + \lambda_i \tau)^2 d\tau
$$

which is separable in variables P_i.

The transpose $F'(P)$ of its gradient and its Hessian $F''(P)$ are

$$F'(P) = (u_0, v_0, u_1, v_1, \ldots, u_n, v_n)$$

$$F''(P) = \begin{bmatrix} F_0 & & & & 0 \\ & F_1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & F_n \end{bmatrix} \tag{3.5}$$

with the R^2 row vectors

$$\begin{aligned} u_i &:= [0, 2\kappa_i\tau_i + \lambda_i\tau_i^2] \\ v_i &:= [\tau_i^2(\kappa_i + \frac{2}{3}\lambda_i\tau_i)(\mathrm{K}_i + \lambda_i\tau_i)^2] \end{aligned} \tag{3.6}$$

and the 4 x 4 square matrices

$$F_i := 2 \left[\begin{array}{cc|cc} 0\,, & 0 & 0\,, & 0 \\ 0\,, & \tau_i & \frac{1}{2}\tau_i^2\,, & \kappa_i + \lambda_i\tau_i \\ \hline 0\,, & \frac{1}{2}\tau_i^2 & \frac{1}{3}\tau_i^3\,, & (\kappa_i + \lambda_i\tau_i)\tau_i \\ 0\,, & \kappa_i + \lambda_i\tau_i & (\kappa_i + \lambda_i\tau_i)\tau_i\,, & (\kappa_i + \lambda_i\tau_i)\lambda_i \end{array}\right] \tag{3.7}$$

Also, the conditions (3.2) to be satisfied by P are highly structured. They have a staircase-like form

$$G(P) \equiv G(a_0, b_0, \ldots, a_n, b_n) \equiv$$

$$\begin{bmatrix} \mathrm{J}(a_0, b_0) + Z_0 - Z_1\,, & & & & & \\ \mathrm{K}(a_0, b_0) & , -a_1 & & & & \\ & , \mathrm{J}(a_1, b_1) + Z_1 - Z_2, & & & & \\ & , \mathrm{K}(a_1\, b_1) & , -a_2 & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & , \mathrm{J}(a_{n-1}, b_{n-1}) + Z_{n-1} - Z_n, & \\ & & & & , \mathrm{K}(a_{n-1}, b_{n-1}) & , -a_n \\ & & & & & , \mathrm{J}(a_n, b_n) + Z_n - Z_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

where

$$J(a, b) := \int_0^\tau \begin{bmatrix} \cos \\ \sin \end{bmatrix} (\phi + x\tau + \frac{\lambda}{2}\tau^2) d\tau , \quad a := \begin{bmatrix} \phi \\ x \end{bmatrix}, b := \begin{bmatrix} \lambda \\ \tau \end{bmatrix}$$

$$K(a, b) := \begin{bmatrix} \phi + x\tau + \frac{\lambda}{2}\tau^2 \\ x + \lambda\tau \end{bmatrix} \tag{3.9}$$

Note that the integral in $J(a,b)$ is easily computed in terms of Fresnel integrals (see 1.2) for $\lambda_i \neq 0$ and elementary integration rules for $\lambda_i = 0$. The Jacobian G' of G has a similar structure

$$G'(P). \equiv G'(a_0, b_0, \ldots, a_n, b_n) \equiv$$

$$\equiv \begin{bmatrix} A_0, B_0, & & & & & 0 \\ C_0, D_0, -I & & & & & \\ & A_1, B_1 & & & & \\ & C_1, D_1, -I & & & & \\ & & \ddots & & & \\ & & & A_{n-1}, B_{n-1} & & \\ & & & C_{n-1}, D_{n-1}, -I & & \\ 0 & & & & A_n, & B_n \end{bmatrix} \tag{3.10}$$

with partial derivative 2 x 2 matrices

$$A_i := D_{(\phi,x)} J(\phi,x,\lambda,\tau)|_{P_i}$$

$$B_i := D_{(\lambda,\tau)} J(\phi,x,\lambda,\tau)|_{P_i} \tag{3.11}$$

$$C_i := \begin{bmatrix} 1 , & \tau_i \\ 0 , & 1 \end{bmatrix} , \quad D_i := \begin{bmatrix} \tau_i^2/2 , & x_i + \lambda_i \tau_i \\ \tau_i , & \lambda_i \end{bmatrix}$$

