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Minimum-Loop Realization of Degree Sequences*

A. J. Goldman† and R. H. Byrd†

The Johns Hopkins University, Baltimore, MD 21218

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Given a finite sequence D of nonnegative integers, let $M(D)$ denote its maximum element and $S(D)$ its sum. It is known that D is realizable as a degree sequence by some graph if and only if $S(D)$ is even, and by a loopless graph if and only if the even integer $S(D) - 2M(D) \geq 0$. Here it is shown that if the even integer $2M(D) - S(D)$ is positive, then one-half this integer is the minimum number of loops in graphs realizing D, and that the minimum-loop realization is unique. These results are extended to a more general loop-cost minimization problem in which loops incident at different vertices can have different costs. The possible numbers of loops, in graphs realizing D, are also determined.

Key words: graph, loopless graph, degree sequence, incidence sequence, partition.

1. Introduction

This note deals with finite undirected graphs. Our usage of "graph" permits both *loops* (edges from a vertex to itself) and *multiple links* (bundles of two or more edges with the same pair of distinct endpoints). The *degree* of vertex v in graph G, denoted $d_G(v)$, is the number of incidences upon v of edges of G; here a loop is considered to be twice-incident upon its single endpoint. Any enumeration of the set $\{v_i\}_1^n$ of the vertices of G gives rise to a sequence $\{d_G(v_i)\}_1^n$ of nonnegative integers which is called a *degree sequence* of G; it is clearly unique up to permutations.

Given any sequence $D = \{d_i\}_1^n$ of nonnegative integers, we set $S(D) = \Sigma_1^n d_i$ and $M(D) = \max_i d_i$. If graph G is such that D is a degree sequence of G, we shall say that G *realizes* D. The theory of such realizations (and their analogs for directed graphs) has a considerable literature including papers [1], [5]–[7][1] on topics close to the present one; an extensive account is given, for example, in Chapter 6 of Chen [2]. Here we require only the two basic results of that theory ([3], [7]):

THEOREM A. *Sequence* D *is realized by some graph* G *if and only if* S(D) *(and hence* S(D) − 2M(D)*) is even; in that case* G *can be chosen free of multiple links.*

THEOREM B. *Sequence* D *is realized by some loopless graph if and only if* S(D) *is even and* 2M(D) ≤ S(D).

Our purpose here is to provide explicit statements and a convenient reference for some elementary results, probably largely of "folklore" nature, related to Theorem B. We shall determine the possible numbers of loops in graphs realizing a given sequence D, and also solve an associated loop-cost minimization problem in which loops incident at different vertices can have different costs.

2. Results and Analyses

Our first objective is to round out the information contained in Theorem B, by presenting the following result.

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† Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, MD 21218, and Center for Applied Mathematics, National Bureau of Standards, Washington, DC 20234.

[1] Numbers in brackets indicate literature references at the end of the paper.

THEOREM 1. *If sequence* D *has* 2M(D) − S(D) = 2L *where* L *is a positive integer, then* D *can be realized by a unique graph with* L *loops, but not by a graph with fewer loops. The "unique graph" has all its loops incident at the unique vertex of maximum degree.*

It will be convenient to base this theorem's proof on the following:

LEMMA. *With* D *as in* Theorem 1, *any graph which realizes* D *has at least* L *loops at its unique vertex of maximum degree.*

PROOF (of Lemma): (a) Let G be a graph which realizes D; choose the numbering so that $M(D) = d_1$. Since $2M(D) > S(D)$, v_1 will be the *only* vertex of maximum degree. Suppose G has L_1 loops at v_1.

(b) By Euler's handshaking lemma, the number of edges of G is $S(D)/2$. Of these, the $S(D)/2 - L_1$ which are not loops at v_1 each have at least one endpoint in $\{v_i\}_2^n$, and so each contributes either 1 or 2 to the sum $\Sigma_2^n d_i = S(D) - M(D)$. It follows that $S(D)/2 - L_1 \leqslant S(D) - M(D)$, yielding

$$L_1 \geqslant M(D) - S(D)/2 = L.$$

PROOF (of Theorem 1): (a) By the Lemma, no graph which realizes D can have fewer than L loops.

