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# On Cauchy-Riemann Equations in Higher Dimensions[1]

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The $n$ linear partial differential equations with constant complex coefficients

$$l_j \equiv \sum_{i,k}^{n} a_{jk}^{i} \frac{\partial u_k}{\partial x_i}=0,$$

($j=1, \ldots, n$) are said to form a system of generalized Cauchy-Riemann equations, if there exist constants $f_{jk}^{h}$ such that

$$\Delta u_j \equiv \sum_{h,k}^{n} f_{jk}^{h} \frac{\partial l_k}{\partial x_h}.$$

It is proved that such systems exist for $n=1,2,4,8$ only. In the cases $n=2,4$ there are three essentially inequivalent systems; $n=8$, only two. If the coefficients are required to be real, there exist only the classic system of two equations, the two systems of Dirac-Fueter equations, and two systems of eight equations.

If two real functions $u_1$, $u_2$ of the real variables $x_1$, $x_2$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}=0, \qquad \frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}=0, \tag{1}$$

they are harmonic, that is, from (1) follows Laplace's equation by differentiation:

$$\Delta u_1=\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}=0, \qquad \Delta u_2=\frac{\partial^2 u_2}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial x_2^2}=0. \tag{2}$$

Introducing the left sides of the Cauchy-Riemann equations

$$l_1=\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}, \qquad l_2=\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}, \tag{3}$$

we observe that this statement is an immediate consequence of the relations

$$\Delta u_1=\frac{\partial l_1}{\partial x_1}+\frac{\partial l_2}{\partial x_2}, \qquad \Delta u_2=\frac{\partial l_2}{\partial x_1}-\frac{\partial l_1}{\partial x_2}, \tag{4}$$

that is to say, that the Laplacians of $u_1$, $u_2$ are linear combinations of the derivatives of the left sides of the Cauchy-Riemann equations.

In 1939 Olga Taussky-Todd[3] studied the following general problem. Let $u_1, u_2, \ldots, u_n$ be functions of the independent variables $x_1, x_2, \ldots, x_n$. Is it possible to find a system of $n$ linear partial differential equations with constant coefficients

$$l_j \equiv \sum_{i,\,k}^{n} a_{jk}^{i} \frac{\partial u_k}{\partial x_i}=0, \tag{5}$$

in such a way that

$$\Delta u_j=\sum_{h,\,k}^{n} b_{jk}^{h} \frac{\partial l_k}{\partial x_h}, \tag{6}$$

the $b_{jk}^{h}$ again being constant coefficients? If a set of functions $u_1, u_2, \ldots, u_n$ satisfies (5), it follows then from (6) that they are harmonic. So we may say that (5) are *Cauchy-Riemann equations in n-dimensional space* and generate a *theory of functions* in this space reasonably *related to potential theory.* O. Taussky proved that this problem can only be solved in spaces of dimension $n=2^m$. In this paper the better result is established that *$n$ must be 1, 2, 4, or 8* and moreover all Cauchy-Riemann systems (5) will be classified. In our discussion we admit that the $x_h$, $u_k$ and the coefficients in (5), (6) are complex. We will use methods of representation theory introduced by Wigner and Eckmann[4] for the solution of problems of an analogous type, but we shall simplify matters a little by dealing with algebras instead of groups.

1. Introducing the $n$-row matrices

$$A_i=(a_{jk}^{i}),\ B_h=(b_{jk}^{h}), \quad i,h=1,2,\ldots,n, \tag{7}$$

and the vectors

$$l=(l_1,l_2,\ldots,l_n), \qquad u=(u_1,u_2,\ldots,u_n), \tag{8}$$

relations (5), (6) can be written

$$l=\sum_{i=1}^{n} A_i \frac{\partial u}{\partial x_i}, \qquad \Delta u=\sum_{h=1}^{n} B_h \frac{\partial l}{\partial x_h}. \tag{9}$$

Inserting the second equation into the first we get

$$\Delta u=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\ldots+\frac{\partial^2 u}{\partial x_n^2}=\sum_{h,i} B_h A_i \frac{\partial^2 u}{\partial x_h \partial x_i}. \tag{10}$$

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[3] O. Taussky, An algebraic property of Laplace's differential equation, Quart. J. Math. **10**, 99 (1939).