In terms of the notation just introduced, (3.3) is equivalent to the minimization problem,

$$\textit{Minimize } F(\mathrm{P}) \textit{ subject to } G(\mathrm{P}) = 0 \tag{3.12}$$

Let

$$L(P, \Lambda) := F(P) + \Lambda^T G(P)$$

be the Lagrangean of (3.12) and suppose that (3.12) satisfies the usual first order necessary and second order sufficient conditions at the optimal point $\bar{P}$ (which we assume to exist):

1. *The Jacobian* $G'(\bar{P})$ *of* G *at* $\bar{P}$ *has full row rank and there exists a* Λ *such that* $(\bar{P}, \bar{\Lambda})$ *is a stationary point of* L:

$$\phi(\bar{P},\bar{\Lambda}) = 0\,, \quad \text{with } \phi(P,\Lambda) := \begin{bmatrix} \Delta_p L(P,\Lambda) \\ G(P) \end{bmatrix} = L'(P,\Lambda)^T \tag{3.13}$$

2. *For the Hessian* $L_{pp}(\bar{P},\bar{\Lambda})$ *of* L *with respect to* P

$$P^T L_{pp}(\bar{P},\bar{\Lambda}) P > 0$$

holds for all $P \neq 0$ *satisfying* $G'(\bar{P})P = 0$.

Then $\bar{P}$ and $\bar{\Lambda}$ can be found as the solution of the nonlinear equations (3.13). The Jacobian ϕ' of ϕ is a highly structured matrix of the form

$$\phi'(P,\Lambda) = \begin{bmatrix} L_{pp}(P,\Lambda) & , & G'(P)^T \\ G'(P) & , & 0 \end{bmatrix} \tag{3.14}$$

where G' is given by (3.10). It is seen from (3.5), (3.10) that L_{pp} has the same block-structure as F'' (3.5). In solving (3.13), Newton's method can be applied to generate iterates $(P^{(k)}, \Lambda^{(k)})$, $k = 0, 1, \ldots$ by solving at each iterate $(P^{(k)}, \Lambda^{(k)})$ the linear equations

$$\phi'(P^{(k)},\Lambda^{(k)}) \begin{bmatrix} \delta P^{(k)} \\ \delta\Lambda^{(k)} \end{bmatrix} = -\phi(P^{(k)},\Lambda^{(k)}) \tag{3.15}$$

for the Newton direction $\begin{bmatrix} \delta P^{(k)} \\ \delta\Lambda^{(k)} \end{bmatrix}$, with ϕ' given by (3.14).

Since computing the Hessian $L_{pp}(P^{(k)},\Lambda^{(k)})$ may be too costly, we may replace L_{pp} within ϕ' by a sufficiently close approximation $H^{(k)}$ as it is done in the minimization algorithms of Han [4, 5] and Powell [11, 12]. One may choose as $H^{(k)}$, e.g. a matrix of the same block structure as L_{pp}, namely (compare 3.5)

$$H^{(k)} = \begin{bmatrix} H_0^{(k)} & & & & 0 \\ & H_1^{(k)} & & & \\ & & \ddots & & \\ & & & & \\ 0 & & & & H_n^{(k)} \end{bmatrix} \tag{3.16}$$

with 4 x 4 blocks $H_i^{(k)}$, $i = 0, 1, \ldots, n$. One then solves (3.15) with L_{pp} *replaced by* $H^{(k)}$, namely

$$\begin{bmatrix} H^{(k)} & , & G'(P^{(k)})^T \\ G'(P^{(k)}) & , & 0 \end{bmatrix} \begin{bmatrix} \delta P^{(k)} \\ \delta\Lambda^{(k)} \end{bmatrix} = -\phi(P^{(k)},\Lambda^{(k)}) \tag{3.17}$$

and computes a new iterate of the form

$$\begin{bmatrix} P^{(k+1)} \\ \Lambda^{(k+1)} \end{bmatrix} = \begin{bmatrix} P^{(k)} \\ \Lambda^{(k)} \end{bmatrix} + \sigma_k \cdot \begin{bmatrix} \delta P^{(k)} \\ \delta\Lambda^{(k)} \end{bmatrix}$$

by choosing a step size σ_k, $0 < \sigma_k \leqslant 1$, for example as in Han [5], by minimizing a certain penalty function along the ray

$$\left\{ \begin{bmatrix} P^{(k)} \\ \Lambda^{(k)} \end{bmatrix} + \sigma \begin{bmatrix} \delta P^{(k)} \\ \delta\Lambda^{(k)} \end{bmatrix} \;\middle|\; \sigma \geqslant 0 \right\} .$$

