ADDITIONS TO THE FORMULAS FOR THE CALCULATION OF MUTUAL AND SELF INDUCTANCE

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CONTENTS

I. INTRODUCTION

Since the appearance of the Bureau of Standards' collection of formulas for the calculation of mutual and self inductance [1] a number of papers have been published upon this subject, some of which have given formulas for cases which had not been previously

[1] This Bulletin, 8, p. 1; Scientific Paper No. 169.

treated, while others have given additional formulas for cases which had already received attention.

The present paper has for its purpose the presentation of the more generally useful of the new formulas with the purpose of supplementing the previous collection. The recent rate of production of formulas has, however, been such as to render it impossible to keep the subject up to date for any length of time. The selection of the material has been necessarily restricted to those cases in which the formulas lend themselves readily to numerical calculations or to those more complicated formulas for which auxiliary tables are available. In certain important cases formulas have been here omitted which may be of great service when suitable tables have been prepared for simplifying the numerical work.

It is to be emphasized, also, that although in the case of certain forms of circuit very complete formulas are available, yet in certain other cases which are important in practical alternating-current work and in wireless telegraphy formulas are still lacking or only imperfectly developed.

In general, the classification of the material here is uniform with that of Scientific Paper No. 169. New formulas are designated by the letter A, while formulas in Scientific Paper No. 169, to which reference is made, bear the numbers by which they are designated in that publication. The nomenclature adopted in the formulas is as far as possible the same in both papers.

II. MUTUAL INDUCTANCE OF PARALLEL COAXIAL CIRCLES

This case is of great practical importance, since it is the fundamental form from which, by integration, formulas for the inductance of solenoids and of circular coils of finite cross section have been derived.

In what follows the radii of the two circles are denoted by A and a, A being greater than a, d is the distance between their planes, while the auxiliary moduli k and k' are given in the equations

$$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}}$$

$$k'=\sqrt{1-k^2}=\frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a)^2+d^2}}$$

BUTTERWORTH'S FORMULAS

Butterworth[2] has recently published five series formulas for the calculation of the mutual inductance of two parallel coaxial circles, and has shown that these furnish all the possible essentially different hypergeometric developments of Maxwell's elliptic integral formula (1).

His formula (A) is the same as (5).[3]

His formula (B) is a much abbreviated form of Havelock's formula (17) and has the further advantage over the latter that the general term is known

$$M=\frac{\pi^2\mu^3}{4}\sqrt{Aa}\left[1-\frac{3^2}{2.6}\mu^2+\frac{3^2.5^2}{2.4.6.8}\mu^4-\frac{3^2.5^2.7^2}{2.4.6^2.8.10}\mu^6+\ .\ .\ .\ .\ \right] \quad (1A)$$

where

$$\mu^2=\frac{k^2}{1-k^2}=\frac{4Aa}{(A-a)^2+d^2} \quad (2A)$$

The general expression for the coefficient of μ^{2n} is the same as that for the coefficient of k^{2n} in the preceding formula, except that the terms are here alternately positive and negative. This formula converges only for $k^2<\frac{1}{2}$, that is, for circles far apart.

The formula (C) of Butterworth's paper is best suited for circles near together, although it converges for circles at all distances. It is written in the form

$$\frac{M}{4\pi\sqrt{Aa}}=k^3\left[\phi_o+\frac{3^2}{2^2}k'^2\phi_1+\frac{3^25^2}{2^24^2}k'^4\phi_2+\ .\ .\ .\ .\ \right] \quad (3A)$$

in which

$$\phi_0=\log_e\frac{4}{k'}-2,\quad \phi_1-\phi_0=\frac{1}{1}-\frac{2}{3},\quad \phi_2-\phi_1=\frac{1}{2}-\frac{2}{5}$$

$$\phi_n-\phi_{n-1}=\frac{1}{n}-\frac{2}{2n+1}=\frac{1}{n(2n+1)}$$

and the general term is

$$\left[\frac{3.5.7\ .\ .\ .\ (2n+1)}{2.4.6\ .\ .\ .\ .\ 2n}\right]^2k'^{2n}\phi_n$$

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[2] Phil. Mag., 31, p. 276; 1916.

[3] An error appears in the expression for the general term of this equation as it appears in the earlier issues of Scientific Paper No. 169. The general term of (5) for the series in the parentheses should read

$$\left[\frac{3.5.7\ .\ .\ .\ (2n+1)}{4.6.8\ .\ .\ .\ .\ 2n}\right]^2\frac{k^{2n}}{(n+2)\ (2n+2)}$$

If we put for the difference of the radii $c=A-a$, and for k and k' their values in terms of A, a, and d, and expand in powers of c/a and d/a, the resulting expression is the Maxwell's series formula (10) or (14). The latter formula has been much used and has been extended at different times by Nagaoka, Rosa, and Cohen, and later still by Coffin (p. 541). The expression (3A) is in a much more favorable form for numerical calculation, the advantage being very marked for the higher order terms.

Butterworth writes his formula (D) in the form

$$\frac{M}{4\pi\sqrt{Aa}}=\frac{1}{k}\left[\phi_0'+\frac{1^2}{2^2}k'^2\phi_1'+\frac{1^2.1^2}{2^2.4^2}k'^4\phi_2'+\frac{1^2 1^2 3^2}{2^2 4^2 6^2}k'^6\phi_3'+\ .\ .\ .\ \right] \tag{4A}$$

where

$$\phi_0'=\log_e\frac{4}{k'}-2,\ \ \phi_1'-\phi_0'=\frac{1}{1}+\frac{2}{1},\ \ \phi_2'-\phi_1'=\frac{1}{2}-\frac{2}{1}$$

$$\phi_n'-\phi'_{n-1}=\frac{1}{n}-\frac{2}{2n-3}=-\frac{3}{n(2n-3)}$$

The general term is

$$\left[\frac{1.1.3\ .\ .\ .\ (2n-3)}{2.4.6\ .\ .\ .\ .\ 2n}\right]^2 k'^{2n}\ \phi'_n$$

Like (3A), this formula converges for circles at all distances, but especially well for circles near together. It is not difficult to prove that (4A) is the same as a new formula previously obtained by Bromwich,[4] but which was not known to Butterworth. This expression, (3′) in Bromwich's paper, was shown by him to be essentially the same as that of Weinstein ((7) in Scientific Paper No. 169). Bromwich's equation is slightly more convergent than that of Weinstein and may be obtained from the latter simply by multiplying by the identity

$$\frac{r_1}{2\sqrt{Aa}}=\frac{r_1}{\sqrt{r_1^2-r_2^2}},$$

in which r_1 and r_2 have the values given under (2) of Scientific Paper No. 169. Bromwich's expression for the general term of (4A) is equivalent to that found by Butterworth, although expressed in a somewhat different form.

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[4] Quart. Jour. Pure and Applied Math., No. 176, p. 381; 1913.

Butterworth's final expression, (E), is

$$\frac{M}{4\pi\sqrt{Aa}}=\varphi_0''+\frac{1\cdot 3}{2^2}\frac{\varphi_1''}{\mu^2}-\frac{1^2\cdot 3\cdot 5}{2^2\cdot 4^2}\frac{\varphi_2''}{\mu^4}+\frac{1^2\cdot 3^2\cdot 5\cdot 7}{2^2\cdot 4^2\cdot 6^2}\frac{\varphi_3''}{\mu^6}-\dots \qquad (5A)$$

in which

$$\varphi_0''=\log_e 4\mu-2 \qquad , \qquad \varphi_1''-\varphi_0''=\frac{1}{1}-\frac{1}{3}+\frac{1}{1}$$

$$\varphi_2''-\varphi_1''=\frac{1}{2}-\frac{1}{5}-\frac{1}{1}$$

$$\varphi_n''-\varphi_{n-1}''=\frac{1}{n}-\frac{1}{2n+1}-\frac{1}{2n-3} \qquad \varphi_3''-\varphi_2''=\frac{1}{3}-\frac{1}{7}-\frac{1}{3}$$

and μ^2 has the same value as in (2A).

The general term is

$$\left[\frac{1\cdot 3\cdot 5\ \dots\ (2n+1)}{2\cdot 4\cdot 6\ \dots\ .\ 2n}\right]^2\frac{\varphi_n''}{(2n-1)(2n+1)\mu^{2n}}.$$

If, in this formula, we substitute the value of μ, it is easy to see that this series is the same as that given as (16), of which the first terms were found by Havelock.[5] Note that $\frac{1}{\mu^2}=\frac{r^2}{4Aa}=\frac{\alpha}{4}$ in (16). The general term of (16) was first found by Bromwich, in the paper already referred to,[4] in terms of the variable $p=\frac{\alpha}{4}$ in his nomenclature. It is advantageous to write the formula in terms of this variable rather than α, since the coefficients are thereby simplified.

Evidently the general term in Coffin's equation (13), which is the form taken by (16), when the circles have equal radii, will be the same as that given above for (5A). For this case, $a=A$, $r=d$, and $\alpha=d^2/a^2$. Like (16), this converges only when $\mu>1$, or, for equal circles, when $d<2a$.

COFFIN'S EXTENSION OF MAXWELL'S FORMULA (14).

Coffin [6] has shown how to obtain further terms in (14). He bases his method on (16). Putting $A=a+c$ and expanding (16) in terms of the small quantities c/a and d/a he has obtained three new terms in each part of (14). These new terms are

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[5] Phil. Mag., 15, p. 332; 1908.

