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A Univariate Inequality for Medians

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An inequality is provided for medians which is an analog of a theorem due to Karamata, dealing with majorization.

Key words: concave; convex; inequality; majorization; median.

There has been a great deal of recent emphasis on majorization and related correlation inequalities, for example Marshall and Olkin [1][1] and Karlin and Rinott [2]. These inequalities have a variety of important applications. However they are based solely on mathematical expectations, for example the result below due to Karamata [3]. After stating this result we give our analog for medians.

Karamata's result (THEOREM): *Let* F *and* G *be distribution functions, on* (c, d). *Let* $\mu = F - G$. *Then* $\int \phi d\mu \geq 0$ *for all convex functions,* $\phi:(c,d) \rightarrow R$, *if and only if*

$$\int_c^d d\mu = \int_c^d x d\mu = 0 \quad \textit{and}$$

$$\mu\,(c,x] \geq 0, \qquad c \leq x \leq d\ .$$

Also see Spiegelman [4], for a different presentation of the direct part.

It should be noted that the direct part of Karamata's result is a generalization of Jensen's inequality. As previously suggested, characterizations such as this hold for medians as well. We define a median M of a random variable X to satisfy

$$P(X \geq M) \geq 1/2 \leq P(X \leq M)\ .$$

It is clear that M may not be uniquely defined. In order to avoid technical difficulties we define a p-median. Suppose a and b are endpoints of the largest closed interval such that every point M

$$a \leq M \leq b \text{ is a median of } X\ .$$

Then for p, such that $0 \leq p \leq 1$, define the p-med $X = pa + (1 - p)b$.

The use of p-medians complicates the statement of the theorem below. If all the random variables in the remainder of this paper have unique medians and, in addition, the class of convex functions is reduced to the class of strictly convex functions, then we get a less complicated analog to Karamata's result. In the new statement of our result the p-median notation is replaced by the word median. However, even under these more restrictive conditions, the converse part of Karamata's theorem has no explicit analog. (This is easy to see for random variables X such that $p(X=\infty) = p(X=-\infty) > .25$, and a convex function defined on the extended real line.) Our analog to Karamata's converse requires the class of functions $\Psi = \{\psi\}$ where each $\psi: R \rightarrow \{0,1\}$ and has the form $\psi = 1 - \chi_I$, where χ_I is the characteristic function of some bounded interval.

The following theorem gives an analog to the theorem of Karamata. The direct part is an extension of Tomkins' [5] version of Jensens' inequality for medians. Tomkins' inequality is for conditional medians and

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[1] Figures in brackets indicate literature references at the end of this paper.

a restatement of his theorem is too lengthy. His result for unconditional medians is cast in our notation at the end of the theorem's statement.

THEOREM: (Direct part) *Let* F *and* G *be distribution functions for the random variables* X *and* Y *respectively. Let* F *and* G *have a common set of medians (the interval)* [a,b].

Also assume:

$$F(t) \geqslant G(t) \qquad t \leqslant a \tag{1a}$$

$$G(t) \geqslant F(t) \qquad t \geqslant b\,. \tag{1b}$$

Then for any convex function ϕ *defined on the support of both* F *and* G *and for every* p, $0 \leqslant p \leqslant 1$, *there exists a* q, $0 \leqslant q \leqslant 1$ *such that*

$$q\text{-}med\ \phi(X) \geqslant p\text{-}med\ \phi(Y). \tag{2}$$

In addition, if ϕ *is monotonic over the range of all* p-*medians for* Y *and* $p \leqslant .5$ *then* q *may be taken equal to* p.

A partial converse of this result holds which requires an extra condition on the class, Ψ·

$$\begin{gathered}\text{For some } p \text{ median of } X,\ M, \\ \psi(M+) \quad \text{or} \quad \psi(M-)=0.\end{gathered} \tag{3}$$

If for all ψ satisfying condition (3) above $\psi(X)$ is stochastically larger than $\psi(Y)$, i.e. $P(\psi(X)>t) \geqslant P(\psi(Y)>t)$, and if for all monotonic functions, ϕ, (2) holds, then (1a) and (1b) hold.

COMMENT: Tomkins' inequality when applied to unconditional medians is a special case of our direct part. Simply take Y to have unit mass at p-med X.
Proof of the direct part: 1a and 1b hold.
Case 1—ϕ is monotone. Then clearly

$$p\text{-med}\ \phi\ (X) = p\text{-med}\ \phi\ (Y).$$

Case 2—ϕ has a minimum at a point t

$$\text{i.e.,} \qquad \phi(z) \geqslant \phi\ (t) \text{ for all } z \varepsilon R.$$

Define $F^{-1}(q) = \inf\{x|F(x) \geqslant q\}$.