After having computed the new iterate $(P^{(k+1)}, \Lambda^{(k+1)})$, one may use a rank-2 update formula, say the PSB-update formula, on each 4 x 4 block $H_i^{(k)}$ in order to generate another matrix $H_i^{(k+1)}$ for each $i = 0, 1, \ldots, n$, and thereby $H^{(k+1)}$, having the same structure (3.16) as $H^{(k)}$ and satisfying the usual Quasi-Newton equation:

$$H_i^{(k+1)}(P_i^{(k+1)} - P_i^{(k)}) = \nabla_{P_i} L(P^{(k+1)}, \Lambda^{(k+1)}) - \nabla_{P_i} L(P^{(k)}, \Lambda^{(+1)}) \tag{3.18}$$

When solving (3.17), the structure of $H^{(k)}$ (3.16) and $G'(P_k)$ (3.10) can be exploited to reduce the number of operations drastically. For ease of notation, let us drop the superscripts and arguments in (3.17) and write briefly

$$\begin{bmatrix} c \\ d \end{bmatrix}$$

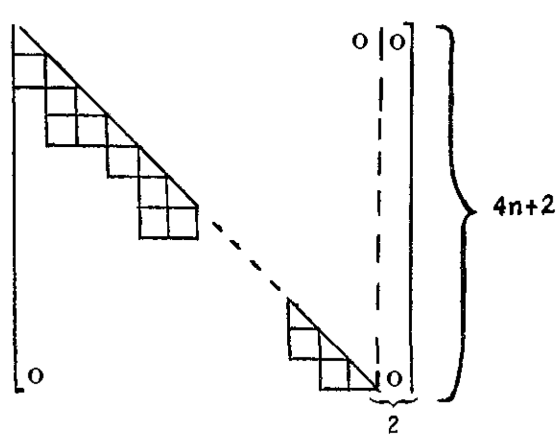
for the right hand side $-\phi(P^{(k)}, \Lambda^{(k)})$ of (3.17). The problem then is to solve an equation of the form

$$\begin{bmatrix} H & , G'^T \\ G' & , 0 \end{bmatrix} \begin{bmatrix} \delta P \\ \delta\Lambda \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \tag{3.19}$$

where H and G' have the block structure (3.16) and (3.10), respectively.

We first reduce G′ by a series of Givens reflexions Ω_j, $\Omega_j^H = \Omega_j$, $\Omega_j^2 = I$, to a lower triangular matrix of the form [compare its structure with (3.10)]:

$$G'{-}_1.\Omega_2 \ldots .\Omega_N = (L, 0) \equiv$$



(3.20)

where all blocks indicated have size 2 x 2 and L is a (4n + 2) x (4n + 2)-lower triangular band matrix. Again, because of the band-structure of (3.10), the number $N = 0(n)$ of Givens reflexions needed is linear in n, so that the unitary matrix

$$\Omega := \Omega_1 \cdot \Omega_2 \cdot \ldots . \Omega_N \tag{3.21}$$

need not be computed explicitly, but can be stored in product form. Partition the matrix

$$\Omega = (\overline{\Omega}, \overline{\overline{\Omega}})$$

where

$$\overline{\overline{\Omega}} = \Omega \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \Omega_1 . \Omega_2 . \ldots . \Omega_N \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are the last two columns of Ω, which are computed using the product form of (3.21), $\overline{\Omega}$ is not needed explicitly. Introduce new variables

$$t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

via $\delta P = \Omega t = \overline{\Omega} t_1 + \overline{\overline{\Omega}} t_2$.

Then because of

$$G' \overline{\Omega} = L , G' \overline{\overline{\Omega}} = 0$$

the second set of equations (3.19)

$$G' \delta P = L t_1 = d \rightarrow t_1$$

can be solved for t_1 in $0(n)$ steps using the structure of L (3.20), and the vector

$$P^1 := \overline{\Omega} t_1 = \Omega_1 \Omega_2 \ldots \Omega_N \quad \begin{bmatrix} t_1 \\ 0 \\ 0 \end{bmatrix}$$

is computed using (3.21).