[6] Phys. Rev. (2), 2, p. 65; 1913.

$$
\left.
\begin{aligned}
&4\pi a\left[\log\frac{8a}{r}\left\{\frac{89c^6+45c^4d^2-345c^2d^4+35d^6}{\overline{128}^2a^6}\right.\right.\\
&\qquad -\frac{109c^7-63c^5d^2-525c^3d^4+175cd^6}{2\cdot\overline{128}^2a^7}\\
&\qquad \left.+\frac{8921c^8-13692c^6d^2-43050c^4d^4+32900c^2d^6-1575d^8}{2\cdot\overline{128}^3a^8}\right\}\\
&\qquad -\left(\frac{2629c^6+12045c^4d^2-17445c^2d^4+1235d^6}{30\cdot\overline{128}^2a^6}\right.\\
&\qquad -\frac{27833c^7+71169c^5d^2-225225c^3d^4+50575cd^6}{420\cdot\overline{128}^2a^7}\\
&\qquad \left.\left.+\frac{5204309c^8+5304852c^6d^2-45499650c^4d^4+21735700c^2d^6-818475d^8}{840\cdot\overline{128}^3a^8}\right)\right]
\end{aligned}
\right\}\text{(6A)}
$$

Now that the general term of (16) is known Coffin's method may be extended to obtain further terms of any desired order. The algebraic work becomes, however, very tedious for the higher order terms, and, as has already been noted, the equivalent formula (3A) is much easier to use for numerical calculations.

SERIES EXPANSION OF ELLIPTIC INTEGRAL FORMULA (4)

Another method of procedure, when k' is too large for (4A) to be conveniently used, is to employ (4), which involves the modulus k_0', which is smaller than k'. We may write (4) in the form

$$M=\frac{2\pi\sqrt{Aa}}{(1-k_0')}\left[(1+k_0')^2F_0-4E_0\right] \tag{7A}$$

and expressing F_0 and E_0 in terms of k_0', by means of the general series formulas (3), we find

$$
\left.
\begin{aligned}
M=&\frac{2\pi\sqrt{Aa}}{(1-k_0')}\left[(1+2k_0')\log_e\frac{4}{k_0'}-4-\frac{3}{4}k_0'^2\left(\log\frac{4}{k_0'}-1\right)\right.\\
&+\frac{k_0'^3}{2}\left(\log\frac{4}{k_0'}-1\right)-\frac{23}{64}k_0'^4\left(\log\frac{4}{k_0'}-\frac{51}{46}\right)\\
&+\frac{9}{32}k_0'^5\left(\log\frac{4}{k_0'}-\frac{7}{6}\right)-\frac{59}{256}k_0'^6\left(\log\frac{4}{k_0'}-\frac{427}{354}\right)\\
&\left.+\frac{25}{128}k_0'^7\left(\log\frac{4}{k_0'}-\frac{37}{30}\right)-\frac{2775}{16384}k_0'^8\left(\log\frac{4}{k_0'}-\frac{2783}{2220}\right)+\cdots\right]
\end{aligned}
\right\}\text{(8A)}
$$

where

$$k_0'=\frac{1-k}{1+k}=\frac{k'^2}{(1+k)^2}.$$

The first two terms of this equation will have to be calculated with a good deal of care, and as the series involves odd powers of the modulus, as well as even, it will be about as easy to use (4A) or (6) and include terms of higher order than are included in (8A).

RANGE OF APPLICATION OF THE SERIES FORMULAS

Since the general term is available in all the series formulas, it is possible to calculate the mutual inductance of any two coaxial circles whatever without having recourse to the elliptical integral formulas. In some cases (see pp. 8 and 9 of Scientific Paper No. 169) the values calculated by the series formulas are more accurate than those obtained by the exact formula, and in any case the series are valuable for obtaining a check, not to mention their adaptability to the carrying out of integrations.

Those cases which it is most difficult to treat by means of a series formula are those where the moduli k and k' are nearly equal. For such cases the elliptical integral formulas will be easy to use, or we may obtain the result with accuracy by means of Nagaoka's series formula (8). Of the hypergeometric formulas (5) and (4A) require only 20 or 30 terms to be calculated (even in the most unfavorable case) to obtain a result correct to the seventh decimal place, and if we previously write down the successive factors which must be multiplied into each term to obtain the next, the calculation is much simplified. For the higher order terms the ratio of successive terms is nearly constant, so that these terms may be obtained with very little labor.

Instead of using (5) or (4A) we may also use (6), which is an expansion of the elliptic integral formula (2), expressed in terms of the modulus $k_1 = \frac{1-k'}{1+k'} = \frac{k^2}{(1+k')^2}$, which is smaller than k. This formula covers very satisfactorily those cases for which k and k' are nearly equal.

As already explained, (8A) gives the mutual inductance in terms of the modulus k_0', which is smaller than k'. This formula may be used when k' is so large as to render the use of (3A) and (4A) difficult.

The following examples are appended to give an idea of the relative advantages of the different formulas in several rather extreme cases:

EXAMPLES

Example 1.— $k = k' = \frac{1}{\sqrt{2}}$. Assuming the radii $A = 25$, $a = 20$, the distance between their planes corresponding to the above condition is $d = \sqrt{1975} = 44.441$. For calculating the mutual inductance in this case the series formulas (4A), (6), and (8A) are applicable. Also for this special case the formula (19) gives a very searching numerical test.

The following table gives the value of $M/4\pi\sqrt{Aa}$ calculated by each of these formulas, together with a record of the number of terms which have to be calculated to obtain a result accurate to seven places. Since, however, six-place logarithms were used in the computation, the values obtained differ in some cases by a little in the seventh place. These discrepancies are of the order of magnitude to be expected in work with six-place logarithms:

Formula	Number of terms	$\frac{M}{4\pi\sqrt{Aa}}$
(8A)	8	0.1128884
(4A)	11	0.1128885
(6)	4	0.1128887
(19)		0.112888542...

Example 2.— $k' = 0.6$, $k = 0.8$. To these corresponds the modulus $k_0' = \frac{1}{9}$. Assuming the same radii as before, namely, $A = 25$, $a = 20$, we find $d = \sqrt{1100} = 33.1663$. The results found for this case are

Formula	Number of terms	$\frac{M}{4\pi\sqrt{Aa}}$
(8A)	7	0.20113986
(4A)	10	0.20113987
(6)	5	0.20113983

Example 3.—Assuming $k^2 = \frac{1}{3}$ and $a/A = \frac{1}{2}$, the distance corresponding is $d = a\sqrt{15}$. One pair of circles satisfying these conditions is $A = 25$, $a = 12.5$, $d = 48.4123$.

Formula	Number of terms	$\frac{M}{4\pi\sqrt{Aa}}$
(2A)	14	0.911653
(1A)	13	0.911653
(6)	3	0.911652

Example 4 (examples 8 and 14, Scientific Paper No. 169).— $A = 25, a = 20, d = 10, \frac{1}{\mu^2} = \frac{1}{16}$

Formula	Number of terms	$\frac{M}{4\pi\sqrt{Aa}}$
(5A)	6	0.885387
(4A)	4	0.885387
(3A)	7	0.885387

III. MUTUAL INDUCTANCE OF PARALLEL ECCENTRIC CIRCLES

A knowledge of the mutual inductance of two circles whose planes are parallel, but whose axes are not coincident, is required in the calculation of the inductance of certain standards whose inductance may be given a continuous variation by moving one coil so that its plane remains constantly parallel to that of another coil, the distance between the planes of the two coils being kept constant. This is the construction employed in the Campbell variable inductance.

On account of the relatively large and uncertain correction necessary to apply to take into account the finite cross section of the coils, the value of the inductance in such a case will best be obtained by measurement rather than by calculation. Nevertheless, formulas allowing of the calculation of the inductance, at least approximately, are valuable for purposes of design. In the case of actual coils the current may be regarded as concentrated in a circular filament at the center of the cross section.

The only formulas yet available for the mutual inductance of eccentric circles are those recently published by Butterworth.[7] He rests his method on the theorem that any formula applicable to this case must be a solution of Laplace's equation, and for the special case that the circles are coaxial must reduce to one of the formulas for coaxial circles given in the previous section.

There follow not only Butterworth's general formulas but also the simpler formulas which hold for the cases of circles of equal radii and coplanar circles.

In these expressions we put

A = radius of the larger circle.
a = radius of the smaller circle.
d = the distance between their planes
ρ = the distance between their axes.
$\mu = \cos\theta = d/r \qquad c = A - a$
$r^2 = d^2 + \rho^2$

$P_n(\mu)$ = the zonal harmonic of the nth order with argument μ.

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[7] Phil. Mag., 31, p. 443; 1916.

For calculating the zonal harmonics we have the well-known expressions

$$\begin{aligned} P_0(\mu) &= 1 \\ P_1(\mu) &= \mu \\ P_2(\mu) &= \frac{1}{2}(3\mu^2 - 1) \\ P_3(\mu) &= \frac{\mu}{2}(5\mu^2 - 3) \\ P_4(\mu) &= \frac{1}{8}(35\mu^4 - 30\mu^2 + 3) \\ P_5(\mu) &= \frac{\mu}{8}(63\mu^4 - 70\mu^2 + 15) \\ P_6(\mu) &= \frac{1}{16}(231\mu^6 - 315\mu^4 + 105\mu^2 - 5) \\ P_7(\mu) &= \frac{\mu}{16}(429\mu^6 - 693\mu^4 + 315\mu^2 - 35) \\ P_8(\mu) &= \frac{1}{128}(6435\mu^8 - 12012\mu^6 + 6930\mu^4 - 1260\mu^2 + 35) \end{aligned} \tag{9A}$$

BUTTERWORTH'S FORMULAS FOR UNEQUAL CIRCLES

For unequal circles far apart

$$M = 2\pi^2 a\left(\frac{a}{A}\right)\left(\frac{A}{r}\right)^3\left[P_2(\mu) - \frac{3}{2}K_1P_4(\mu)\frac{A^2}{r^2} + \frac{3.5}{2.4}K_2P_6(\mu)\frac{A^4}{r^4} - \frac{3.5.7}{2.4.6}K_3P_8(\mu)\frac{A^6}{r^6} + \cdots\right]$$

where

$$\begin{aligned} &K_1 = 1 + \frac{a^2}{A^2},\ K_2 = 1 + 3\frac{a^2}{A^2} + \frac{a^4}{A^4} \\ &K_3 = 1 + 6\frac{a^2}{A^2} + 6\frac{a^4}{A^4} + \frac{a^6}{A^6},\ K_n = F\left(-n-1,\ -n,\ 2,\ \frac{a^2}{A^2}\right) \end{aligned} \tag{10A}$$

F is the hypergeometric series. (See p. 17, Scientific Paper No. 169.)

