CALCULATION OF THE CONSTANTS OF PLANCK'S RADIATION EQUATION: AN EXTENSION OF THE THEORY OF LEAST SQUARES

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During the past few years several methods have been proposed for determining from experimental measurements the constants of Planck's equation for the distribution of energy in the spectrum of a uniformly heated inclosure or black body. The equation is as follows:

$$E = c_1 \lambda^{-5} (e^{\frac{c_2}{\lambda\theta}} - 1)^{-1}$$

In this equation E is a measure of the energy, λ the wave length, θ the absolute temperature, c_1 is a constant depending on the units in which E is measured and gives the scale of ordinates, c_2 is a constant which affects the shape of the curve and is sometimes called the "constant of spectral radiation." On account of the ease with which it may be eliminated from computations and its minor importance in indicating the actual distribution of energy, passing mention is usually deemed sufficient for c_1. The determination of c_2 has properly been the center of most concern, and several methods of computing it from observed data on E and λ at constant temperature or E and θ at constant wave length have been offered.[1] Recent discussions have been devoted to observations at constant temperature, the curve in this case being called an "isothermal," and the methods of solution differ mainly in the manner of combining observations and in the use of correction terms to the transcendental expression for c_2 in terms of the other magnitudes involved. Some variations have appeared in the results given by these different methods. C. E. Van Orstrand has suggested as a remedy a solution by least squares. The carrying out of this suggestion is one of the purposes of this paper. A

[1] A concise summary of current methods and a detailed explanation of a new one may be found in this Bulletin, **13**, p. 535, 1916 (Scientific Paper No. 287), by J. H. Dellinger; see also Buckingham and Dellinger, this Bulletin, **7**, p. 393, 1911; W. W. Coblentz, this bulletin, **10**, p. 2, 1914; W. W. Coblentz, this Bulletin, **13**, p. 459, 1916 (Scientific Paper No. 284). Copious references to early German methods of attack are given in Coblentz's first paper. Phys. Rev. **28**, p. 466, 1909; **32**, p. 591, 1911; Ann. d. Phys., **309**, p. 277, and p. 649, 1901; **311**, p. 192, 1901.

method of attack is indicated and a least-square solution for the constants made, using some of the data that have been used in the past to determine c_2 by other methods. Theoretical reasons why the results of some of the previous investigations differ will also be given.

THE METHOD OF ATTACK

The rationale of the least-square reduction employed herein is as follows. Consider the Planck equation as stated above,

$$E = c_1 \lambda^{-5} (e^{\frac{c_2}{\lambda\theta}} - 1)^{-1}$$

A set of n observations being given on E and λ at constant temperature θ, it is required to determine the most probable values of the constants c_1 and c_2.

The function may be written,

$$E = c_1 \lambda^{-5} e^{\frac{c_2}{\lambda\theta}} (1 - e^{-\frac{c_2}{\lambda\theta}})^{-1}$$

If E_1, λ_1; E_2, λ_2; * * * E_n, λ_n are the n observations all of equal weight, the observation equations are

$$\begin{aligned} E_1 &= c_1 \lambda_1^{-5} e^{\frac{c_2}{\lambda_1\theta}} (1 - e^{-\frac{c_2}{\lambda_1\theta}})^{-1} \\ E_2 &= c_1 \lambda_2^{-5} e^{\frac{c_2}{\lambda_2\theta}} (1 - e^{-\frac{c_2}{\lambda_2\theta}})^{-1} \\ &\quad * \qquad\qquad * \qquad\qquad * \\ E_n &= c_1 \lambda_n^{-5} e^{\frac{c_2}{\lambda_n\theta}} (1 - e^{-\frac{c_2}{\lambda_n\theta}})^{-1} \end{aligned} \tag{1}$$

In these equations the parentheses on the right-hand side containing the exponential is relatively insensitive to a change in c_2 as long as c_2 is a fairly close approximation to its proper value. The equations may be made linear and explicit in log c_1 and c_2 by substituting in the parentheses a reasonably approximate value c_2' for c_2 which may be determined by trial or by any method whatever, taking the logarithms of both sides and properly weighting the transformed observation equations.