Now we turn to the first set of equations (3.19)

$$H \delta P + G'^{\mathrm{T}} \delta \Lambda = c \tag{3.22}$$

Multiplying these equations by $\overline{\overline{\Omega}}^T$ and introducing t_1 and t_2 instead of δP, we get because of $\overline{\overline{\Omega}}^T G'^{\,T} = 0$

$$\overline{\overline{\Omega}}^T H \overline{\Omega} t_1 + \overline{\overline{\Omega}}^T H \overline{\overline{\Omega}} t_2 = \overline{\overline{\Omega}}^T c$$

or

$$(\overline{\overline{\Omega}}^T H \overline{\overline{\Omega}}) t_2 = \overline{\overline{\Omega}}^T c - \overline{\overline{\Omega}}^T H P^1 \rightarrow t_2 \tag{3.23}$$

Again, the 2 x 2 matrix $\bar{\bar{\Omega}}^T H \bar{\bar{\Omega}}$ and the vectors $\bar{\bar{\Omega}}^T H P^1$ can be computed with $0(n)$ operations using the block structure of H (3.16). t_2 is obtained by solving the two linear equations (3.23) and δP is calculated by

$$P^2 := \bar{\bar{\Omega}} t_2 \, , \delta P := P^1 + P^2 \quad .$$

Finally, we multiply (3.22) by $\bar{\Omega}^T$ in order to get $\delta\Lambda$. Observing (3.20) we obtain a triangular system of linear equations

$$L^T t \delta\Lambda = \bar{\Omega}^T c - \bar{\Omega}^T H \delta P$$

the right hand side of which can be easily computed with $0(n)$ operations using the structure of H and the product form of $\bar{\Omega}^T$:

$$\bar{\Omega}^T = \begin{bmatrix} 1 & & 0\,0\,0 \\ & \ddots & \\ 0 & & 1\,0\,0 \end{bmatrix} \quad \Omega N \Omega_{N-1} \dots \Omega_1$$

All in all, we can compute the solution of (3.19) with 0(n) arithmetic operations, so that the Han-Powell method is quite effective in our case. The method has been realized and successively tested by Huckle [6]. With respect to a convergence analysis of the above method (the method converges locally superlinearly under some mild assumptions) we refer to the literature Han [4,5], Powell [12], Tapia [16].

4. Smoothing by Clothoidal Splines

We consider the following generalization of (3.3) (compare Reinsch [13]):

PROBLEM: *For a given family* $\{\overline{Z_i}\}_{i=0,1,\dots,n+1}$ *of different points*

$$\overline{Z_i} = \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix} \varepsilon \, R^2$$

and numbers $S \geqslant 0, \Delta x_i > 0, \Delta y_i > 0, i = 0,1,\dots,n+1$, *find parameters*

$$\{\{(\phi_i, \varkappa_i, \lambda_i, \tau_i)\}_{i=0,1,\dots,n}, \{Z_i\}_{i=0,1,\dots,n+1}, z\}, Z_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \varepsilon \, R^2 \quad , \tag{4.1}$$

which determine a clothoidal spline Z(s) *via (3.1) satisfying the conditions*

$$\text{a)} \quad (3.2) \text{ and } Z(s_{n+1}) = Z_{n+1} \tag{4.2}$$

$$\text{b)} \quad \sum_{i=0}^{n+1} \left(\left(\frac{x_i - \bar{x}_i}{\Delta x_i} \right)^2 + \left(\frac{y_i - \bar{y}_i}{\Delta y_i} \right)^2 \right) + z^2 = S$$

(z is a slack variable) such that $\int_0^{s_{n+1}} \lambda(s)^2 ds$ *is minimal.*