This formula converges for values of r greater than $(A+a)$. For the limiting case that the circles are coaxial it goes over into (17), Havelock's formula for circles far apart. The general term of (10A) is known, but the higher-order terms become tedious to calculate. Table A, taken from Byerleys "Fourier's Series and Spherical Harmonics," will be found helpful in making calculations with (10A) and the succeeding formulas.

For unequal circles near together

$$M = 4\pi\sqrt{Aa}\left[(\lambda - 2) + \frac{3}{16}\frac{c^2}{Aa}\left\{\left(\lambda - \frac{1}{3}\right)\varphi_2 - \chi_2\right\} - \frac{15}{1024}\frac{c^4}{A^2a^2}\left\{\left(\lambda - \frac{31}{30}\right)\varphi_4 - \chi_4\right\} + \cdots\right] \quad (11A)$$

in which

$$\lambda = \log_e \frac{16\sqrt{Aa}}{c(1+\mu)\sqrt{1+\nu^2}}$$

and μ^2 and $-\nu^2$ are the roots of the quadratic

$$t^2 - t\left(1 - \frac{r^2}{c^2}\right) - \frac{d^2}{c^2} = 0$$

and

$$\varphi_2 = \frac{2}{3} - \frac{2}{3}P_2(\mu)P_2(i\nu) = \frac{1}{2}\{(1+\mu^2) - \nu^2(1-3\mu^2)\}$$

$$\varphi_4 = \frac{8}{15} - \frac{16}{21}P_2(\mu)P_2(i\nu) + \frac{8}{35}P_4(\mu)P_4(i\nu)$$

$$= \frac{1}{8}\{(3 + 2\mu^2 + 3\mu^4) - 2\nu^2(1 + 6\mu^2 - 15\mu^4) + \nu^4(3 - 30\mu^2 + 35\mu^4)\}$$

$$\chi_2 = \frac{1}{4}(1-\mu)\{(1-\mu) - \nu^2(1+7\mu)\}$$

$$\chi_4 = \frac{1}{96}(1-\mu)\{3(1-\mu)(7 + 2\mu + 7\mu^2) - 6\nu^2(5 + \mu - \mu^2 + 59\mu^3) + \nu^4(21 + 241\mu - 113\mu^2 - 533\mu^3)\}$$

The general solution of the quadratic is

$$t = \frac{1}{2}\left(1 - \frac{r^2}{c^2}\right) \pm \frac{1}{2}\sqrt{\left(1 - \frac{r^2}{c^2}\right)^2 + 4\frac{d^2}{c^2}}$$

the positive root is taken as μ^2, while the negative root is $-\nu^2$.

Butterworth gives the method for obtaining the general term of (11A), but the calculation of the quantities φ and χ for the higher-order terms becomes very tedious. An idea of the importance of the terms omitted in (11A) may be obtained from an examination of the convergence of the formula (16) with the same values of A, a, and d. The convergence for the eccentric circles will be at least as good as that of (16), which holds for the limiting case of coaxial circles.

Similar considerations hold for (10A) and (17) if in the latter d be taken as of the same value as r in the former.

BUTTERWORTH'S FORMULAS FOR EQUAL ECCENTRIC CIRCLES

If the circles have the same radius a and their distance apart is large compared with a,

$$M = 2\pi^2 a\left(\frac{a}{r}\right)^3 \left[P_2(\mu) - 3\,P_4(\mu)\frac{a^2}{r^2} + \frac{75}{8}\,P_6(\mu)\frac{a^4}{r^4} - \frac{245}{8}\,P_8(\mu)\,\frac{a^6}{r^6} + \ . \ . \ .\right] \tag{12A}$$

For equal circles near together

$$M = 4\pi a\left[(\lambda_1 - 2) + \frac{3}{16}\,\frac{r^2}{a^2}\left\{\left(\lambda_1 - \frac{1}{3}\right) P_2\,(\mu) - \psi_2\right\} - \frac{15}{1024}\,\frac{r^4}{a^4}\left\{\left(\lambda_1 \ - \frac{31}{30}\right)P_4(\mu) - \psi_4\right\} + \frac{35}{(128)^2}\,\frac{r^6}{a^6}\left\{\left(\lambda_1 - \frac{247}{210}\right)P_6(\mu) - \psi_6\right\} - \ . \ . \ .\right] \tag{13A}$$

in which

$$\lambda_1 = \log_e \frac{16a}{r(1+\mu)}, \quad \mu = \frac{d}{r}$$

$$\psi_2 = -\frac{1}{4}\,(1-\mu)\,(1+7\mu)$$

$$\psi_4 = \frac{1}{96}\,(1-\mu)\,(21 + 241\mu - 113\mu^2 - 533\mu^3)$$

$$\psi_6 = -\frac{1}{960}\,(1-\mu)\,(185 - 2957\mu + 3728\mu^2 + 18008\mu^3 - 3247\mu^4 - 18107\mu^5)$$

BUTTERWORTH'S FORMULAS FOR COPLANAR CIRCLES

(a) Unequal coplanar circles far apart.

$$M_0 = -\pi^2 a\left(\frac{a}{A}\right)\left(\frac{A}{r}\right)^3\left[1 + \frac{3^2}{2.4}K_1\frac{A^2}{r^2} + \frac{3^2.5^2}{2.4^2.6}K_2\frac{A^4}{r^4} + \frac{3^2.5^2.7^2}{2.4^2.6^2.8}K_3\frac{A^6}{r^6} + \ . \ . \ .\right] \tag{14A}$$

the values of the K's being the same as in (10A).

(b) Unequal coplanar circles near together, but $r > (A-a)$; i. e., the distance between their centers greater than the difference of their radii.

$$M_0 = 4\pi\sqrt{Aa}\left[(\lambda_0 - 2) - \frac{3}{32}\,\frac{r^2}{Aa}\left(\lambda_0 - \frac{5}{6}\right)\left(1 - 2\frac{c^2}{r^2}\right) - \frac{45}{8192}\,\frac{r^4}{A^2\,a^2}\left\{\lambda_0\left(1 - \frac{8}{3}\frac{c^2}{r^2} + \frac{8}{3}\frac{c^4}{r^4}\right) - \left(\frac{97}{60} - \frac{214}{45}\frac{c^2}{r^2} + \frac{214}{45}\frac{c^4}{r^4}\right)\right\} - \ . \ . \ .\right] \tag{15A}$$

with

$$\lambda_0 = \log_e \frac{16\sqrt{Aa}}{r}, \quad c = A - a$$

This formula holds even when the circles intersect. A negative sign in the formulas for the mutual inductance in this section indicates that the electromotive force induced in either coil, due to a change of current in the other, is of opposite sign to that which would be induced, under the same circumstances, if the coils were in the coaxial position.

(*c*) unequal coplanar circles for which $r<(A-a)$—that is, where one circle is entirely within the other.

$$M_0=4\pi\sqrt{Aa}\left[(\lambda_2-2)+\frac{3}{32}\frac{c^2}{Aa}\left\{\left(\lambda_2-\frac{1}{3}\right)(1+\mu^2)-\frac{1}{2}(1-\mu)^2\right\}-\frac{15}{8192}\frac{c^4}{A^2a^2}\left\{\left(\lambda_2-\frac{31}{30}\right)(3+2\mu^2+3\mu^4)-\frac{1}{4}(1-\mu)^2(7+2\mu+7\mu^2)\right\}+\ldots\right] \quad (16A)$$

$$\lambda_2=\log_e\frac{16\sqrt{Aa}}{c(1+\mu)},\quad \mu^2=1-\frac{r^2}{c^2}$$

(*d*) unequal coplanar circles, one touching the other internally; that is, $r=A-a=c$.

$$M_0=4\pi\sqrt{Aa}\left[(\lambda_3-2)+\frac{3}{32}\frac{c^2}{Aa}\left(\lambda_3-\frac{5}{6}\right)-\frac{45}{8192}\frac{c^4}{A^2a^2}\left(\lambda_3-\frac{97}{60}\right)+\ldots\right] \quad (17A)$$

$$\lambda_3=\log_e\frac{16\sqrt{Aa}}{c}$$

(*e*) equal coplanar circles far apart.

$$M_0=-\pi^2a\left(\frac{a}{r}\right)^3\left[1+\frac{9}{4}\frac{a^2}{r^2}+\frac{375}{64}\frac{a^4}{r^4}+\frac{8575}{512}\frac{a^6}{r^6}+\ldots\right] \quad (18A)$$

(*f*) equal coplanar circles near together.

$$M_0=4\pi a\left[(\lambda_0-2)-\frac{3}{32}\frac{r^2}{a^2}\left(\lambda_0-\frac{5}{6}\right)-\frac{45}{8192}\frac{r^4}{a^4}\left(\lambda_0-\frac{97}{60}\right)-\frac{175}{(512)^2}\frac{r^6}{a^6}\left(\lambda_0-\frac{251}{140}\right)-\ldots\right] \quad (19A)$$

$$\lambda_0=\log_e\frac{16a}{r}$$

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EXAMPLES

Example 5 (Formula (10A)).—$A=25,\ a=20,\ d=40,\ r=100.$

Thence $$c=5,\ \mu=0.4,\ A/r=\frac{1}{4},\ \frac{a}{A}=0.8$$

$$\begin{aligned}
P_2(\mu) &= -0.260000 & K_1 &= 1.64\\
P_4(\mu) &= -\ \ .113000 & K_2 &= 3.3296\\
P_6(\mu) &= \ \ \ \ .2926 & K_3 &= 7.5597\\
P_8(\mu) &= -\ \ .2670
\end{aligned}$$

Thence

$$M=2\pi^2\times 0.8\times 20\times\frac{1}{64}[-0.26000+0.01737+.00484+.00108]$$

$$=\frac{\pi^2}{2}[-.23671]$$

The corresponding coaxial case $A=25,\ a=20,\ d=100$ gives, using (17)

$$M=2\pi^2\times 0.8\times 20\times\frac{1}{64}[1-0.15375+0.02439-.00404]$$

$$=\frac{\pi^2}{2}\ (0.86660)$$

The convergence of (10A) is about as good as (17), and an estimate of the neglected terms may be gained from an examination of the same terms in (17).

