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A Note on the Bounds of the Real Parts of the Characteristic Roots of a Matrix[1]

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Two theorems are given for the bounds of the real parts of the characteristic roots of an $n\times n$ matrix, depending on the use of an arbitrary set of n positive numbers. The set is then specialized in several ways so as to lead to theorems for the bounds in terms of the elements of the matrix.

1. Fundamental Theorems

Let $A=(a_{ij})$ be an $n\times n$ matrix. Let M and m be the maximum and minimum real parts of its characteristic roots.

Let $u_1, u_2, \ldots, u_n$, be a set of positive numbers. In this note we prove the following two theorems:

Theorem A.

$$\text{(i)}\quad M\leqq\max_r\left\{\frac{1}{2}(a_{rr}+\bar{a}_{rr})+\frac{1}{2u_r}\sum_{\substack{s=1\\ s\neq r}}^{n}|a_{rs}+\bar{a}_{sr}|u_s\right\}$$

$$\text{(ii)}\quad m\geqq\min_r\left\{\frac{1}{2}(a_{rr}+\bar{a}_{rr})-\frac{1}{2u_r}\sum_{\substack{s=1\\ s\neq r}}^{n}|a_{rs}+\bar{a}_{sr}|u_s\right\}.$$

Theorem B. If the elements a_{rr} are real and the elements a_{rs}, $r\neq s$ are real and non-negative, then

$$\min_r\left(a_{rr}+\frac{1}{u_r}\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}u_s\right)\leqq M\leqq\max_r\left(a_{rr}+\frac{1}{u_r}\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}u_s\right).$$

Using these theorems and giving specific values to the set u_r, we obtain some further inequalities for M and m. These inequalities are numbered theorems 1 to 4. Theorem B is substantially a theorem of L. Collatz.[3]

To prove theorem A(i), we write $A=H+iK$, where H and K are Hermitian matrices whose elements h_{rs} and k_{rs} are given by

$$h_{rs}=\frac{a_{rs}+\bar{a}_{sr}}{2},\quad k_{rs}=\frac{i}{2}(\bar{a}_{sr}-a_{rs}). \tag{1}$$

It is known that the real parts of the characteristic roots of A are bounded above and below by the maximum and minimum characteristic roots (real) of H. Hence to prove A(i) it is sufficient to prove that the maximum characteristic root of H is less than the right-hand side of (i), and to prove A(ii) it is sufficient to prove that the minimum characteristic root is greater than the right-hand side of (ii).

Let λ and μ be the maximum and minimum characteristic roots of H. If x is a vector with n components, then h_{rr} is one value of x^*Hx/x^*x for all values of r, where x^* is the transpose of the conjugate of x. Hence $\lambda\geqq h_{rr}$, for all r, and $\mu\leqq h_{rr}$ for all r.

Let U be the diagonal matrix with elements $u_1, u_2, \ldots, u_n$. Let $B=U^{-1}HU$ and let its elements be b_{rs}. Then

$$b_{rs}=\frac{1}{u_r}h_{rs}u_s,\quad b_{rr}=h_{rr}. \tag{2}$$

As B is a transform of H, B and H have the same characteristic roots, so that λ and μ are characteristic roots of B. If α is any characteristic root of B, then α lies in at least one circle with center b_{rr} and radius

$$\sum_{\substack{s=1\\ s\neq r}}^{n}|b_{rs}|.$$

Thus we have

$$|\alpha-b_{rr}|\leqq\sum_{\substack{s=1\\ s\neq r}}^{n}|b_{rs}|$$

for at least one value of r.

As $\lambda-b_{rr}\geqq 0$, and $\mu-b_{rr}\leqq 0$, we have

$$\lambda\leqq b_{rr}+\sum_{\substack{s=1\\ s\neq r}}^{n}|b_{rs}|$$

for a least one value of r, and

$$\mu\geqq b_{rr}-\sum_{\substack{s=1\\ s\neq r}}^{n}|b_{rs}|$$

for at least one value of r. By (1) and (2), this completes the proof of theorem A.