Again with the notation [compare (3.4)]

$$\begin{aligned} a_i^T &= (\phi_i, x_i),\ b_i^T = (\lambda_i, \tau_i) \\ P_i^T &= (\phi_i, x_i, \lambda_i, \tau_i) \end{aligned} \qquad i = 0,1,\ldots,n$$

$$\begin{aligned} P_{n+1}^T &= [Z_0^t, Z_1^T, \ldots, Z_{n+1}^T, z] \\ &= (x_0, y_0, x_1, y_1, \ldots, x_{n+1}, y_{n+1}, z) \,\varepsilon\, R^{2n+5} \\ P^T &= [P_0^T, P_1^T, \ldots, P_n^T, P_{n+1}^T] \,\varepsilon\, R^{6n+9} \end{aligned}$$

the objective function $F(P)$ to be minimized is separable in the P_i

$$F(P) := \sum_{i=0}^{n} \int_0^{\tau_i} (x_i + \lambda_i t)^2 dt \tag{4.3}$$

and has a Hessian of the form [compare (3.5)]

$$F''(P) = \begin{bmatrix} F_0 & & & & & 0 \\ & F_1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & .F_n & \\ 0 & & & & & F_{n+1} \end{bmatrix} \tag{4.4}$$

with the same 4 x 4 matrices $F_0, \ldots, F_n$ as in (3.7) and a $(2n+5)$ by $(2n+5)$ matrix $F_{n+1} := 0$.

The constraints (4.2) now have the structure [see (3.8)]:

$$G(P) \equiv \begin{bmatrix} J(a_0,b_0) + Z_0 - Z_1 \\ K(a_0,b_0) - a_1 \\ J(a_1,b_1) + Z_1 - Z_2 \\ K(a_1,b_1) - a_2 \\ \cdot \\ \cdot \\ \cdot \\ J(a_{n-1},b_{n-1}) + Z_{n-1} - Z_n \\ K(a_{n-1},b_{n-1}) - a_n \\ J(a_n,b_n) + Z_n - Z_{n+1} \\ \varrho(Z_0,Z_1,\ldots,Z_{n+1},z) \end{bmatrix} = 0 \tag{4.5}$$

where L and K are again given by (3.9) and the scalar function ϱ is defined by [compare (4.2) b)]

$$\varrho(Z_0, \ldots, Z_{n+1}, z) := \left(\frac{1}{2} \sum_{i=0}^{n+1} \left(\frac{x_i - \bar{x}_i}{\Delta x_i}\right)^2 + \left(\frac{y_i - \bar{y}_i}{\Delta y_i}\right)^2 \right) + z^2 - S \ .$$

With these new definitions of F and G, problem (4.1) has the same structure as (3.12), namely

$$\textit{minimize } F(P) \textit{ subject to } G(P) = 0 \tag{4.6}$$

Consider again the Lagrangean of (4.6)

$$L(P,\Lambda) := F(P) + \Lambda^T G(P)$$

We again assume that (4.6) has an optimal solution $\bar{P}$ and that at $\bar{P}$ the optimality conditions (3.13) are satisfied.

By (3.13), (3.14) the optimal solution $(\bar{P}, \bar{\Lambda})$ solves

$$\phi(P,\Lambda) := L'(P,\Lambda) \equiv \begin{bmatrix} \Delta_P L(P,\Lambda) \\ G(P) \end{bmatrix} = 0 \tag{4.7}$$

whose Jacobian is again

$$\phi'(P,\Lambda) = \begin{bmatrix} L_{PP}(P,\Lambda) & , & G'^T \\ G' & , & 0 \end{bmatrix} \tag{4.8}$$

but its structure is slightly more complicated than in section 3 because of our new definitions of $F(P)$ (4.3) and $G(P)$ (4.5). It is easily verified that in the present case $\phi'(P,\Lambda)$ has the following form (illustrated for $n=2$)

$$G'(P) = \begin{array}{c} \left[\begin{array}{ccc|cccc|c} A_0,B_0 & & 0 & I, & -I, & 0, & 0 & 0 \\ C_0,D_0,-I & & & 0, & 0, & 0, & 0 & 0 \\ & A_1,B_1 & & 0, & I, & -I, & 0 & 0 \\ & C_1,D_1, -I & & 0, & 0, & 0, & 0 & 0 \\ 0 & & A_2,B_2 & 0, & 0, & I, & -I & 0 \\ \hline 0 & & 0 & & & r & & z \end{array}\right] \begin{array}{l} \}\ 2 \\ \\ \\ \\ \}\ 2 \\ \}\ 1 \end{array} \\ \underbrace{\ \ }_{2}\ \underbrace{\ \ }_{2} \qquad\qquad \underbrace{\ \ }_{2} \qquad\qquad \underbrace{\ \ }_{1} \end{array} \tag{4.9}$$