Example 6 (Formula (11A)).—$A=25,\ a=12.5,$ and $\rho=d=\frac{12.5}{\sqrt{2}}$ $=8.8375.$

Thence $c=12.5,\ \frac{d^2}{c^2}=\frac{1}{2},\ \frac{c^2}{Aa}=\frac{1}{2}$

$$\begin{aligned}
\varphi_2 &= 1.25 & \chi_2 &= -0.18735\\
\varphi_4 &= 1.0935 & \chi_4 &= -\ \ .41549\\
\mu^2=\nu^2 &= \frac{1}{\sqrt{2}} & \lambda &= \log_e 9.40754=2.24151
\end{aligned}$$

Thence by (11A)

$$\frac{M}{4\pi\sqrt{Aa}}=0.24151+0.24118-0.00636+\cdot\cdot\cdot\cdot$$

$$=.47633$$

For the coaxial case $A=25,\ a=12.5,\ \frac{d}{a}=\frac{1}{\sqrt{2}}$ we find in (16),

$\alpha=\frac{3}{4},\ \log_e\frac{8\sqrt{Aa}}{\sqrt{c^2+d^2}}=2.22453$ and

$$\frac{M}{4\pi\sqrt{Aa}}=0.22453+0.26595-0.00982+0.00095$$

$$=0.48161$$

Here, again, the importance of higher-order terms neglected in the eccentric case may be estimated by calculating these terms for the coaxial case. Thus, the next term in (11A) should amount in this problem to about $+0.0006$.

Example 7 (Formula (13A)).—Equal circles near together $A = a = 25$, $r = 16$, and $\frac{d}{r} = 0.7$

Thence $\frac{r}{a} = \frac{16}{25}$, $\mu = 0.7$, $d = 11.2$

$$P_2(\mu) = 0.23500 \qquad \psi_2 = -0.4425$$
$$P_4(\mu) = -.41206 \qquad \psi_4 = -.15153$$
$$P_6(\mu) = -.1253 \qquad \psi_6 = -.71742$$
$$P_8(\mu) = .3067 \qquad \lambda = \log_e \frac{16 \times 25}{16 \times 1.7} = 2.68825$$

Accordingly

$$\frac{M}{4\pi a} = 0.68825 + 0.07648 + 0.00130 + 0.00008 = 0.76611$$

Formula (13) gives for the coaxial case $A = a = 25$, $d = 16$,

$$\frac{M}{4\pi a} = 0.52573 + 0.16838 - 0.00367 + 0.00020 = 0.69064$$

IV. SELF-INDUCTANCE OF A SINGLE-LAYER COIL OR SOLENOID

The inductance of a single-layer coil or solenoid is most easily obtained by basing the solution of the problem on that for a cylindrical current sheet. The latter may be regarded as being equivalent to a single-layer helical winding of flat tape of negligible thickness, the adjacent turns being separated by insulation of infinitesimal thickness.

Rosa [8] has shown how the difference between the inductance of such an ideal winding and that of an actual winding of round wire may be computed and has prepared tables for facilitating the calculation of this correction.[9]

The following formulas of this section apply only to cylindrical current sheets, and the results obtained by their use require correction by Rosa's method to reduce to the case of an actual winding.

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[8] This Bulletin, 2, pp. 161–187; 1906.

[9] Scientific Paper No. 169, pp. 122, 197, and 199.

BUTTERWORTH'S FORMULAS FOR INDUCTANCE OF A CYLINDRICAL CURRENT SHEET

In a recent article Butterworth[10] has obtained the differential equation of which Lorenz's absolute formula (72) for the inductance of a solenoid is a solution, and has developed from this the eight possible hypergeometrical series expansions. Of these, his formulas G, I, J, and M resemble closely those denoted by the letters F, H, K, and L, respectively, but the latter formulas are the more convergent, and these only will be considered here. Formulas F and K are new, and H and L are expressed by Butterworth in a form especially convenient for calculation.

There is an advantage in expressing the inductance of a cylindrical current sheet in such a manner as to make clear the connection with the well-known formula for the inductance when the length of the current sheet is infinite; that is, $L_1 = 4\pi^2 a^2 n^2/b$, and each of the following formulas is so written as to indicate this relation. The quantities k and k' have already been defined on page 120, Scientific Paper No. 169.

Butterworth's formula (F) gives

$$L = L_1 \frac{\sqrt{4a^2+b^2}}{b}\left[1 - \frac{4}{3\pi}k - \frac{3}{8}k^2 - \frac{5}{64}k^4 - \frac{35}{1024}k^6 - \cdot\cdot\cdot\cdot\cdot\right] \qquad (20A)$$

the general term in the parentheses being

$$-\frac{[1\cdot 3\cdot 5\cdot\cdot\cdot\cdot(2n+1)][1\cdot 1\cdot 3\cdot 5\cdot\cdot\cdot\cdot(2n-3)]}{2^{2n}n!\,(n+1)!}k^{2n}$$

This converges for coils of all lengths, but most rapidly for long coils (k small).

Formula (H) of Butterworth's paper is

$$L = L_1\left[1 - \frac{4}{3\pi}\cdot\frac{2a}{b} + \frac{1}{8}\left(\frac{2a}{b}\right)^2 - \frac{1}{64}\left(\frac{2a}{b}\right)^4 + \frac{5}{1024}\left(\frac{2a}{b}\right)^6 - \ .\ .\ .\right] \qquad (21A)$$

in which the general term is

$$(-1)^{n+1}\frac{[1.3.5\ldots.(2n-1)]\,[1.1.3.5\ldots.(2n-3)]}{2^{2n}\,n!\quad(n+1)!}\left(\frac{2a}{b}\right)^{2n}$$

an expression which agrees with that given by Webster and Havelock, formula (79), but which appears here in a somewhat simpler form than that given by them.

[10] Phil. Mag., 31, p. 276; 1916.

The formula (21A) converges only for coils whose length b is greater than the diameter $2a$.

Butterworth's formula (K) may be written

$$L = L_1 \cdot \frac{2}{\pi k^3}\left(\frac{2a}{b}\right)\left[\frac{2}{3}(1-k^3) + k'^2\psi_1 - \frac{1.3}{2.4}k'^4\psi_2 - \frac{1.1.3.5}{2.4.4.6}k'^6\psi_3 - \frac{1.1.3.3.5.7}{2.4.6.4.6.8}k'^8\psi_4 - \ldots\right] \tag{22A}$$

in which

$$\psi_1 = \log_e \frac{4}{k'} - \frac{3}{2}, \quad \psi_4 - \psi_3 = \frac{1}{6} + \frac{1}{8} - \frac{1}{3} - \frac{1}{7}$$

$$\psi_2 - \psi_1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{1} - \frac{1}{3}, \quad \psi_n - \psi_{n-1} = \frac{1}{2n-2} + \frac{1}{2n} - \frac{1}{2n-5} - \frac{1}{2n-1}$$

$$\psi_3 - \psi_2 = \frac{1}{4} + \frac{1}{6} - \frac{1}{1} - \frac{1}{5}$$

and the general term is

$$-\frac{[1.1.3.5\ \ldots\ (2n-5)]\,[3.5.7\ \ldots\ (2n-1)]}{[2.4.6\ \ldots\ (2n-2)]\,[4.6.8\ \ldots\ (2n)]}k'^{2n}\psi_n$$

This formula converges for all values of k', but especially well for short solenoids (k' small).

Finally, we have Butterworth's formula (L), which gives for the inductance the value

$$L = L_1 \cdot \frac{2}{\pi}\left(\frac{b}{2a}\right)\left[\psi_1' + \frac{1.1}{2.4}\psi_2'\left(\frac{b}{2a}\right)^2 - \frac{1.1.1.3}{2.4.4.6}\left(\frac{b}{2a}\right)^4\psi_3' + \frac{1.1.3.1.3.5}{2.4.6.4.6.8}\left(\frac{b}{2a}\right)^6\psi_4' - \ldots\right] \tag{23A}$$

in which

$$\psi_1' = \log_e \frac{8a}{b} - \frac{1}{2}, \quad \psi_4' - \psi_3' = \frac{1}{6} + \frac{1}{8} - \frac{1}{3} - \frac{1}{5}$$

$$\psi_2' - \psi_1' = \frac{1}{2} + \frac{1}{4} + \frac{1}{1} - \frac{1}{1}, \quad \psi_n' - \psi_{n-1}' = \frac{1}{2n-2} + \frac{1}{2n} - \frac{1}{2n-5} - \frac{1}{2n-3}$$

$$\psi_3' - \psi_2' = \frac{1}{4} + \frac{1}{6} - \frac{1}{1} - \frac{1}{3}, \qquad = -\frac{3}{n(2n-5)}\,\frac{[(2n-3)(2n-2)-1]}{(2n-3)(2n-2)}$$

and the general term is

$$\frac{[1.1.3\ \ldots\ (2n-5)]\,\{1.3.5\ \ldots\ (2n-3)]}{[2.4.6\ \ldots\ (2n-2)]\,[4.6.8\ \ldots\ 2n]}\psi_n'\left(\frac{b}{2a}\right)^{2n-2}$$

This is Rayleigh and Niven's formula (69), which was extended by Coffin, formula (71), but the general term has not previously been given. The formula converges only for coils whose length is less than the diameter.

The preceding four formulas cover between them the whole range of possible solenoids, although in the case of coils in which the length and diameter are nearly equal the number of terms necessary to be calculated is not inconsiderable, as may be noted from the following examples. In such cases the calculation of the higher-order terms is materially simplified, if one obtains the expression for the factor necessary to apply to each term in order to obtain the term next following.

EXAMPLE

Example 8.—For the case in which the length equals the diameter, we have $k = k' = \frac{1}{\sqrt{2}}$.

Using formula (20A), 14 terms gives $L = 0.6884229\ L_1$
Using formula (22A), 13 terms gives $L = 0.6884230\ L_1$
where $L_1 = 4\pi^2 a^2 n^2 / b$

For the coil of example 60 of Scientific Paper No. 169, $a = 27.0862$ $b = 30.551$, we find from eight terms of (23A) the value $L = 0.5546957\ L_1$, and seven terms in (22A) give $L = 0.554697\ L_1$. The values found by Nagaoka's formulas (77) and (76) differ from the first of these values by only one and three units, respectively, in the last place.