Thus, after taking logarithms and transposing,

$$\begin{aligned} &\log c_1 - \frac{c_2}{\lambda_1\theta} = \log E_1 + 5 \log \lambda_1 + \log (1 - e^{-\frac{c_2'}{\lambda\theta}}), \text{ weight } E_1^2 \\ &\log c_1 - \frac{c_2}{\lambda_2\theta} = \log E_2 + 5 \log \lambda_2 + \log (1 - e^{-\frac{c_2'}{\lambda_2\theta}}), \text{ weight } E_2^2 \\ &* \qquad * \qquad * \qquad * \qquad * \qquad * \\ &\log c_1 - \frac{c_2}{\lambda_n\theta} = \log E_n + 5 \log \lambda_n + \log (1 - e^{-\frac{c_2'}{\lambda_n\theta}}), \text{ weight } E_n^2 \end{aligned} \tag{2}$$

The logarithmic transformation employed here is by no means new and is quite frequently used as a medium for reducing data by least squares. However, while weighting the observation equations so transformed is evidently necessary in order to make a proper least-square solution, the principle is generally neglected in texts and papers dealing with practical means of handling observation equations to make them linear in the quantities sought, and consequently a sort of inherent inaccuracy depending on the scheme of reduction exists in many commonly accepted current methods of which the above is one.

It may be well to justify the principle here. Consider any observed value E_i. Let its probable error be r_i. Let $\varphi(E_i)$ be any function of E_i, and R_i be the probable error of $\varphi(E_i)$.

Then [2]

$$R_i^2 = \left(\frac{\partial \varphi(E_i)}{\partial E_i}\right)^2 r_i^2 \tag{3}$$

Now the relative weights of two quantities are inversely as the squares of their probable errors.[3] That is, if P_i be the weight of $\varphi(E_i)$ and p_i be the weight of E_i, then

$$P_i : p_i :: \frac{1}{R_i^2} : \frac{1}{r_i^2}$$

or, (4)

$$P_i = \frac{p_i}{\left[\frac{\partial \varphi(E_i)}{\partial E_i}\right]^2}$$

In the particular case above where equations I are all of unit weight or reduced to a unit weight basis,

$$p_i = 1, \text{ and } \varphi(E_i) = \log E_i$$

thus P_i, which is the weight of any one of equations II becomes,

$$P_i = \frac{1}{\left(\frac{\partial \log E_i}{\partial E_i}\right)^2} = E_i^2$$

which was to be shown.

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[2] Merriman, Least Squares, pp. 78–79; 1913.

[3] Merriman, Least Squares, p. 60, 1913; Johnson, W. W., Theory of Errors and Method of Least Squares p. 53, 1892.

Equations 2 may be condensed into the following form:

$$\begin{aligned} &\log c_1 - m_1 c_2 = M_1, \text{ weight } E_1^2 \\ &\log c_1 - m_2 c_2 = M_2, \text{ weight } E_2^2 \\ &\quad * \qquad * \qquad * \qquad * \\ &\log c_1 - m_n c_2 = M_n, \text{ weight } E_n^2 \end{aligned} \tag{5}$$

in which all quantities except $\log c_1$ and c_2 are either known or may be determined from the observations.

The normal equations whose solution gives the most probable values of $\log c_1$ and c_2 are

$$\begin{aligned} \log c_1 \Sigma E^2 - c_2 \Sigma E^2 m &= \Sigma E^2 M \\ -\log c_1 \Sigma E^2 m + c_2 \Sigma E^2 m^2 &= -\Sigma E^2 m M \end{aligned} \tag{6}$$

Whence,

$$\log c_1 = \frac{\Sigma E^2 M \cdot \Sigma E^2 m^2 - \Sigma E^2 m M \cdot \Sigma E^2 m}{\Sigma E^2 \cdot \Sigma E^2 m^2 - (\Sigma E^2 m)^2}, \tag{7}$$

and if common logarithms are used

$$c_2 = 2.3026 \cdot \frac{\Sigma E^2 \Sigma E^2 m M - \Sigma E^2 m \cdot \Sigma E^2 M}{\Sigma E^2 \cdot \Sigma E^2 m^2 - (\Sigma E^2 m)^2}. \tag{8}$$

If c_2 does not agree well with the approximate value, c'_2, of equations 2, another solution can be made using the computed value as c'_2.