Theorem B may be proved by a modification of the proof given by L. Collatz (see footnote 3), or alternatively, as follows.

Let $C=A+NE$, where E is the unit matrix, and N is a positive number so chosen that $a_{rr}+N>0$ for

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all r. Let $D=U^{-1}CU$, where U is the diagonal matrix with elements $u_1, u_2, \ldots, u_n$.

The characteristic roots of C and D are the same, whereas the characteristic roots of A are those of C diminished by N.

If d_{rs} is an element of D, then $d_{rs}=(1/u_r)a_{rs}u_s$. Again the maximum positive root of a matrix with nonnegative elements is bounded above and below by the maximum and minimum values of the sums of elements of a row. The maximum positive root of such a matrix is also a root of maximum modulus, and so greater than the real part of any other root.

The sum of the rth row of the matrix D is

$$a_{rr}+N+\frac{1}{u_r}\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}u_s.$$

Hence theorem B follows by subtracting N from each row.

2. The Case of Real Elements

In this section we shall suppose all the elements of A to be real. Let

$$R_r=\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs},\quad C_r=\sum_{\substack{s=1\\ s\neq r}}^{n}a_{sr}.$$

Theorem 1. (i) *If* $a_{rs}+a_{sr}\geqq 0,\ r\neq s$, *if* $R_r+C_r>0$, *for all* r, *if* $a_{ii}\geqq a_{rr}$ *for all* r, *and* $a_{jj}\geqq a_{rr},\ r\neq i$, *if* $R_p+C_p\geqq(R_r+C_r)$ *for all* r *and* $R_q+C_q\geqq R_r+C_r$, $r\neq p$, *then*

$$M\leqq\max\left(a_{ii}+\frac{R_q+C_q}{2},\quad a_{jj}+\frac{R_p+C_p}{2}\right).$$

(ii) *If* $a_{mm}\leqq a_{rr}$ *for all* r *and* $a_{nn}\leqq a_{rr},\quad r\neq m$, *then*

$$m\geqq\min\left(a_{mm}-\frac{R_q+C_q}{2},\quad a_{nn}-\frac{R_p+C_q}{2}\right).$$

To prove (i), we may suppose $i=1$, and let

$$\sigma=\frac{R_p+C_p}{2},\quad \sigma^1=\frac{R_q+C_q}{2}.$$

We apply A (i), where we may leave out the modulus sign. We take $u_1=\sigma$ and

$$u_r=\frac{R_r+C_r}{2},\qquad r\neq 1$$

$$a_{11}+\frac{1}{\sigma}\sum_{s=2}^{n}\frac{a_{1s}+a_{s1}}{2}\,u_s\leqq a_{11}+\frac{R_1+C_1}{2\sigma}\max_{s\neq 1}u_s.$$

If

$$\frac{R_1+C_1}{2}=\sigma,$$

then

$$\max_{s\neq 1}u_s=\sigma';$$

if

$$\frac{R_1+C_1}{2}\neq\sigma,$$

then

$$\frac{R_1+C_1}{2}\leqq\frac{R_q+C_q}{2}=\sigma'$$

and

$$\max_{s\neq 1}u_s=\sigma.$$

Hence in either case

$$a_{11}+\frac{1}{\sigma}\sum_{s=2}^{n}\frac{a_{1s}+a_{s1}}{2}\,u_s\leqq a_{11}+\sigma'. \tag{3}$$

Again for $r\neq 1$, we have

$$a_{rr}+\frac{1}{u_r}\sum_{\substack{s=1\\ s\neq r}}^{n}\left(\frac{a_{rs}+a_{sr}}{2}\right)u_s\leqq a_{rr}+\max_{s\neq r}u_s\cdot\frac{1}{u_r}\sum_{\substack{s=1\\ s\neq r}}^{n}\left(\frac{a_{rs}+a_{sr}}{2}\right)$$
$$\leqq a_{rr}+\sigma. \tag{4}$$

From (3) and (4), part (i) of theorem 1 follows. Part (ii) of theorem 1 follows similarly from A(ii).