DISK COILS

An extension to the Rayleigh and Niven formula (70) for a disk coil of radial dimension c and mean radius a has been found by Lyle. The additional terms are

$$4\pi n^2 a\left[\left(\frac{11}{46080}\frac{c^4}{a^4}+\frac{103}{105.256^2}\frac{c^6}{a^6}\right)\log\frac{8a}{c} + \frac{1}{2400}\frac{c^4}{a^4}+\frac{98579}{(131072)(44100)}\frac{c^6}{a^6}\right] \tag{70A}$$

Formula (70) with the additional terms of (70A) suffices for values of c as great as the mean radius a.

For those coils where c is large compared with a—that is, for disk coils in which the inner radius is small compared with the outer radius—a formula has been developed by Spielrein [11] who puts

$$L = n^2 A f(\alpha) \tag{24A}$$

[11] Arch. für Elektrotechnik, 3, p. 187; 1915.

in which

A = The outer radius of the disk

α = the ratio of inner radius to outer radius

$$G=\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\ .\ .\ .\right)=0.9159656\ .\ .\ .\ .$$

$$u_n=\frac{\pi}{2}\left[\frac{1\cdot 3\cdot 5\ .\ .\ .\ .\ .\ .\ (2n-1)}{2\cdot 4\cdot 6\ .\ .\ .\ .\ .\ .\ \ 2n}\right]^2$$

and

$$f(\alpha)=\frac{8\pi}{3}\frac{1}{(1-\alpha)^2}\left[(2G-1)-\sum_1^\infty\frac{2u_n\ 2n\ \alpha^{2n+1}}{(2n-1)\ (2n+1)}\right.$$
$$-\alpha^3\left\{\pi\log 2-(2G-1)-\frac{\pi}{4}+\frac{\pi}{2}\log\frac{1}{\alpha}\right.$$
$$\left.\left.-\sum_1^\infty\frac{2u_n\ (2n+1)}{2n\ (2n+2)}\alpha^{2n}\right\}\right]$$
$$=\frac{1}{(1-\alpha)^2}\left[6.96957-\alpha^3\left(30.3008\log\frac{1}{\alpha}+9.08008\right)\right.$$
$$+1.48044\ \alpha^5+0.33045\ \alpha^7$$
$$+0.12494\ \alpha^9+0.06038\ \alpha^{11}+0.0337\ \alpha^{13}+\ .\ .\ .)$$

Spielrein gives a second formula for values of α between 0.5 and 1, but if we remember that $\frac{c}{2a}=\frac{1-\alpha}{1+\alpha}$, it is easy to show that this is equivalent to (70) and (70A), lacking the term in c^6/a^6.

In Table E is reproduced a collection of values of $f(\alpha)$ calculated by Spielrein for a number of values of α.

V. SELF-INDUCTANCE OF A CIRCULAR COIL OF RECTANGULAR CROSS SECTION

For the calculation of the inductance of a circular coil of rectangular cross section, whose dimensions b and c are relatively small compared with the mean radius a, the most accurate formula previously available has been that of Weinstein,[12] which appears as (88) in Scientific Paper No. 169. In the next volume of Wiedemann's Annalen, after that containing Weinstein's article, Stefan [13] published what is the same formula, but so arranged as to facilitate numerical calculations. (See (90) in Scientific Paper No. 169.) There is nothing in Stefan's article to show that he was acquainted with Weinstein's work.

[12] Wied. Ann., 21, p. 329; 1884.

[13] Wied. Ann., 22, p. 113; 1884.

The formula given by Stefan is reproduced here.

$$L=4\pi a n^2\left\{\left(1+\frac{3b^2+c^2}{96a^2}\right)\log\frac{8a}{\sqrt{b^2+c^2}}-y_1+\frac{b^2}{16a^2}y_2\right\} \tag{90}$$

Values of the two quantities y_1 and y_2 are given by Stefan in tables which he prepared to make his formula more generally useful. Changing his nomenclature to agree with that of this paper, the two equations given by him for the calculation of y_1 and y_2 are

$$y_1=\frac{\pi x}{3}-\frac{1}{12x^2}\log(1+x^2)-\frac{x^2}{12}\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\left(x-\frac{1}{x}\right)\tan^{-1}x-\frac{1}{12}$$

$$\left.\begin{aligned}y_2=\frac{1}{6}\Big[\frac{69}{20}+\frac{221}{60}\cdot\frac{1}{x^2}-\frac{1}{10x^4}\log(1+x^2)+\frac{x^2}{2}\log\left(1+\frac{1}{x^2}\right)\\ -\frac{8\pi x}{5}+\frac{16x}{5}\tan^{-1}x\Big]\end{aligned}\right. \tag{25A}$$

in which $x=b/c$, the ratio of the axial dimension of the cross section to the radial.

Stefan's formula and tables have been reproduced in a number of handbooks, including the Bureau of Standards' collection, with the statement that both y_1 and y_2 are unchanged, when b and c are interchanged; that is, that y_1 and y_2 are the same functions of c/b that they are of b/c.

This statement is true of y_1, but does not hold for y_2, as may be seen from the defining equations (25A).

MODIFICATION OF STEFAN'S FORMULA FOR THE CASE $(c>b)$

The formula (25A) shows that y_2 grows rapidly larger as c is increased relatively to b, and approaches infinity as its limit when the ratio b/c approaches zero. In such cases interpolation of the values of y_2 becomes difficult. This difficulty may, however, be avoided if, for the case $c>b$, we write Stefan's formula in the form [14]

$$L=4\pi a n^2\left\{\left(1+\frac{3b^2+c^2}{96a^2}\right)\log\frac{8a}{\sqrt{b^2+c^2}}-y_1+\frac{c^2}{16a^2}y_3\right\} \tag{26A}$$

In this equation the quantity y_3 is related to y_2 by the equation $y_3=b^2/c^2.y_2$, and sufficient values of y_3 are included in Appendix B to allow of accurate interpolation. The defining equation for y_3 is

$$\left.\begin{aligned}y_3=\frac{1}{6}\Big[\frac{69x^2}{20}+\frac{221}{60}-\frac{8\pi x^3}{5}+\frac{16x^3}{5}\tan^{-1}x\\ -\frac{1}{10x^2}\log(1+x^2)+\frac{x^4}{2}\log\left(1+\frac{1}{x^2}\right)\Big]\end{aligned}\right\} \tag{27A}$$

[14] This form of the equation, with tables for computation, was first given in the 1916 revision of Scientific Paper No. 169.

LYLE'S FORMULA

The Weinstein-Stefan formula was obtained by integrating the series expression (10), for the mutual inductance of two coaxial circles, over the retangular cross section of the coil in question. In this integration are included terms of second order only in c/a and b/a. If the dimensions of the cross section are small, relatively to the mean radius of the coil, this approximation will suffice. In a good many cases, however, the further terms are not negligible, and in any case it is desirable to be able to prove that they are negligible.

To carry out the integration of (14) so as to include higher-order terms is a difficult matter on account of the large number of terms which must be treated. In a recent paper Lyle[15] has shown how to simplify the work so that further terms may be obtained in Weinstein's formula, and has published the expressions for the terms of fourth and sixth order, together with tables for calculating the fourth-order term.

The author of the present paper has called Prof. Lyle's attention to an error in one of the coefficients of the sixth-order terms of the extension of (14) upon which the integration was performed, and he has very kindly repeated his work and supplied the correction of the single term affected.

In addition Prof. Lyle has been so good as to communicate to us additional tables, not heretofore published, with his permission for their incorporation with this paper. The following is quoted from Prof. Lyle's letter:

In a former paper I have extended Maxwell's and Weinstein's formula for the self-inductance of a circular coil of rectangular section to the sixth, and, following Stefan, have given tables by means of which the result, up to the fourth order, may easily be applied to the calculation of inductances. I have lately recalculated the figures given in one of these sets of tables and extended the latter to the sixth order.

Thus, if uniform current density over the whole section of the coil be assumed, its self-inductance may be written in the form

$$\left.\begin{aligned} L=4\pi a n^2\Big[\Big(1+m_1\frac{d^2}{a^2}+m_2\frac{d^4}{a^4}+m_3\frac{d^6}{a^6}\Big)\log\frac{8a}{d} \\ -l_0+l_1\frac{d^2}{a^2}+l_2\frac{d^4}{a^4}+l_3\frac{d^6}{a^6}\Big] \end{aligned}\right\} \qquad (28A)$$

in which

a is the mean radius of the coil
n the number of turns
$d^2=b^2+c^2$, where
b=the axial width of the coil
c=the radial depth of the coil

[15] Phil. Trans., 213A, pp. 421–435; 1914.

In Appendix C Lyle's values of $m_1, m_2, m_3, l_0, l_1, l_2,$ and l_3 are given for different values of c/b for thick coils—that is, those in which b is greater than c—and in Appendix D are given their values for different values of b/c for thin coils—that is, those in which b is less than c.

The following relations exist between Lyle's constants and the quantities y_1, y_2, and y_3 of formulas (90) and (26A):

$$l_0 = y_1$$
$$y_2 = 16\ l_1\ (1 + c^2/b^2)$$
$$y_3 = 16\ l_1\ (1 + b^2/c^2)$$

A second form was given by Lyle to his formula in his original paper. Formula (28A) has, however, the advantage that it differs from Stefan's formula only in that the fourth and sixth order terms are added. Therefore, in any given case, a rough preliminary calculation will suffice to show whether the higher-order terms are of importance. In a great many cases it may thus be shown that Stefan's formula is sufficient, and only in extreme cases (coils of relatively very large cross section) will the sixth and higher order terms be important. In the latter case no other formula is yet available for obtaining such an accurate value with so little labor. Lyle's formula, however, fails for the case of coils whose length b is considerably greater than the mean radius a.

BUTTERWORTH'S FORMULA FOR SELF-INDUCTANCE OF A LONG MULTIPLE-LAYER COIL

Butterworth [16] has developed a series formula for the case of a long coil whose winding depth is rather large. Its region of convergence, for coils whose length is greater than four times the outer radius and whose winding depth is greater than about one-fifth the mean radius, covers the case of coils whose cross section is so great that Stefan's and Lyle's formulas are not sufficiently convergent.