APPLICATION OF THE METHOD

The application of the above process is straightforward and will not require an excessive amount of time or trouble of one possessing even a moderate amount of skill as a computer. The following 12 observations taken with a vacuum bolometer, fluorite prism, and mirror spectrometer were furnished by Dr. W. W. Coblentz.[4]

TABLE 1

Observations of Energy, Measured by Galvanometer Deflection, and Wave Length

[Absolute temperature 1350° C]

E (galvanometer deflection)	λ (microns)	E (galvanometer deflection)	λ (microns)	E (galvanometer deflection)	λ (microns)
3.12	0.9338	23.95	1.526	12.10	4.446
6.75	1.054	19.20	3.569	11.02	4.638
11.80	1.197	17.45	3.760	9.10	5.001
18.03	1.357	15.05	4.031	7.70	5.329

[4] These data have been used for other determinations of c_2. See Coblentz, this Bulletin, 13, p. 474; 1916.

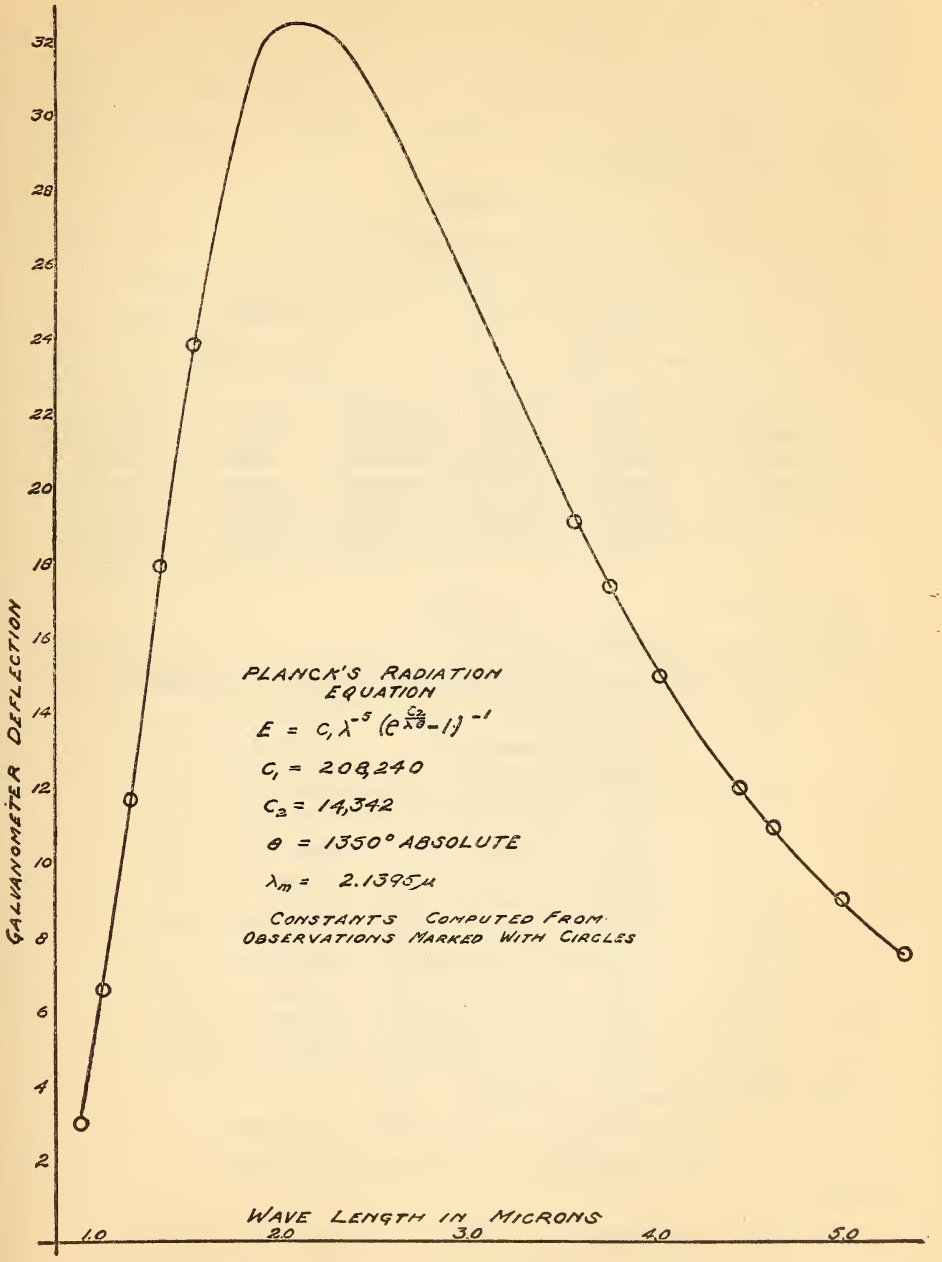

FIG. 1.—*Planck's radiation curve, computed from the data of Table I.*

The author carried out the computations to seven significant figures, using a seven-place Vega logarithm table and an eight-place computing machine. Five significant figures will amply suffice for practical purposes and will materially reduce the amount of labor required. The value of c'_2 used in equations 2 to reduce them to equations 5 was $c'_2 = 14\ 340$. A table of computed coefficients is herewith given.

TABLE 2

Coefficients for Equations 5

$m=\frac{1}{\lambda\theta}$	$M=\log E_1 + 5\log\lambda_1 + \log(1-e^{-\frac{c'_2}{\lambda\theta}})$	E^2	$m=\frac{1}{\lambda\theta}$	$M=\log E_1 + 5\log\lambda_1 + \log(1-e^{-\frac{c'_2}{\lambda\theta}})$	E^2
0.0007932542	0.3454193	9.7344	0.0001970055	4.0911939	304.5025
.0007027901	.9434885	45.5625	.0001837610	4.1722971	226.5025
.0006188310	1.4622922	139.2400	.0001666458	4.2804091	146.4100
.0005458664	1.9187218	325.0809	.0001597112	4.3274943	121.4404
.0004854133	2.2966665	573.6025	.0001481185	4.3990489	82.8100
.0002075485	4.0233168	368.6400	.0001390018	4.4561279	59.2900

Let $c_2 = 1{,}000\ v$.

Equations 5 become

$$\log c_1 - 0.7932542\ v = 0.3454193, \text{ weight } 9.7344$$
$$\log c_1 - .7027901\ v = 0.9434885, \text{ weight } 45.5625$$