where the 2 by 2 matrices A_i, B_i, C_i, D_i are again given by (3.11) and the vector r is

$$\left(r := \frac{x_0 - \bar{x}_0}{(\Delta x_0)^2}, \frac{y_0 - \bar{y}_0}{(\Delta y_0)^2}, \ldots, \frac{x_{n+1} - \bar{x}_{n+1}}{(\Delta x_{n+1})^2}, \frac{y_{n+1} - \bar{y}_{n+1}}{(\Delta y_{n+1})^2} \right)$$

Likewise $L_{PP}(P,\Lambda)$ has the structure [compare (4.3), (4.4)]

$$L_{PP}(P,\Lambda) = \begin{bmatrix} L_0 & & & & & & 0 \\ & L_1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & L_n & \\ 0 & & & & & & L_{n+1} \end{bmatrix} \tag{4.10}$$

where the L_i, $i \leqslant n$, are symmetric 4 by 4 matrices and L_{n+1} is the $(2n+5)$ by $(2n+5)$ diagonal matrix.

$$L_{n+1} := \Lambda_z \cdot \mathrm{diag}(\Delta x_0, \Delta y_0, \ldots, \Delta x_{n+1}, 1)^{-2} \quad , \tag{4.11}$$

where Λ_z is the last component of Λ.

As in the previous section, one has to solve (4.8) by Newton's method (compare (3.15) – (3.17) where at each iteration point $[P^{(k)}, \Lambda^{(k)}]$ the Hessian L_{PP} is approximated by a positive definite matrix $H^{(k)}$ having the same structure as L_{PP} (4.10),

$$H^{(k)} = \begin{bmatrix} H_0^{(k)} & & & & & & 0 \\ & H_1^{(k)} & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & H_n^{(k)} & \\ 0 & & & & & & H_{n+1}^{(k)} \end{bmatrix} \tag{4.12}$$

with certain 4 by 4 matrices $H_i^{(k)}$ for $i \leqslant n$ and the diagonal matrix (see 4.11)

$$H_{n+1}^{(k)} = \Lambda_z^{(k)} \cdot \mathrm{diag}(\Delta x_0, \Delta y_0, \ldots, \Delta x_{n+1}, \Delta y_{n+1}, 1)^{-2} \quad . \tag{4.13}$$

After having computed $P^{(k+1)}$, $\Lambda^{(k+1)}$ (see previous section) $H^{(k+1)}$ is obtained from $H^{(k)}$ by updating each $H_i^{(k)}$, $i \leqslant n$, individually by some update method (e.g., the PSB-method) which guarantees the same quasi-Newton relation (3.18) as in section 4; $H_{(n+1)}^{(k+1)}$ is computed by (4.13).

Of course, for large numbers n the efficiency of the algorithm outlined crucially depends on the number of operations needed to perform one Newton step $[P^{(k)}, \Lambda^{(k)}] \rightarrow [P^{(k+1)}, \Lambda^{(k+1)}]$, that is to solve a linear system of equations [see (3.17), (3.19)] of the form

$$\begin{bmatrix} H & , G'^T \\ G' & , 0 \end{bmatrix} \begin{bmatrix} \delta P \\ \delta \Lambda \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \tag{4.14}$$

for δP, $\delta \Lambda$, where H and G' are given matrices with the structure (4.12) and (4.9), respectively. An algorithm of the type considered at the end of the previous section leads to difficulties inasmuch as it would take $0(n_3)$ operations to solve (4.14) because it requires the computation and storage of a large dense matrix of the order $0(n)$.

Another numerically stable way to solve the linear system (4.14), which exploits the symmetry of the matrix

$$\begin{bmatrix} H & , & G'^T \\ G' & , & 0 \end{bmatrix} \tag{4.15}$$

would be to use the Bunch-Parlett decomposition of (4.15) (see Bunch, Parlett [4]). However, this method requires a pivot selection in each basic elimination step, which, though preserving the symmetry, will in general destroy the specific block structure of the matrix in (4.15). This method, therefore, also requires $0(n^3)$ operations to solve (4.14). A cheaper method for solving (4.14) might be a variant of the conjugate gradient algorithm for solving linear equations

$$\mathrm{Ax} = \mathrm{b}$$

with a symmetric nonsingular, but perhaps indefinite matrix A, which is described in Paige and Saunders [9]. This method can take the block structure (4.12), (4.9) of H and G' into account and therefore requires only $0(n^2)$ operations and $0(n)$ storage to solve (4.14).