Changing Butterworth's nomenclature to agree with that previously employed in this paper, his method may be summarized as follows:

Writing L_1 = the inductance of an infinite cylindrical current sheet of the same mean radius as the coil, the inductance of a finite solenoid having the same length as the given coil is $L_2 = KL_1$,[17]

[16] Proc. Phys. Soc., London, 27, p. 371; 1915. [17] Scientific Paper No. 169, p. 119, formula (75).

and we may write for the inductance of the actual coil, $L = L_2 + \Delta L$, in which

$$\left.\begin{aligned}\frac{\Delta L}{L_1} = -\frac{1}{3}\frac{c}{a}\Bigg[1 - \frac{c}{4a} - \frac{1}{2\pi}\frac{c}{b}\left(\log\frac{8a}{c} - \frac{23}{12}\right) + \frac{1}{160\pi}\frac{c^3}{a^3}\frac{a}{b}\Bigg(\log\frac{8a}{c} \\ -\frac{1}{20}\Bigg) - \frac{1}{4}\frac{c}{a}\frac{a^2}{b^2}\left(1 - \frac{7}{4}\frac{a^2}{b^2} + \frac{17}{4}\frac{a^4}{b^4} - \cdots\right) \\ -\frac{1}{96}\frac{c^3}{a^3}\frac{a^2}{b^2}\left(1 - \frac{39}{10}\frac{a^2}{b^2} + \cdots\right) - \cdots\Bigg]\end{aligned}\right\}\quad(29A)$$

EXAMPLES

Example 9.—As an example, we may consider one of the coils treated by Butterworth, viz, $b/a = 4$, $c/a = 0.2$, the value of the mean radius and the number of turns being so chosen, for simplicity, that $\frac{\pi^2 n^2 a^3}{b^2} = 1\ 000\ 000$. This gives $L_1 = 16$ millihenrys.

The value of K for $2a/b = 0.5$ is 0.818136,[18] and, therefore, $L_2 = 13.09017$.

$$\frac{\Delta L}{L_1} = -\frac{1}{15}\Bigg[1 - .05 - \frac{1}{40\pi}\left(\log_e 40 - \frac{23}{12}\right) + \frac{1}{80000\pi}\left(\log_e 40 - \frac{1}{20}\right) \\ -\frac{1}{320}\left(1 - \frac{7}{64} + \frac{17}{1024}\right) - \frac{1}{192000}\left(1 - \frac{39}{160}\right)\Bigg]$$

$$= -\frac{1}{15}\left(1 - .05 - .014103 + .000014 - .002845 - .000005\right) = -.062204$$

or $\Delta L = -0.99526$ mh.

The only other formula available as a check is Rosa's formula (91), which gives the result $\Delta L = -1.0207$ mh, or a difference from Butterworth's formula of 0.0255 mh, or about two parts in a thousand of the total inductance. Lyle's formula can not be used in this case.

Butterworth explains the above difference as being due to the neglect of the curvature in the geometric mean-distance formulas used in obtaining B_s in Rosa's formula (91).

For the coil $b = 10$, $c = 1$, $a = 10$, $n = 1000$ we find

$\frac{L}{4\pi} = 15.53984$ Lyle's formula to fourth order

$= 15.54071$ Lyle's formula to sixth order

$= 15.5361$ Rosa's formula (91) (see example 66 of Scientific Paper No. 169).

Butterworth's formula is not applicable to a coil as short as this.

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[18] Scientific Paper No. 169, Table XXI.

Here, again, Rosa's formula gives a result somewhat too small, although the difference in this case is only 3 in 10 000.

These checks on Rosa's formula are valuable, since it is the only formula yet available in the region where neither Lyle's formula (28A) nor Butterworth's formula (29A) converges well.

The error due to the neglect of the curvature in applying the geometric mean-distance formulas in obtaining a result by Rosa's formula will not usually be regarded as important. It would not, however, be difficult to obtain a correction for this effect, although the formula thus obtained would not be so simple to use as (91).

NOTE ON COHEN'S APPROXIMATE FORMULA (92)

Cohen's formula (92) is applicable to a coil of several layers. The formula presupposes that the rectangular cross section is divided into a number of equal axial rectangles equal to the number of layers, and the formula for the inductance involves the radii of the layers.

Butterworth has shown that assuming a coil of given cross-sectional dimensions the inductance as calculated by (92) comes out quite different according to the number of layers assumed in the cross section. He goes on to show that this may be explained by the fact that in the derivation of (92) the approximations made at certain points of the demonstration are not sufficient to give the accuracy claimed by Cohen. However, for a certain choice of the number of sections, different in each case, and not a priori determinate, the result may lie quite close to the true result.

As an example of these points Butterworth has calculated, by means of Cohen's formula, the inductance of the coil in the example next preceding but one for different assumptions with regard to the quantity m in (92).

$m=$	1	2	3	4	5	10	infinite.	
$L=$	12.70	12.11	12.07	12.06	12.09	12.14	12.19	millihenrys.

The correct value of the inductance for this case is, to four significant figures, 12.09 millihenrys.

VI. SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS

Formulas are given in section 8 of Scientific Paper No. 169 for the calculation of the self-inductance of straight wires of different cross section and for the mutual inductance of two such conductors when placed parallel to one another.

Such cases are easily treated by the method of the geometric mean distance. For the calculation of the self-inductance of a straight conductor of any desired cross section we have only to calculate the mutual inductance of two parallel straight filaments placed at a distance apart equal to the geometric mean distance of the cross section from itself.

Similarly the mutual inductance of two parallel straight conductors is equal to the mutual inductance of two parallel straight filaments whose distance apart is taken equal to the geometric mean distance of the area of cross section of one conductor from the cross section of the other.

The calculation of the self-inductance of any straight conductor or any pair of parallel straight conductors may, therefore, be accomplished by substituting the proper geometric mean distance for R in the formula

$$M = 2l\left[\log\frac{2l}{R} - 1 + \frac{R}{l}\right] \tag{30A}$$

which is the expression (99) of Scientific Paper No. 169 for the mutual inductance of two filaments of length l, at a distance R apart, which is small compared with their length. In most practical cases the last term of (30A) may safely be neglected.

To aid in making calculations by this method, the formulas for geometric mean distance, in a number of important cases, are presented in section 9 of Scientific Paper No. 169.

The inductance of a circuit composed of a number of linear conductors may, in general, be found by taking the sum of the self-inductances of the individual conductors and the mutual inductances of each wire on all the others. In the case of a return circuit—that is, a circuit consisting of two parallel wires in which the direction of the current in one is opposed to the direction of the current in the other—the inductance of the remainder of the circuit being negligible in comparison, $L = L_1 + L_2 - 2M$, in which L_1 and L_2 are the self-inductances of the two wires and M is their mutual inductance.

This equation, taken in connection with (30A), if the last term in the latter be neglected, gives as a general formula for a return circuit

$$L = 2l\,[2\log R_{12} - \log R_1 - \log R_2] \tag{31A}$$

in which R_1 and R_2 are, respectively, the geometric mean distances of the cross sections of the two wires on themselves, and R_{12} is the geometric mean distance of the cross sections of the two wires.

If the cross sections of the two wires are the same, this formula becomes

$$L = 4l \log \frac{R_{12}}{R_1} \tag{32A}$$

These formulas have been employed, in conjunction with those of section 8 of Scientific Paper No. 169, to obtain the inductance of a considerable number of the special circuits treated in that section.

INDUCTANCE OF SHUNTS

In recent years the use of shunts of large carrying capacity for measuring the current in alternating-current circuits has lent a very practical importance to a knowledge of the inductance in such cases.

As such shunts are constructed, it is true, the inductance is very small (of the order of a few abhenrys), but since the resistance is often less than a thousandth of an ohm, the phase angle between electromotive force and current may, even with such a small inductance, depart widely from zero, so that the assumption that such apparatus is noninductive may cause very serious error in the measurement of current and power.

We will consider here shunts of two main types—(*a*) shunts of flat metal strip, bent so as to form a return circuit whose parallel elements are very close together, and (*b*) tubular shunts.

(*a*) *Shunts of flat strip.*—If we neglect the thickness of the strip, in comparison with its width and the distance apart of the two parallel conductors, we may calculate $\log R_1$ and $\log R_2$ from (123) and $\log R_{12}$ by (132). The expression resulting from the substitution of these quantities in (31A) may, however, be put in a more serviceable form, if we expand the logarithmic and inverse trigonometric functions. Putting w for the width of the strip and g for the distance between the strips, then, if g/w is small,

$$L = 4l\left[\frac{\pi g}{w} + \frac{g^2}{w^2}\log\frac{g}{w} - \frac{3}{2}\frac{g^2}{w^2} - \frac{1}{12}\frac{g^4}{w^4} - \cdot\ \cdot\ \cdot\right] \tag{33A}$$

Since g/w is not always small, we should, in the case of strips at some distance apart, use the exact expressions for $\log R_1$ and $\log R_{12}$.

More often, however, we will be unable to neglect the thickness of the strip. Silsbee [19] has recently treated this case by calculating $\log R_1$ by (124), the formula for the geometric mean distance of a

[19] Scientific Paper No. 281, pp. 375–421; 1916.

rectangle from itself, and log R_{12} by Gray's formula for two parallel rectangles.[20] Expanding these quantities in series involving b/w, and g/w, the thickness of the strip being denoted by b and the thickness of the insulating space between the two strips by g, he finds finally

$$L=4l\left[\frac{\pi}{3}(3\beta-\delta)-\frac{25}{12}\beta^2-\frac{1}{12}\beta^4-\frac{1}{12}\beta^2\delta^2\right.$$
$$\left.+\frac{1}{12\delta^2}(\alpha^4\log\alpha-2\beta^4\log\beta+\gamma^4\log\gamma-2\delta^4\log\delta)\right] \qquad (34A)$$

in which

$$\alpha=\frac{2b+g}{w},\ \beta=\frac{b+g}{w},\ \gamma=\frac{g}{w},\ \delta=\frac{b}{w}.$$

This expression reduces to the preceding expression (33A) if we let δ approach zero.