Let r-med $\phi(Y) = \phi(z_{rj})\ j=1,2$ with $z_{r1} \leqslant z_{r2}$. It follows immediately from Tomkins' result that there exists an r such that $z_{r1} \leqslant p$-med $Y \leqslant z_{r2}$.

If all the p-medians of Y lie on one side of the value, t, where $\phi\ (z)$ takes its minimum and $p \leqslant .5$, it follows that $\phi(p(0\text{-med } Y) + (1-p)\ (1\text{-med } Y)) \leqslant p\ \phi\ ((0\text{-med } Y) + (1-p)\ \phi\ (1\text{-med } Y))$. Since ϕ is monotonic in the interval between the 0 and 1 medians, it follows that $p\ \phi\ (0\text{-med } Y) + (1-p)\ \phi\ (1\text{-med } Y) \leqslant p\text{-med}\ \phi(Y)$. Thus in this case we may take $r=p$. Otherwise:

for $z \leqslant z_{r1}$

$$F(z) \geqslant G(z)$$

which implies

$$z \geqslant F^{-1}(G(z)).$$

Similarly for $z \geqslant z_{r2}$

$$z \leqslant F^{-1}(G(z)).$$

Note that if G is continuous and strictly monotonic $F^{-1}G(Y)$ may be taken equal to X. By definition

$$\begin{aligned} 1/2 &\leqslant P(\phi(Y) \geqslant r\text{-med } \phi(Y)) \\ &= P(Y \leqslant z_{r1}) + P(Y \geqslant z_{r2}) . \end{aligned}$$

Notice that

$$P(Y \leqslant z_{r1}) \leqslant P(F^{-1}G(Y) \leqslant z_{r1})$$

$$\text{and} \quad P(Y \geqslant z_{r2}) \leqslant P(F^{-1}G(Y) \geqslant z_{r2}) \quad \text{(by assumptions 1a and 1b).}$$

Therefore

$$P(\phi(X) \geqslant r\text{-med } \phi(Y)) \geqslant 1/2 .$$

If Y does not have a continuous distribution standard, approximation procedures can be applied to complete the proof.

The converse case:

Since $\phi(x) = x$ and $\phi(x) = -x$ are convex functions, if follows that p-med $X = p$-med Y. Suppose at some point to the left of the smallest median of X, $G(t) > F(t)$, then if

$$\psi_1(z) = \begin{cases} 1 \text{ if } z \leqslant t \\ 0 \text{ if } t < z \leqslant \sup_p \text{ (p-median of X)} \\ 1 \text{ if } z > \sup_p \text{ (p-median of X)} \end{cases}$$

$$\text{and} \quad \psi_2(z) = \begin{cases} 1 \text{ if } z \leqslant t \\ 0 \text{ if } t < z < \sup_p \text{ (p-median of X)} . \\ 1 \text{ if } z \geqslant \sup_p \text{ (p-median of X)} \end{cases}$$

Then either $\psi_1(Y)$ or $\psi_2(Y)$ is stochastically larger than $\psi_1(X)$ or $\psi_2(X)$, respectively.

Q.E.D.

COMMENT: A. Marshall in a private communication has pointed out that the proof of the direct part of the above theorem is easily expanded to include functions ϕ *s.t.* $\{x|\phi(x)<c\}$ is an interval.

Churchill Eisenhart, Editor, asked if the definition of median given by Dunham Jackson; Bulletin of the American Math Society, 1921 160–164, "Note on the Median of a Set of Numbers," also 1923, 17–20, "Note on Quartiles and Allied Measures," could be used as a unique choice of median in the preceding theorem? The answer is yes; I conjecture more is true. Let $0 < c < 1$ and let $L(X) = (1-c)X^p$ for $X \geqslant 0$, $L(X) = c(-X)^p$ for $X \leqslant 0$, $p \geqslant 1$. Let $A_p(L,X)$ denote the minimizer of $EL(X-a)$ with respect to a. Then for any convex function ϕ; $\lim_{p\to 1} A_p(L,\phi(X)) \geqslant \lim_{p \to 1} \phi(A_p(L,X))$. A detailed proof will appear later. Thus, Jackson's definition of quantiles also satisfies a version of Jensen's inequality. Versions of Karamata's theorem also hold.

References

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