It is interesting to note in this context that the system (4.14) can be solved with only $0(n)$ operations, if the block-diagonal matrix $L_{PP}(\overline{P}, \overline{\Lambda})$ (4.10) would be positive definite at the solution $(\overline{P}, \overline{\Lambda})$ of (4.7). In this case, it can be shown that the matrices $H^{(k)}$ (4.12) generated by the usual update techniques (PSP-, DFP-, or BFGS-methods) will be positive definite, at least locally, if the starting values $[P^{(0)}, \Lambda^{(0)}$, and $H^{(0)}]$ are sufficiently close to $(\overline{P}, \overline{\Lambda})$ and $L_{PP}(\overline{P}, \overline{\Lambda})$, respectively.

If H is positive definite, then a numerically stable method of solving (4.14) requiring only $0(n)$ operations runs as follows:

In a first step compute the Cholesky decomposition of

$$H = R^T R$$

which requires $0(n)$ operations and gives an upper triangular R of the form [compare (4.12)]

$$R = \begin{bmatrix} R_0 & & & & & & 0 \\ & R_1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & R_n & \\ 0 & & & & & & R_{n+1} \end{bmatrix} \tag{4.16}$$

with 4 x 4 upper triangular R_i for $i \leqslant n$ and diagonal R_{n+1}. Premultiplying (4.14) by

$$\begin{bmatrix} R^{-T} & , & 0 \\ -G'R^{-1}R^{-T} & , & I \end{bmatrix}$$

gives the equivalent system

$$\begin{bmatrix} R & , (G'R^{-1})^T \\ 0 & ,-(G'R^{-1})(G'R^{-1})^T \end{bmatrix} \begin{bmatrix} \delta P \\ \delta\Lambda \end{bmatrix} = \begin{bmatrix} R^{-\mathrm{T}}c \\ d-G'R^{-1}R^{-\mathrm{T}}c \end{bmatrix} \tag{4.17}$$

So the next step is to compute

$$c' := R^{-\mathrm{T}}c \, , A := G'R^{-1}$$

which again requires only $0(n)$ operations because of the simple structure of R (4.16) and G' (4.9). Note, moreover, that the product matrix $A = G'R^{-1}$ has a form very similar to (4.9), namely (illustration for $n=2$):

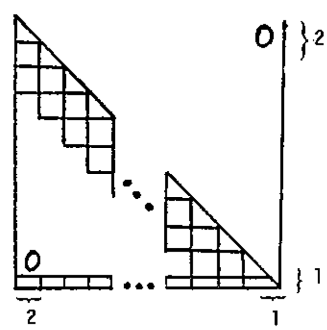
$$
A = \left[\begin{array}{cccccccccccc|cccccccc|c}
x & x & x & x & & & & & & & & 0 & x & o & x & o & & & & 0 & 0 \\
x & x & x & x & & & & & & & & & o & x & o & x & & & & & \\
x & x & x & x & x & o & & & & & & & & & & & & & & & \\
x & x & x & x & x & x & & & & & & & & & & & & & & & \\
 & & & & x & x & x & x & & & & & & & x & o & x & o & & & \\
 & & & & x & x & x & x & & & & & & & o & x & o & x & & & \\
 & & & & x & x & x & x & x & o & & & & & & & & & & & \\
 & & & & x & x & x & x & x & x & & & & & & & & & & & \\
 & & & & & & & & x & x & x & x & & & & & x & o & x & o & o \\
0 & & & & & & & & x & x & x & x & 0 & & & & o & x & o & x & o \\
\hline
\multicolumn{12}{c|}{0} & x & x & x & x & x & x & x & x & x
\end{array}\right] \tag{4.18}
$$