To show that, in practice, the difference between the two formulas (33A) and (34A) may be large, we may consider an example given by Silsbee:

$$l=35.62,\ b=0.1064,\ w=4.986,\ g=0.0336.$$

Here, although the metal used is only about 1 mm thick, the thickness of the insulation between the two legs of the shunt is only about one-third of this, so that b/w is about three times as great as g/w, instead of being negligible, as (33A) supposes.

Making the calculation by Silsbee's formula, the terms taken in order are as follows:

$$4l\ (0.065867-0.001642-7\times10^{-8}-3\times10^{-8}-0.003282$$
$$+0.001288-0.000002+0.000292)$$
$$=4l\ (0.062521)=8.91\times10^{-9}\text{ henry}$$

The value found by (33A) is

$$4l\ (0.021171-0.000227-0.000068)=4l\ (0.020876)$$

which is less than one-third of the correct value. This example, then, illustrates the fact that formula (33A) should be used only in those cases where the quantity b/w can be shown to be negligible in comparison with g/w. Such cases are likely to be rare, except when the distance of the strips apart is comparable with the width of the strip, a condition not conducive to good design. Formula (33A) is, therefore, of limited usefulness.

Silsbee has also treated the case where the return circuit is made up of two strips of different thickness. Suppose two strips

[20] This Bulletin, 3, p. 6, 1907, formula (8).

of the same width w, but of different thicknesses b and c, the thickness of the insulating space between them being g, then if we denote by L_1 the inductance of a length l of the conductor of thickness b, the other conductor serving as a return, and by L_2 the inductance of a length l of the conductor of thickness c, with its return through the other conductor, Silsbee shows (page 378 of his article) that $L_1 = 2l\ (\log R_{12} - \log R_1)$, where R_1 is the geometric mean distance of strip b from itself and R_{12} is the geometric mean distance of the cross sections of the two strips.

Calculating these geometric mean distances by the formulas used in the previous case, and expanding the resulting expressions, he finds

$$\left.\begin{aligned} L_1 = 2l\Big[\pi\left(\gamma + \frac{\delta}{6} + \frac{\eta}{2}\right) - 25\left(\frac{\eta\delta}{24} + \frac{\gamma\alpha}{12} + \frac{\eta^2}{36} + \frac{\delta^2}{72}\right) + \\ \frac{1}{12\delta\eta}\left(\alpha^4 \log \alpha - \kappa^4 \log \kappa - \lambda^4 \log \lambda + \gamma^4 \log \gamma - 2\delta^3\eta \log \delta\right)\Big] \end{aligned}\right\} \quad (35A)$$

where

$$\alpha = \frac{b+c+g}{w},\ \gamma = \frac{g}{w},\ \delta = \frac{b}{w},\ \eta = \frac{c}{w},\ \kappa = \frac{b+g}{w},\ \lambda = \frac{c+g}{w}$$

The inductance L_2 is found by interchanging the letters δ and η, and the inductance of the complete circuit is $L = L_1 + L_2$. Equation (34A) can be derived from (35A) by letting $b = c$ in the complete expression for $(L_1 + L_2)$.

(*b*) *Tubular shunts.*—These are generally constructed of two concentric tubes of resistance metal, one of which forms a return for the other, the two potential leads being attached at points which differ in different designs. In case one or both of the potential leads are so disposed that an electromotive force is induced in the lead, this will change the effective reactance of the shunt, which may be defined as the ratio of the quadrature component of the voltage between those ends of the potential leads which are attached to the measuring apparatus, to the current in the shunt. For a full treatment of this question the reader is referred to page 378 of Silsbee's article.

Silsbee has attacked the problem by calculating directly the linkages of the magnetic flux with the different elements of the shunt, and has given in series form the inductance for four practical designs of tubular shunt. In accordance with his suggestion that these formulas more clearly illustrate the procedure which may be adopted in deriving formulas for similar cases not included in these examples, if their relation to the geometric mean distances involved be made clear, this method of treatment will here be outlined.

The geometric mean distance of an annulus of inner radius a_2 and outer radius a_1 is given by formula (129)

$$\log R_1 = \log a_1 - \frac{a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2} + \frac{1}{4} \frac{(3a_2^2 - a_1^2)}{(a_1^2 - a_2^2)}$$

which Silsbee develops in the series form

$$\log R_1 = \log a_1 - \frac{t}{3} + \frac{t^3}{30} + \frac{t^4}{40} + \cdots \tag{36A}$$

useful in the case when the ratio $t = (a_1 - a_2)/a_1$ of the thickness to the outer radius is small.

The geometric mean distance of an annulus from any area entirely inside of it is by (135), if for the outer radius we put a_3 and for the inner radius a_4,

$$\log R_{12} = \frac{a_3^2 \log a_3 - a_4^2 \log a_4}{a_3^2 - a_4^2} - \frac{1}{2}$$

or

$$\log R_{12} = \log a_3 - \frac{s}{2} - \frac{s^2}{12} + \frac{s^4}{60} + \cdots \tag{37A}$$

in terms of the quantity $s = \frac{a_3 - a_4}{a_3}$. This formula gives the geometric mean distance of the cross sections of two concentric rings.

For the design a of tubular shunt, treated by Silsbee (p. 400), in which one tube forms a return circuit for the other, and the potential leads are brought out in the same plane at a distance l from the junction of the tubes,

$$\begin{aligned} L &= L_1 + L_2 - 2M \\ &= 2l[2 \log R_{12} - \log R_1 - \log R_2) \\ &= 2l\left[2\left(\log a_3 - \frac{s}{2} - \frac{s^2}{12} + \frac{s^4}{60} + \cdots\right)\right. \\ &\qquad -\left(\log a_1 - \frac{t}{3} + \frac{t^3}{30} + \frac{t^4}{40} + \cdots\right) \\ &\qquad \left.-\left(\log a_3 - \frac{s}{3} + \frac{s^3}{30} + \frac{s^4}{40} + \cdots\right)\right] \end{aligned}$$

This may be reduced by the relations

$$\log a_3 - \log a_1 = \log \frac{a_3}{a_4} + \log \frac{a_4}{a_1}$$

$$\log \frac{a_4}{a_3} = \log \frac{[a_3 - (a_3 - a_4)]}{a_3} = \log (1-s)$$

$$\log \frac{a_1}{a_4} = \log \frac{[a_4 - (a_4 - a_1)]}{a_4} = \log (1-u)$$

and expanding $\log (1-s)$ and $\log (1-u)$, we obtain Silsbee's equation

$$L = l\left[\frac{2}{3}t + 2u + \frac{2}{3}s + u^2 + \frac{2}{3}s^2 + \frac{3}{5}s^3 + \frac{2}{3}u^3 - \frac{t^3}{15} + \cdots\right] \tag{38A}$$

For his case b, in which the potential leads are attached to the outer tube at points a distance l apart and are carried away at right angles to the axis of the tube

$$\left.\begin{aligned} L &= L_2 - M = 2l(\log R_{12} - \log R_2) \\ &= 2l\left[\left(\log a_3 - \frac{s}{2} - \frac{s^2}{12} + \frac{s^4}{60} + \cdots\right) - \left(\log a_3 - \frac{s}{3} + \frac{s^3}{30} + \frac{s^4}{40}\right)\right] \\ &= l\left(-\frac{s}{3} - \frac{s^2}{6} - \frac{s^3}{15} - \frac{s^4}{60} + \cdots\right) \end{aligned}\right\} \tag{39A}$$

Silsbee's case c is like the preceding, except that the potential leads are attached to the inner tube and are brought perpendicularly out through holes in the outer tube. For this arrangement

$$\begin{aligned} L &= L_1 - M = 2l[\log R_{12} - \log R_1] \\ &= 2l\left[\left(\log a_3 - \frac{s}{2} - \frac{s^2}{12} + \frac{s^4}{60}\right) - \left(\log a_1 - \frac{t}{3} + \frac{t^3}{30} + \frac{t^4}{40}\right)\right] \end{aligned}$$

which, remembering that

$$\log \frac{a_1}{a_3} = \log \frac{a_1}{a_4} + \log \frac{a_4}{a_3} = \log (1-u) + \log (1-s)$$

gives on expansion of the logarithms

$$L = l\left[2u + s + \frac{2}{3}t + u^2 + \frac{5}{6}s^2 + \cdots\right] \tag{40A}$$

Silsbees's final case d uses potential leads attached to the inner tube at a distance l apart, one of them being carried away inside the inner tube parallel to its axis. It is necessary, therefore, in this case to take into account the electromotive force induced in this lead, which depends upon the geometric mean distance of the inner tube on an area inside of it.

The inductive effect of the outer tube on the inner tube is equal and opposite to the effect of the outer tube on the potential lead, so that we find for the inductance simply

$$\left.\begin{aligned} L &= L_1 - M_e = 2l\,[\log R_e - \log R_1] \\ &= 2l\left[\left(\log a_1 - \frac{t}{2} - \frac{t^2}{12} + \frac{t^4}{60} + \cdots\right) - \left(\log a_1 - \frac{t}{3} + \frac{t^3}{30} + \frac{t^4}{40} + \cdots\right)\right] \\ &= l\left[-\frac{t}{3} - \frac{t^2}{6} - \frac{1}{15}t^3 - \frac{t^4}{60} + \cdots\right] \end{aligned}\right\} \quad (41\text{A})$$

It is to be noted that in cases *b* and *d* the inductance comes out negative; that is, that the potential between the terminals lags behind the current in phase.

The method used in deriving formulas (38A) to (41A), inclusive, may be used to derive the inductance in other cases not here treated.

WASHINGTON, July 27, 1917.