(b) Choose the numbering so that $M(D) = d_1$. Then a graph with L loops, which realizes D, is obtained by placing L loops at v_1 and drawing d_i edges from v_1 to v_i for $2 \leqslant i \leqslant n$; the correctness of this graph's degree at v_1 follows from

$$\sum_2^n d_i + 2L = S(D) - M(D) + 2L = M(D) = d_1.$$

(c) Now let G be any graph which realizes D and has exactly L loops; by the Lemma, all these loops are incident at v_1. Let d_i^* be the number of edges from v_1 to v_i in G, for $2 \leqslant i \leqslant n$. Then on the one hand $\Sigma_2^n d_i^* = d_1 - 2L = \Sigma_2^n d_i$, and on the other hand $d_i^* \leqslant d_i$ for $2 \leqslant i \leqslant n$. It follows that $d_i^* = d_i$ for $2 \leqslant i \leqslant n$, so that G coincides with the graph constructed in (b). This completes the proof of Theorem 1.

We turn now to a more general problem. Suppose given a sequence $C = \{c_i\}_1^n$ of nonnegative real numbers, and interpret c_i as the "cost" per loop incident at v_i; i.e., if graph G with vertex-set $\{v_i\}_1^n$ has λ_i loops attached at v_i, then the total loop-cost of G is $\Sigma_1^n c_i \lambda_i$. We seek a graph G which realizes a given sequence D as degree-sequence, and does so at minimum total loop-cost. (Theorem 1 treated the special case in which all $c_i = 1$.)

By Theorem A, this problem has a solution if and only if $S(D) - 2M(D)$ is even. When this is the case and $2M(D) \leqslant S(D)$, it follows from Theorem B that the optimal solution is found as a loopless graph realizing D. The remaining possibility is resolved by the following theorem, which shows that the solution is essentially independent of the cost-structure C.

THEOREM 2. *Suppose* 2M(D) − S(D) = 2L *where* L *is a positive integer. Then an optimal solution, unique if all* $c_i > 0$, *is given by the "unique graph" of* THEOREM 1.

PROOF: This is an immediate consequence of the Lemma and Theorem 1.

The uniqueness assertions in Theorems 1 and 2 bear the same relation to uniqueness results by Hakimi [4], Owens and Trent [7] and Senior [8], as do the remaining assertions of Theorems 1 and 2 to Theorem B.

Finally, we wish to determine the possible numbers of loops in graphs which realize a given sequence D. Theorem B and Theorem 1 specify the minimum of these numbers; it remains to specify their maximum, and to ascertain which values between the two can actually arise. To this end it is convenient to define, for $D = \{d_i\}_1^n$, Odd (D) to be the cardinality of {i: d_i is odd}.

THEOREM 3. *Sequence* D, *with its number of positive entries different from* 2 *and with* S(D) *even, is realized by a graph with precisely* k *loops if and only if* 2k *lies between* max (0, 2M(D) − S(D)) *and* S(D) − Odd(D) *inclusive*.

PROOF: (a) Theorems B and 1 give max(0, $2M(D) - S(D)$) as the minimum possible value for $2k$.

(b) From $D = \{d_i\}_1^n$ we determine a sequenee $\Delta = \{\delta_i\}_1^n$ of nonnegative integers as follows: by requiring $d_i = 2\delta_i + 1$ if d_i is odd, $d_i = 2\delta_i$ if d_i is even. Then $S(D) = 2S(\Delta) + \text{Odd}(D)$, so that Odd($D$) is even. Clearly any graph with vertex-set $\{v_i\}_1^n$ that realizes D can have at most δ_i loops ineident at v_i, thus at most $S(\Delta) = (S(D) - \text{Odd}(D))/2$ loops in all. This upper bound is aehieved by attaching δ_i loops to v_i for $1 \leq i \leq n$, pairing off in any way the members of the even-eardinality set $\{v_i\colon d_i \text{ is odd}\}$ counted by Odd(D), and joining the vertiees in each pair by a single edge. So $S(D) - \text{Odd}(D)$ is indeed the maximum value for $2k$.

(c) Beginning with the graph constructed in (b), repeat the following step as long as possible, producing a sequence of graphs each realizing D and having one fewer loop than its predecessor: if the current graph has three distinct vertices, v_i, v_p, v_q such that v_i bears a loop ℓ and some edge e joins v_p and v_q, then replace ℓ and e by a pair of edges from v_i to v_p and to v_q respectively.