[4] B. Eckmann, Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen, Comment Math. Helv. **15**, 358 (1942).

This identity holds for every vector $u$. Comparing coefficients it turns out that

$$B_iA_i=I, \quad B_hA_i+B_iA_h=0, \qquad i\neq h, \tag{11}$$

$I$ being the $n$-row unit matrix. Thus the matrix $B_i$ is the inverse of $A_i$, and our question is reduced to the problem of constructing $n$ matrices $A_i$ of $n$ rows satisfying the relation

$$A_h^{-1}A_i+A_i^{-1}A_h=0, \qquad i\neq h, \tag{12}$$

2. In order to solve this problem we observe that (12) is invariant under a general equivalence transformation

$$A_i\rightarrow SA_iT, \qquad i=1,2,\ldots,n, \tag{13}$$

where $S$, $T$ are two matrices with nonvanishing determinants. We do not distinguish between two Cauchy-Riemann systems (5) related to each other by such a transformation but call them essentially equal. Using this equivalence, we may transform by (13) one of our matrices—say $A_n$—into the unit matrix. For $h=n$ we have then

$$A_i+A_i^{-1}=0 \quad \text{or} \quad A_i^{-1}=-A_i \quad \text{or} \quad A_i^2=-I$$

and for $h=1, 2, \ldots, n-1$ this gives

$$A_hA_i+A_iA_h=0.$$

Thus we may restrict ourselves to the problem of constructing $(n-1)$ matrices $A_1,A_2,\ldots,A_{n-1}$ having the property

$$A_i^2=-I, \quad A_hA_i+A_iA_h=0, \qquad i\neq h. \tag{14}$$

If we have only these special systems under consideration the general equivalence transformation (13) will be restricted to a similarity transformation

$$A_i\rightarrow SA_iS^{-1}, \tag{15}$$

because the unit matrix must be left invariant. Matrices of the type (14) have been studied at first by Hurwitz[5] in the special case where the $A_i$ are real and orthogonal.

3. From the basic relations (14) it follows that the $2^{n-1}$ matrices

$$\left.\begin{array}{l} I,\ A_1,\ A_2,\ \ldots,\ A_{n-1},\\ A_1A_2,\ A_1A_3,\ \ldots,\ A_{n-2}A_{n-1},\\ A_1A_2A_3,\ \ldots,\ A_{n-3}A_{n-2}A_{n-1},\\ \ldots\\ A_1A_2,\ \ldots,\ A_{n-1}, \end{array}\right\} \tag{16}$$

[5] A. Hurwitz, Über die Komposition der quadratischen Formen, collected papers 2, 641 (1933).

this is to say, all products with increasing subscripts of the factors, form a matrix algebra of order $2^{n-1}$. Indeed, the product of two matrices of the set (16) is (up to the sign) again a matrix of the set. Let us now consider the abstract associative algebra $H$ of order $2^{n-1}$ over the complex field given by the basic elements

$$\left.\begin{array}{l} 1,\ e_1,\ e_2,\ \ldots,\ e_{n-1},\\ e_1e_2,\ e_1e_3,\ \ldots,\ e_{n-2}e_{n-1},\\ e_1e_2e_3,\ \ldots,\ e_{n-3}e_{n-2}e_{n-1},\\ \ldots\\ e_1e_2,\ \ldots,\ e_{n-1}, \end{array}\right\} \tag{17}$$

and the multiplication rules

$$e_i^2=-1, \qquad e_he_i+e_ie_h=0, \qquad i\neq h. \tag{18}$$

Thus our problem is finally to construct a representation of the algebra $H$ by $n$-row matrices. In order to do this we use the following well-known theorems of representation theory:

*Theorem* I. *There is—up to similarity transformation (15)—only a finite number m of irreducible representations, where m is the order of the center of the given algebra H. Any representation is the sum of irreducible representations.*