APPENDIX

TABLE A.—Values of Zonal Harmonics for Use with Formulas (10A) to (19A)

μ	$P_2(\mu)$	Δ_1	Δ_2	$P_4(\mu)$	Δ_1	Δ_2	Δ_3
0.00	−0.5000	+ 38	74	0.3750	− 93	−185	+12
.05	− .4962	112	76	.3657	− 278	−173	16
.10	− .4850	188	74	.3379	− 451	−157	22
.15	− .4662	262	76	.2928	− 608	−135	30
.20	− .4400	338	74	.2320	− 743	−105	37
.25	− .4062	+ 412	76	.1577	− 848	− 68	41
.30	− .3650	488	74	+ .0729	− 916	− 27	50
.35	− .3162	562	76	− .0187	− 943	+ 23	56
.40	− .2600	638	74	− .1130	− 920	+ 79	63
.45	− .1962	712	76	− .2050	− 841	142	67
.50	− .1250	+ 788	74	− .2891	− 699	209	77
.55	− .0462	862	76	− .3590	− 490	286	81
.60	+ .0400	938	74	− .4080	− 204	367	90
.65	.1338	1012	76	− .4284	+ 163	457	94
.70	.2350	1088	74	− .4121	+ 620	551	102
.75	.3438	+1162	76	− .3501	1171	653	108
.80	.4600	1238	74	− .2330	1824	761	116
.85	.5838	1312	76	− .0506	2585	878	120
.90	.7150	1388	74	+ .2079	3462	997	....
.95	.8538	+1462	....	.5541	4459	....	....
1.00	1.0000	....	....	+1.0000	....	....	....

μ	$P_6(\mu)$	Δ_1	Δ_2	Δ_3	Δ_4	μ	$P_6(\mu)$	μ	$P_6(\mu)$
0.00	−0.3125	+ 163	+ 311	− 43	− 27	0.70	−0.1253	0.90	−0.2412
.05	− .2962	474	268	− 70	− 19	.71	− .1578	.91	− .1802
.10	− .2488	742	198	− 89	− 20	.72	− .1899	.92	− .1077
.15	− .1746	940	109	−109	− 7	.73	− .2214	.93	− .0229
.20	− .0806	1049	0	−116	0	.74	− .2518	.94	+ .0751
.25	+ .0243	1049	−116	−116	+11	.75	− .2808	.95	.1875
.30	.1292	933	−232	−105	+20	.76	− .3081	.96	.3151
.35	.2225	701	−337	− 85	+41	.77	− .3333	.97	.4590
.40	.2926	+ 364	−422	− 44	47	.78	− .3559	.98	.6204
.45	.3290	− 58	−466	+ 3	73	.79	− .3756	.99	.8003
.50	.3232	− 524	−463	76	85	.80	− .3918	1.00	+1.0000
.55	.2708	− 987	−387	161	110	.81	− .4041	....	....
.60	.1721	−1374	−226	271	129	.82	− .4119	....	....
.65	+ .0347	−1600	+ 45	400	153	.83	− .4147	....	....
.70	− .1253	−1555	445	553	179	.84	− .4120	....	....
.75	− .2808	−1110	998	732	207	.85	− .4030	....	....
.80	− .3918	− 112	1730	939	280	.86	− .3872	....	....
.85	− .4030	+1618	2669	1219	....	.87	− .3638	....	....
.90	− .2412	4287	3888	....	....	.88	− .3322	....	....
.95	+ .1875	+8125	....	....	....	.89	− .2916	....	....
1.00	+1.0000	....	....	....	....	....	....	....	....

TABLE B.—Values of y_2 and y_3 in Formulas (90) and (26A)

[Radial Depth of Cross Section Greater than the Axial Breadth]

$\frac{b}{c}$	y_2	y_3	Δ_1	Δ_2
0	∞	0.59722	136	237
0.05	239.43	.59858	373	191
.10	60.231	.60231	564	153
.15	27.020	.60795	717	124
.20	15.378	.61512	841	100
.25	9.9765	.62353	941	80
.30	7.0327	.63294	1021	67
.35	5.2502	.64315	1088	54
.40	4.0876	.65403	1142	47
.45	3.2861	.66545	1189	41
.50	2.7093	.67734	1230	37
.55	2.2798	.68964	1267	35
.60	1.9509	.70231	1302	35
.65	1.6931	.71533	1337	33
.70	1.4871	.72870	1370	34
.75	1.3198	.74240	1404	36
.80	1.1819	.75644	1440	34
.85	1.0669	.77084	1474	39
.90	.9698	.78558	1513	38
.95	.8872	.80071	1551	
1.00	.8162	.81622		

TABLE C.—Constants in Lyle's Formula (28A), Thick Coils, b>c

Communicated by Prof. Lyle.

c/b	100 m_1	10^4 m_2	10^6 m_3	l_0	100 l_1	10^4 l_2	10^6 l_3	c/b
0.00	3.125000	−9.7656	76.29	0.5000000	0.781250	6.5104	−69.30	0.00
.025	3.123699	−9.7463	76.01	.5252663	.783689	6.4896	−68.94	.025
.05	3.119805	−9.6886	75.14	.5489951	.790984	6.4274	−67.88	.05
.10	3.104373	−9.4613	71.79	.5924342	.819830	6.1838	−63.77	.10
.15	3.079157	−9.0942	66.50	.6310248	.866769	5.7944	−57.35	.15
.20	3.044872	−8.6040	59.67	.6652018	.930230	5.2827	−49.20	.20
.25	3.002451	−8.0115	51.78	.6953236	1.008207	4.6774	−39.99	.25
.30	2.952982	−7.3402	43.36	.7217163	1.098406	4.0102	−30.44	.30
.35	2.897643	−6.6144	34.88	.7446891	1.198386	3.3128	−21.17	.35
.40	2.837644	−5.8573	26.77	.7645392	1.305696	2.6145	−12.70	.40
.45	2.774168	−5.0903	19.34	.7815523	1.417987	1.9404	− 5.39	.45
.50	2.708333	−4.3316	12.82	.7960019	1.533097	1.3107	+ .55	.50
.55	2.641155	−3.5961	7.32	.8081473	1.649113	.7400	+ 5.05	.55
.60	2.573529	−2.8951	+ 2.89	.8182324	1.764399	+ .2378	8.17	.60
.65	2.506224	−2.2366	− .51	.8264842	1.877606	− .1912	10.03	.65
.70	2.439877	−1.6260	− 2.94	.8331124	1.987664	− .5460	10.82	.70
.75	2.375000	−1.0656	− 4.52	.8383088	2.093763	− .8287	10.73	.75
.80	2.311992	− .5563	− 5.37	.8422476	2.195318	−1.0437	9.97	.80
.85	2.251149	− .0970	− 5.61	.8450864	2.291944	−1.1966	8.73	.85
.90	2.192680	+ .3141	− 5.37	.8469663	2.383421	−1.2939	7.16	.90
.95	2.136717	+ .6800	− 4.75	.8480134	2.469663	−1.3425	5.42	.95
1.00	2.083333	+1.0037	− 3.85	.8483397	2.550686	−1.3490	3.62	1.00
1.05	2.032551	+1.2888	− 2.75	.8480444	2.626593	−1.3199	1.83	1.05
1.10	1.984351	+1.5387	− 1.53	.8472152	2.697542	−1.2613	+ .13	1.10

TABLE D.—Constants in Lyle's Formula (28A), Thin Coils, c>b

Communicated by Prof. Lyle.

b/c	$100\,m_1$	$10^4\,m_2$	$10^6\,m_3$	l_0	$100\,l_1$	$10^4\,l_2$	$10^6\,l_3$	b/c
0.00	1.041667	2.3872	14.97	0.5000000	3.732639	4.1667	17.05	0.00
.025	1.042978	2.3913	15.01	.5252663	3.732506	4.1614	17.00	.025
.05	1.046862	2.4035	15.13	.5489951	3.731810	4.1434	16.81	.05
.10	1.062294	2.4508	15.59	.5924342	3.727159	4.0584	15.88	.10
.15	1.087510	2.5237	16.25	.6310248	3.716052	3.8971	14.13	.15
.20	1.121795	2.6140	16.96	.6652018	3.696644	3.6550	11.60	.20
.25	1.164216	2.7115	17.56	.6953236	3.667845	3.3359	8.45	.25
.30	1.213685	2.8057	17.91	.7217163	3.629250	2.9510	+4.98	.30
.35	1.269024	2.8859	17.89	.7446891	3.581036	2.5161	+1.49	.35
.40	1.329023	2.9430	17.42	.7645392	3.523847	2.0499	−1.73	.40
.45	1.392498	2.9694	16.49	.7815523	3.458662	1.5715	−4.42	.45
.50	1.458333	2.9601	15.11	.7960019	3.386676	1.0990	−6.44	.50
.55	1.525512	2.9119	13.36	.8081473	3.309190	+ .6479	−7.68	.55
.60	1.593137	2.8239	11.33	.8182324	3.227522	+ .2308	−8.16	.60
.65	1.660442	2.6971	9.11	.8264842	3.142942	− .1431	−7.93	.65
.70	1.726790	2.5337	6.83	.8331124	3.056619	− .4677	−7.09	.70
.75	1.791667	2.3372	4.58	.8383088	2.969599	− .7400	−5.77	.75
.80	1.854675	2.1114	+2.44	.8422476	2.882783	− .9592	−4.09	.80
.85	1.915518	1.8608	+ .48	.8450864	2.796929	−1.1265	−2.21	.85
.90	1.973987	1.5898	−1.24	.8469663	2.712655	−1.2448	− .23	.90
.95	2.029950	1.3028	−2.69	.8480134	2.630449	−1.3175	+1.74	.95
1.00	2.083333	1.0037	−3.85	.8483397	2.550686	−1.3490	+3.62	1.00
1.05	2.134116	.6963	−4.71	.8480444	2.473638	−1.3437	+5.34	1.05
1.10	2.182315	.3839	−5.28	.8472152	2.399492	−1.3062	+6.86	1.10

TABLE E.—Values of f (α) in Formula (24A)

α	f (α)	α	f (α)	α	f (α)
0	6.969573	0.35	14.19157	0.70	28.08799
0.05	7.715806	.40	15.64577	.75	31.20646
.10	8.555811	.45	17.2339	.80	34.91552
.15	9.487594	.50	18.9740	.85	39.52880
.20	10.51246	.55	20.8897	.90	45.74241
.25	11.63398	.60	23.01363	1.00	∞
.30	12.85776	.65	25.39097		