$$\log c_1 - .6188310\ v = 1.462292, \text{ weight } 139.2400$$
$$\log c_1 - .5458664\ v = 1.918722, \text{ weight } 325.0809$$
$$\log c_1 - .4854133\ v = 2.296666, \text{ weight } 573.6025$$
$$\log c_1 - .2075485\ v = 4.023317, \text{ weight } 368.6400$$
$$\log c_1 - .1970055\ v = 4.091194, \text{ weight } 304.5025$$
$$\log c_1 - .1837610\ v = 4.172297, \text{ weight } 226.5025$$
$$\log c_1 - .1666458\ v = 4.280409, \text{ weight } 146.4100$$
$$\log c_1 - .1597112\ v = 4.327494, \text{ weight } 121.4404$$
$$\log c_1 - .1481185\ v = 4.399049, \text{ weight } 82.8100$$
$$\log c_1 - .1390018\ v = 4.456128, \text{ weight } 59.2900$$

The normal equations 6 become

$$2402.816 \log c_1 - 824.2166\ v = 7645.762$$
$$824.2166 \log c_1 - 359.4441\ v = 2144.792$$

whence

$$\log c_1 = 5.3185650,\ c_1 = 208{,}240,\ c_2 = 1000\ v = 14{,}342.1$$

This value of c_2 agrees very closely with the assumed value, 14 340, in equations 5. Had there been a wide difference, as much as 1 per cent, say, the computed value of c_2 could be used

for the c'_2 of equations 5 and the computations repeated until a satisfactory agreement was attained.

Should a second solution be deemed advisable, it may be accomplished very rapidly and with little labor. The coefficients of the unknowns in the normal equations will not be altered by a change in c'_2. The known terms will be altered slightly, due to slight changes in the last few of equations 5 from the right-hand side of the distribution. However, the normal equations are so insensitive to changes in c'_2 that after a reasonably judicious selection of a first value a second solution will rarely be necessary.

In regard to units, c_2 is expressed in micron degrees. The scale of ordinates is affected by c_1, which depends upon the units in which E, the emissivity, is measured. In the above observations from which c_1 is calculated, E is measured by the deflections of a galvanometer whose sensitivity varies under different experimental conditions. The value of c_1 is, therefore, of an arbitrary nature and will hold only for data measured by this same galvanometer scale under similar experimental conditions.