*Theorem* II. *Let f be the degree of a representation (number of rows of the representing matrices). Then the degrees $f_1, f_2, \ldots, f_m$ of the irreducible representations satisfy the relation*

$$f_1^2+f_2^2+\ \ldots\ +f_m^2=\text{order of } H=2^{n-1}.$$

4. Let us discuss first the case that the number $n$ of Cauchy-Riemann eq (5) is even. Then the last element $(e_1e_2\ \ldots\ e_{n-1})$ of the sequence (17) commutes with $e_1,e_2,\ldots,e_{n-1}$ and is therefore a center element of the algebra $H$. It is not difficult to show that the elements 1 and $(e_1e_2\ \ldots\ e_{n-1})$ span the center of $H$, that is to say that the general center element is of the form $\alpha+\beta(e_1e_2\ \ldots\ e_{n-1})$, where $\alpha$, $\beta$ are complex numbers. The order of this center being $m=2$, we learn from theorem I, that our algebra $H$ has exactly two irreducible representations, $D_1$ and $D_2$. They are related in the following way. If $D_1$ is given by

$$D_1\colon \qquad 1\rightarrow I, \qquad e_i\rightarrow E_i, \tag{19}$$

($E_i$ being the representing matrices) then $D_2$ is given by

$$D_2\colon \qquad 1\rightarrow I, \qquad e_i\rightarrow -E_i. \tag{20}$$

In order to prove this let us observe that if the $E_i$ satisfy the basic relations (14)

$$E_i^2=-I, \qquad E_hE_i+E_iE_h=0,$$

the same is true for the matrices $(-E_i)$. Hence, if (19) is a representation, then (20) is another. Furthermore, if $D_1$ is irreducible, the same is true for $D_2$ and, finally, $D_1$, $D_2$ are not similar. Indeed, the center element $(e_1e_2 \ldots e_{n-1})$ is represented in $D_1$ by the matrix $E_1E_2 \ldots E_{n-1}$, which must be a multiple $cI$ of the unit matrix, since it commutes with the whole irreducible set of representing matrices. In $D_2$ the representing matrix is

$$-E_1E_2 \ldots E_{n-1}=-cI.$$

But $(cI)$ and $(-cI)$ are not similar because, certainly, $c\neq 0$. This finishes our proof.

This discussion shows, in particular, that $D_1$ and $D_2$ have the same degree $f$. From theorem II we have

$$2f^2=2^{n-1} \quad \text{and so} \quad f=2^{\frac{n-2}{2}}. \tag{21}$$

As stated in section 3, our basic problem is to find a representation $D$ of the algebra $H$ by $n$-row matrices. From theorem I it follows that $D$ must be the sum of the representations $D_1$, $D_2$, each of them perhaps repeated several times. So the degree $n$ of $D$ is a multiple of $f$:

$$n=k\cdot 2^{\frac{n-2}{2}}. \tag{22}$$

But this is only possible for $n=2,4,8$, and the multiplicities $k$ are 2, 2, 1, respectively. In the cases $n=2,4$ we have for $D$ the three inequivalent possibilities

$$D_1+D_1, \qquad D_1+D_2, \qquad D_2+D_2 \tag{23}$$

and in the case $n=8$ the two possibilities $D_1$, $D_2$.

5. The case of odd $n$ is rather trivial because the center of $H$ in this case is formed only by the multiples of the unity element. So we have only one irreducible representation. Its degree $f$ is given by

$$f^2=2^{n-1}, \quad f=2^{\frac{n-1}{2}},$$

and the wanted representation $D$ must be a multiple of this unique irreducible representation:

$$n=k\cdot 2^{\frac{n-1}{2}}.$$

This leads to $n=1$.