Theorem 2. *Let* a_{rr} *be real*, $a_{rs}\geqq 0,\ r\neq s$. (i) *If* $a_{ii}\geqq a_{rr}$, *for all* r, $a_{jj}\geqq a_{rr},\ r\neq i$, and $R_p\geqq R_r$ *for all* r, $R_q\geqq R_r,\ r\neq p$, *then* $M\leqq\max\,(a_{ii}+R_q,\ a_{jj}+R_p)$. (ii) *If* $a_{mm}\leqq a_{rr}$ *for all* r, $a_{nn}\leqq a_{rr},\ r\neq m$, and $R_p^1\leqq R_r$, *for all* r, $R_q^1\leqq R_r,\quad r\neq p^1$, *then* $M\geqq\min\ (a_{mm}+R_q^1,\ a_{nn}+R_p^1)$.

The proof of this is similar to the proof of theorem 1 using theorem B, and may be omitted.

Theorem 3. *If* a_{rr} *is real and* $a_{rs}\geqq 0,\ r\neq s$, *then*

$$M\leqq\max_r\left(a_{rr}+\left\{\sum_{\substack{ij\\ i\neq j}}^{n}a_{ij}^2-\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}^2\right\}^{\frac{1}{2}}\right).$$

We apply theorem B and take

$$u_r=\left(\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}^2\right)^{\frac{1}{2}},$$

where we suppose $u_r\neq 0$. By the Hölder-Schwartz inequality

$$\frac{1}{u_r}\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}u_s\leqq\frac{1}{u_r}\left(\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}^2\right)^{\frac{1}{2}}\left(\sum_{\substack{s=1\\ s\neq r}}^{n}u_s^2\right)^{\frac{1}{2}}$$
$$=\left(\sum_{s=1}^{n}u_s^2-\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}^2\right)^{\frac{1}{2}}.$$

The theorem follows from the definition of u_r.

Theorem 4. *If* a_{rr} *is real and* $a_{rs}\geqq 0,\ r\neq s$, *and if*

$$L_r=\left(\sum_{\substack{s=1\\ s\neq r}}^{n}a_{rs}^2\right)^{\frac{1}{2}}\neq 0.$$

then

$$M\leqq\max_r a_{rr}$$
$$+\min\left\{\left[\left(\sum_{s=1}^{n}L_s\right)^2-\max_r L_r^2\right]^{\frac{1}{2}}\left(\sum_{s=1}^{n}L_s\max_r L_r\right)^{\frac{1}{2}}\right\}.$$

We take $u_r = L_r^{\frac{1}{2}}$ and apply the Hölder-Schwartz inequality and obtain

$$\frac{1}{u_r} \sum_{\substack{s=1 \\ s \neq r}}^{n} a_{rs} u_s \leqq L_r^{\frac{1}{2}} \left(\sum_{\substack{s=1 \\ s \neq r}}^{n} L_s \right)^{\frac{1}{2}}$$

$$= \left\{ L_r \left(\sum_{s=1}^{n} L_s - L_r \right) \right\}^{\frac{1}{2}} \tag{5}$$

$$< \max_r \left\{ L_r \sum_{s=1}^{n} L_s \right\}^{\frac{1}{2}} \tag{6}$$

If x, $y > 0$, $x + y = a$, we have

$$a^2 - y^2 \geq a^2 - (a-x)^2 = 2ax - x^2 > ax - x^2.$$

Applying this to (5) we get

$$\left\{ L_r \left(\sum_{s=1}^{n} L_s - L_r \right) \right\}^{\frac{1}{2}} \leqq \left(\sum_{s=1}^{n} L_s \right)^2 - \max_i L_i^2 \Big\}^{\frac{1}{2}} \tag{7}$$

From (6) and (7) and theorem B the theorem follows.

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