A table of residuals follows:

TABLE 3

List of Residuals from Observations Given in Table 1. Computed from Planck's Equation, Using the Constants Obtained by the Least-Square Solution

E		Residuals, observed minus computed	(Residuals)^2
Observed	Computed		
3.12	3.360	−0.240	0.057600
6.75	6.713	.037	.001369
11.80	11.849	− .049	.002401
18.03	18.023	.007	.000049
23.95	23.863	.087	.007569
19.20	19.311	− .111	.012321
17.45	17.461	− .011	.000121
15.05	15.109	− .059	.003481
12.10	12.105	− .005	.000025
11.02	10.926	.094	.008836
9.10	9.036	.064	.004096
7.70	7.641	.059	.003481
			Σv^2=.101349

It may be of interest to note here the probable errors, R_{c_1} and R_{c_2}, of c_1 and c_2, which are an indication of the precision of the values as determined. The probable errors are,

$$R_{c_1} = c_1 \times R \log c_1 = 0.6745\sqrt{\frac{\Sigma v^2 \qquad \Sigma E^2 m^2}{(n-2)\Sigma E^2 \cdot \Sigma E^2 m^2 - (\Sigma E^2 m)^2}} = 624$$

(See equation 3.)

$$R_{c_2} = 1000\, R_v = 674.5\sqrt{\frac{\Sigma v^2 \qquad \Sigma E^2}{(n-2)\Sigma E^2 \Sigma E^2 m^2 - (\Sigma E^2 m)^2}} = 8.0$$

n being the number of observations. The other coefficients are those as found in equations 6.

The adjusted values of the constants written in the conventional manner are then

$$c_1 = 208{,}240 \pm 624, \text{ in arbitrary units.}$$

$$c_2 = 14{,}342 \pm 8.0, \text{ in micron degrees.}$$

DISCUSSION OF SOME CURRENT METHODS[5]

Two methods about which discussion has centered will now be briefly considered.

In the first or "equal ordinate" method, a curve is fitted by eye to the plotted observations and a line distant E_1 and parallel to the λ axis drawn. c_2 in terms of the abscissas λ_1 and λ_2, corresponding to the points of intersection of the line with the curve has been shown[6] to be

$$c_2 = \frac{\theta\, \lambda_2\, \lambda_1}{\lambda_2 - \lambda_1}\left[5 \log \frac{\lambda_2}{\lambda_1} - e^{-\frac{c_2}{\lambda_2 \theta}}\right] \qquad (9)$$

to a high degree of approximation. In this manner a number of values of c_2 are computed and the arithmetic mean taken as a final value.

A second method, which has come to be called the "two point" method, has been recently developed.[7] A high approximation for c_2 in terms of any two observations, E_1, λ_1: E_2, λ_2, is

$$c_2 = \theta\, \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1}\left[\log \frac{E_2}{E_1} + 5 \log \frac{\lambda_2}{\lambda_1} - e^{-\frac{c_2}{\lambda_2 \theta}}\right] \qquad (10)$$

[5] See note 1, p. 237.

[6] Buckingham and Dellinger, this Bulletin, 7, p. 393, 1911; W. W. Coblentz, Phys. Rev., 32, p. 591, 1911.

[7] J. H. Dellinger, this Bulletin, 13, p. 540; 1916.

and the rigorous solution is

$$c_2=\theta\frac{\lambda_2\lambda_1}{\lambda_2=\lambda_1}\left[\log\frac{E_2}{E_1}+5\log\frac{\lambda_2}{\lambda_1}+\log\frac{1-e^{-\frac{c_2}{\lambda_1\theta}}}{1-e^{\frac{c_1}{\lambda_1\theta}}}\right] \tag{11}$$

The former equation is accurate enough for all cases except where both observations are on the right side of the maximum.

A discussion follows to show why these two methods as they have been applied necessarily give somewhat inconsistent results.