Let G be the graph with which this process terminates and let λ_j be the number of loops of G at v_j ($1 \leq j \leq n$), for a total of k loops. If $k=0$, we are done, so assume $k>0$. For any vertex v_i sueh that $\lambda_i>0$, it follows from the construction of G that

$$d_i - 2\lambda_i = \sum_{j \neq i} (d_j - 2\lambda_j). \qquad (*)$$

It follows that there is either just one such vertex, say v_1, or else exactly two, say v_1 and v_2, with $d_1 - 2\lambda_1 = d_2 - 2\lambda_2$ and with $d_j = 2\lambda_j = 0$ for all $j>2$. Thc latter case is ruled out by the theorem's hypothesis on D. In the former case, (*) yields

$$d_1 - 2\lambda_1 = S(D) - d_1,$$

from which it readily follows that

$$2k = 2\lambda_1 = 2d_1 - S(D) = 2M(D) - S(D);$$

again we are done.

It only remains to treat the exceptional case excluded by the hypothesis of Theorem 3.

THEOREM 4. *Sequence* D, *with exactly two positive entries and with* S(D) *even, is realized by a graph with precisely* k *loops if and only if* 2k *lies between* 2M(D) − S(D) *and* S(D) − Odd(D) *inclusive and* 2k ≡ 2M(D) − S(D) (mod 4).

PROOF: (a) Number so that d_1 and d_2, with $d_1 \geq d_2$, are the two positive entries of D. Since $S(D) = d_1 + d_2$ is even, d_1 and d_2 have the same parity; Odd(D) is 2 or 0 according as the parity is odd or even.

(b) The arguments in (a) and (b) of Theorem 3's proof still apply, to show that $S(D) - \text{Odd}(D)$ and $2M(D) - S(D)$ are respectively double the maximum and minimum numbers of loops in graphs that realize D. These extreme values of $2k$ differ by $2d_2 - \text{Odd}(D)$, a multiple of 4.

(c) Define $\Delta = \{\delta_i\}_1^n$ as in the proof of Theorem 3. Form a graph realizing D which has δ_1 loops incident at v_1, δ_2 loops incident at v_2, and Odd(D) edges between v_1 and v_2. Then repeat the following step as long as possible, producing a sequence of graphs each realizing D and having two fewer loops (hence, a value of $2k$ less by 4) than its predecessor: if the current graph has loops at both v_1 and v_2, then delete one loop at each of these vertices and replaee them by two new edges between v_1 and v_2.

The final graph in this process has a number k of loops (all at v_1) given by

$$2k = 2(\delta_1 - \delta_2) = d_1 - d_2 = 2M(D) - S(D).$$

Thus all values of $2k$ identified in Theorem 4's statement are indeed achieved.

(d) To show that no other values can be achieved, consider any graph realizing D, with λ_j loops at v_j ($j = 1,2$) for a total of $k = \lambda_1 + \lambda_2$ loops. Counting the edges from v_1 to v_2 in two different ways (by incidences on v_1 and by incidences on v_2) yields the relation $d_1 - 2\lambda_1 = d_2 - 2\lambda_2$, so that

$$2k = 2(\lambda_1 - \lambda_2) + 4\lambda_2 = (d_1 - d_2) + 4\lambda_2 = 2M(D) - S(D) + 4\lambda_2;$$

thus the residue of $2k$ (mod 4) is as stated in the theorem.

The following observation is included for completeness. Let $\lambda_G(v)$ denote the number of loops incident on vertex v in graph G with vertex-set $\{v_i\}_1^n$; then $\Lambda(G) = \{\lambda_G(v_i)\}_1^n$ is the *loop-sequence* of G corresponding to this enumeration of the vertices. Given a pair (D,Λ) of sequences $D = \{d_i\}_1^n$ and $\Lambda = \{\lambda_i\}_1^n$ of nonnegative integers, it is natural to ask whether there exists a graph G with $D = D(G)$ and $\Lambda = \Lambda(G)$. But this is the case if and only if $D - 2\Lambda = \{d_i - 2\lambda_i\}_1^n$ is the degree sequence of a *loopless* graph, and a necessary and sufficient condition for that to hold is found by applying Theorem B to $D - 2\Lambda$.

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