Collecting the results we get the theorem:

*A system of n Cauchy-Riemann equations of the type (5), (6) is only possible for $n=1,2,4,8$. In the cases $n=2,4$, there are three inequivalent systems, in the case $n=8$, only two.*

6. In this section we establish the Cauchy-Riemann systems explicitly and discuss especially the real ones.

(a) In the case $n=2$ the degree $f$ of the irreducible representation $D_1$ is $f=1$ according to (21). So $D_1$, $D_2$ may be given by

$$D_1: \quad e_1\to i, \qquad D_2: e_1\to -i, \tag{24}$$

$i$ being the imaginary unit. Our first possibility (23) $D=D_1+D_1$ is then

$$e_1\to\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}=A_1, \qquad A_2=I \tag{25}$$

and yields the Cauchy-Riemann system (5)

$$i\,\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_1}=0, \qquad i\,\frac{\partial u_2}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=0 \tag{26}$$

consisting of two separate equations for $u_1$ and $u_2$. Hence, we may restrict ourselves to the single equation

$$i\,\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}=0, \tag{27}$$

which expresses the fact (if $x_1$, $x_2$ are real variables) that $u$ is a complex analytic function of $x_1-ix_2$. By differentiation of (27) it follows of course $\Delta u=0$. The two other possibilities (23) may be established by changing $i$ into $(-i)$ in one or both equations (26).

In order to find the real Cauchy-Riemann systems—that is to say real representations of $H$—we must form the sum of one of our irreducible representations and its complex conjugate representation. Taking into account that $D_1$ and $D_2$ are complex conjugates, we have finally only the unique real representation $D=D_1+D_2$ and only one real Cauchy-Riemann system, which is, of course, the system (1).

(b) For $n=4$ the representation $D_1$ may be given by the so-called Pauli-matrices

$$D_1: \quad e_1\to E_1=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}, \quad e_2\to E_2=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},$$

$$e_3\to E_3=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix} \tag{28}$$

with

$$e_1e_2e_3\to\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}. \tag{29}$$

$D_2$ is obtained by changing the sign of those matrices The Cauchy-Riemann system corresponding to $D=D_1+D_1$ splits again into the equations

$$\left.\begin{aligned} i\,\frac{\partial u_2}{\partial x_1}-\frac{\partial u_2}{\partial x_2}+i\,\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_4}=0\\ i\,\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_2}-i\,\frac{\partial u_2}{\partial x_3}+\frac{\partial u_2}{\partial x_4}=0\end{aligned}\right\} \tag{30}$$

for $u_1$, $u_2$ alone and the same equations for $u_3$, $u_4$. In this case, however, $D_1$ and $D_2$ are *not* conjugate complex, but $D_1$ is similar to its own conjugate complex and the same is true for $D_2$. This follows from the fact that in $D_1$ the center-element $e_1e_2e_3$ is represented by a real matrix according to (29). So we have two nonequivalent real Cauchy-Riemann systems corresponding to the representations $D_1 + D_1$ and $D_2 + D_2$. They are R. Fueter's equations for right and left regular functions of a quaternion variable, and they are closely related to Dirac's equations in quantum mechanics. The first system may be derived from our equations (30) in the following way. Let $x_1$, $x_2$, $x_3$, $x_4$ be real variables and $u_1$, $u_2$ complex functions:

$$u_1=v_1+iv_2, \qquad u_2=v_3+iv_4.$$

Splitting the eq (30) into their real and imaginary parts we get the four equations wanted for $v_1$, $v_2$, $v_3$, $v_4$. The second system follows in the same way if we replace (30) by the equations corresponding to $D_2+D_2$.

(c) The case $n=8$ is entirely different from the previous cases, because we found in section 4 that the representation $D$ solving our problem is either $D_1$ or $D_2$ and hence irreducible. Thus the Cauchy-Riemann systems of this case will not split into systems of fewer equations. We omit the computation of the matrices of $D_1$, which is closely related to the socalled Cayley numbers building a nonassociative algebra and mention only the result that those matrices may be constructed as real matrices. They yield two Cauchy-Riemann systems.

As a final result we have the following statement: *The only generalized real Cauchy-Riemann systems are* 1. *the classic system of two equations*; 2. *the two systems of Dirac-Fueter equations, each system having four equations*; *and* 3. *two systems of eight equations.*

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