We next reduce A to "lower triangular" form by multiplying A from the right by suitable Givens reflexions $\Omega_1, \Omega_2, \ldots, \Omega_N = 0(n)$ matrices Ω_i and only $0(n)$ operations are needed and the structure of (4.16) is essentially preserved and fill in will occur at most $0(n)$ places. Each will annihilate a particular above diagonal element of A; the resulting matrix is of the form

$$
A\Omega_1\Omega_2 \ldots \Omega_N = (L, O) \tag{4.19}
$$

where "O" denotes a $(4n+3)$ by $(2n+6)$ zero matrix and L is a $(4n+3)$ by $(4n+3)$ lower triangular matrix with the structure

L =



Note that the dense $(6n+9)$ by $(6n+9)$ product matrix $\Omega = \Omega_1\Omega_2, \ldots, \Omega_N$ need not be computed. Its storage in product form requires only $0(n)$ places. Concurrently with the elimination process for finding L, we can compute the vector

$$
c'' := \Omega_N \ldots \Omega_2\Omega_1 c'
$$

Now it is easy to solve (4.17) for δP and $\delta\Lambda$. The second equation (4.17) gives by (4.19) at once

$$
AA^T\delta\Lambda = LL^T\delta\Lambda = -d + Ac' = -d + (L,0)c'' \tag{4.20}
$$

so that

$$
\delta\Lambda = -L^{-T}L^{-1}d + (L^{-T},0)c'' \tag{4.21}
$$

i.e. $\delta\Lambda$ can be found by solving three linear equations with triangular matrices. The first equation(4.17) now gives by (4.21)

$$
\begin{aligned}
R\delta P &= R^{-T}c - A^T\delta\Lambda \\
&= c' - \Omega \begin{bmatrix} L^T \\ 0 \end{bmatrix} \delta\Lambda \\
&= c' - \Omega \begin{bmatrix} -L^{-1}d + (I,0)c'' \\ 0 \end{bmatrix}
\end{aligned}
\tag{4.22}
$$

Unfortunately enough, the computation of

$$
c''' := \Omega \begin{bmatrix} L^{-1}d - (I,0)c'' \\ 0 \end{bmatrix} = \Omega_1\Omega_2, \ldots, \Omega_N \begin{bmatrix} L^{-1}d - (I,0)c'' \\ 0 \end{bmatrix}
$$

requires the storage of all Ω_i (this was not needed in computing c''). Note that $L^{-1}d$ has already been obtained during the calculation of $\delta\Lambda$ (4.21). Finally, by (4.22), δP is obtained by solving one more triangular system of linear equations

$$
R\delta P = c' - c''' \rightarrow \delta P \quad , \tag{4.23}
$$

again requiring only $0(n)$ operations.

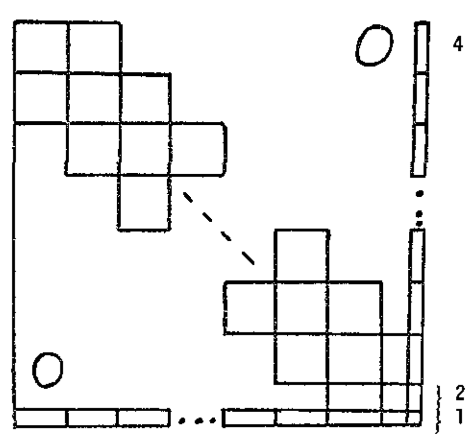
At the expense of numerical stability one may get around the elimination process to find L and the storage of the orthogonal matrices Ω_i in the following way:

Having computed the Cholesky decomposition of $H = R^TR$, the matrix $A = G'R^{-1}$, the product AA^T and its Cholesky decomposition $AA^T = LL^T$, computing $\delta\Lambda$ and δP from (4.20), (4.22) is straightforward:

$$
\begin{aligned}
LL^T\delta\Lambda &= -d + Ac' \rightarrow \delta\Lambda \rightarrow A^T\delta\Lambda \\
R\delta P &= c' - A^T\delta\Lambda \rightarrow \delta P
\end{aligned}
\tag{4.24}
$$

Note in this context that the product AA^T has a simple sparse structure needing only $0(n)$ places for storage:

$AA^T =$



Both algorithms require only $0(n)$ operations for solving (4.14) in each Newton step, but the former will be numerically more stable, as it avoids the calculation of AA^T and cancels products such as LL^{-1}, RR^{-1}, which arise inherently during the solution of (4.24), as often as possible.

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5. References

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