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A Note on Bounds of Multiple Characteristic Roots of a Matrix [1]

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If $A=(a_{is})$ is an $n\times n$ matrix, and if C_i are the circles, center a_{ii} and radii $\sum_{\substack{s=1\\s\neq i}}^{n}|a_{is}|$, and if λ is a characteristic root with m independent characteristic vectors, Olga Taussky proved the following two results:

(1) If λ lies outside all but one circle C_i, then m cannot be greater than 1.

(2) If $m=n-1$, then λ is an inner or boundary point of at least m circles C_i.

In this note the gap between these two results is closed, and it is shown that λ lies in at least m circles C_i, for all finite values of m and n, $m\leqq n$.

If $A=(a_{ij})$ is an $n\times n$ matrix and if C_i are the circles with centres a_{ii} and radii $\sum_{\substack{s=1\\s\neq i}}^{n}|a_{is}|$, Olga Taussky [3] proves these two theorems.

Theorem A. A characteristic root λ, which is an inner or boundary point of only one C_i, cannot have two independent characteristic vectors corresponding to it.

Theorem B. If A has a characteristic root λ of multiplicity $n-1$, with $n-1$ independent characteristic vectors, the λ lies in at least $n-1$ circles C_i.

In this note it is proved that

Theorem C. If λ is a characteristic root of A with $m\leq n$ independent characteristic vectors corresponding to it, then λ lies in at least m circles C_i.

Theorem C is a generalization of both Theorems A and B and closes the gap between them.

Theorem C contains the following generalization of a well-known theorem about determinants (for definitions and references, see, O. Taussky, A recurring theorem on determinants, Am. Math. Monthly **56,** 672 (1949)).

Theorem D.[4] Let A be a matrix that cannot be transformed to the form $\begin{pmatrix} P & U \\ O & Q \end{pmatrix}$ by the same permutation of the rows and columns, where O consists of zeros, and P and Q are square matrices. Let further $|a_{ii}|\neq\sum_{\substack{s=1\\s\neq i}}^{n}|a_{is}|$ for at least one value of i. If the rank of the matrix A is $n-m$, where $0\leq m\leq n$ then there must be at least m values of i for which the inequalities

$$|a_{ii}|<\sum_{\substack{s=1\\s\neq i}}^{n}|a_{is}|$$

hold.

We require this lemma:

Lemma. If X_i, $i=1, 2, \ldots, m$, are m independent vectors with components x_{is}, $s=1, 2, \ldots, n$, $n\geq m$, we may construct a set of m independent vectors Y_i, with components y_{is}, which are linear combinations of the vectors X_i and which have the property that we may select components of maxima moduli corresponding to each Y_i, so that no two such selected components have the same subscripts.

We may suppose $m\geq 2$.

We choose $Y_1=X_1$. Let $y_{1s_1}=x_{1s_1}$ be a component of maximum modulus of Y_1. Choose α_1 and α_2 so that

$$\alpha_1 y_{1s_1}+\alpha_2 x_{2s_1}=0. \tag{1}$$

and

$$Y_2=\alpha_1 Y_1+\alpha_2 X_2. \tag{2}$$

Since $y_{is_1}\neq 0$, $\alpha_2\neq 0$, and so since X_1 and X_2 are linearly independent, $Y_2\neq 0$. Let y_{2s_2} be a component of maximum modulus of Y_2. By (1) and (2) $y_{2s_1}=0$, hence $s_2\neq s_1$. Further, Y_1 and Y_2 are linear combinations of the vectors X_1 and X_2 and are independent. The construction is thus complete for two independent vectors Y_i. If $m\geq 3$, we choose three numbers β_1, β_2, β_3 so that

$$\beta_1 y_{1s_1}+\beta_2 y_{2s_1}+\beta_3 x_{3s_1}=0$$

$$\beta_1 y_{1s_2}+\beta_2 y_{2s_2}+\beta_3 x_{3s_2}=0$$

and

$$Y_3=\beta_1 Y_1+\beta_2 Y_2+\beta_2 X_3.$$

Since

$$\begin{vmatrix} y_{1s_1} & y_{2s_1} \\ y_{1s_2} & y_{2s_2} \end{vmatrix} = y_{1s_1}y_{2s_2}\neq 0,$$

$\beta_3\neq 0$, and $Y_3\neq 0$. The argument used above may now be repeated to show that if y_{3s_3} is a component of maximum modulus of Y_3, then $s_3\neq s_1$, $s_3\neq s_2$. Further, Y_3 has the other properties required of Y_i.

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[3] O. Taussky, Bounds for characteristic roots of matrices II, J. Research NBS **46,** 124 (1951) RP2184.

[4] This was pointed out by O. Taussky.

This would complete the construction for three vectors X_i.

If $m \geq 4$, the other properties may be continued step by step till all the independent vectors Y_i are exhausted. This completes the proof of the Lemma. To complete the proof of Theorem C, given the set of m independent characteristic vectors X_i, corresponding to the characteristic root λ, we construct the set Y_i of the Lemma. Since Y_i are linear combinations of X_i, they are also characteristic vectors corresponding to the characteristic root λ. Hence we have the system of equations

$$\sum_{s=1}^{n} a_{ts} y_{is} = \lambda y_{it}, i=1, 2, \ldots, m; t=1, 2, \ldots, n.$$

In particular we have

$$\sum_{s=1}^{n} a_{s_i s} y_{is} = \lambda y_{is_i}$$

and so

$$(\lambda - a_{s_i s_i}) y_{is_i} = \sum_{\substack{s=1 \\ s \neq s_i}}^{n} a_{s_i s} y_{is}.$$

Dividing through by y_{is_i}, and taking the moduli of the two sides, since $|y_{is_i}| \geq |y_{is}|$, $s=1, 2, \ldots, n$ we get that λ lies in the circle C_{s_i}.

Since $s_i \neq s_j$, $i \neq j$, we conclude that λ lies in m different circles C_i. This concludes the proof of the theorem.

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