Consider the conditions. There are given n observations all of equal weight on the quantities E and λ. These observations are operated on a pair at a time to give

$$\begin{aligned} c_2&=\varphi_1\ (E_1,\ E_2,\ \lambda_1,\ \lambda_2)\\ &\ \ *\qquad\quad *\qquad\quad *\\ c_2&=\varphi_i\ (E_i,\ E_k,\ \lambda,\ \lambda_k)\end{aligned} \tag{12}$$

Herein lies the difficulty that has previously been met. It has been taken for granted that the values of c_2 obtained from the expressions 12 are of equal weight, and to get a final value of c_2 an arithmetic mean is taken. Since the probable error of a qauntity, which is proportional to the square root of the reciprocal of its weight,[8] computed from a function of observed quantities depends upon the nature of the function and the probable errors of the observed quantities, this step is obviously at fault unless the change in weight due to changes in the function is inappreciable as observations are taken from place to place in the distribution.

In what follows a system of weighting the separately computed values of c_2 is indicated that will tend to make the results by the two-point method agree with the results given by least squares.

Consider any pair of equations 5, say the i'th and the k'th, (k>i). Let them be reduced to a unit weight basis [9] and solved for c_2.

Thus,

$$\begin{aligned} E_i\log c_1-E_im_ic_2&=E_iM_i\\ E_k\log c_1-E_km_kc_2&=E_kM_k\end{aligned}$$

Whence,

$$c_2(m_k-m_i)E_iE_k=(M_k-M_i)E_iE_k \tag{13}$$

or,

$$c_2=\frac{M_k-M_i}{m_k-m_i} \tag{14}$$

[8] Merriman, Least Squares, p. 69; 1913.

[9] Wright and Hayford, Adjustment of Observations, p. 95; 1906.

which, when proper substitutions are made, is the same as equation 11.

If $E_i = E_k$, equation 14, after certain approximations,[10] becomes identical with 9.

If there are n of equations 5, there will be possible $\frac{n(n-1)}{2}$ expressions like 13, from each of which may be obtained an expression for c_2 like 14, which is a possible (but all are not equally probable) expression for the true value of c_2. A brief elementary consideration of the error theory will show that the most probable value of c_2 is not the mean of the $\frac{n(n-1)}{2}$ equations 14. The most probable value is given by a least-square solution of the $\frac{n(n-1)}{2}$ equations 13, or,

$$c_2 = \frac{\Sigma E^2_i E^2_k (M_k - M_i)(m_k - m_i)}{\Sigma E^2_i E^2_k (m_k - m_i)^2} \tag{15}$$

In which $k > i$, k takes any value between 2 and n, inclusive, and i takes every corresponding possible value between 1 and $n-1$, inclusive. It is evident from inspection that this solution will be given by a reduction of the $\frac{n(n-1)}{2}$ equations 14, if each is weighted as

$$E_i^2 E_k^2 (m_k - m_i)^2 \tag{16}$$

It will now be shown that if equations 5 be reduced by the two-point method, giving each expression for c_2 the weight indicated by 16, and if every possible combination of the observations a pair at a time be made, the resulting value of c_2 will be identical with that given by the method of least squares.[11]

For the sake of brevity in notation and manipulation, let four observation equations similar in type to equations 5 be considered, viz,

$$\begin{aligned} a + bx_1 &= y_1, \text{ weight } p_1 \\ a + bx_2 &= y_2, \text{ weight } p_2 \\ a + bx_3 &= y_3, \text{ weight } p_3 \\ a + bx_4 &= y_4, \text{ weight } p_4 \end{aligned} \tag{17}$$

[10] See note 1, p. 237, Buckingham and Dellinger.

[11] Wright, Adjustment of Observations, p. 141; 1884.

In which the b of these equations corresponds to the c_2 of equations 5.

Equations 17 may be reduced to an equal-weight basis by multiplying each through by the square root of its weight. Thus,

$$\sqrt{p_1}a + \sqrt{p_1}bx_1 = \sqrt{p_1}y_1$$
$$\sqrt{p_2}a + \sqrt{p_2}x_2b = \sqrt{p_2}y_2$$
$$\sqrt{p_3}a + \sqrt{p_3}x_3b = \sqrt{p_3}y_3$$
$$\sqrt{p_4}a + \sqrt{p_4}x_4b = \sqrt{p_4}y_4$$

Let the i'th and the k'th $(k>i)$ equations be solved simultaneously for b. Then

$$\sqrt{p_i}\sqrt{p_k}(x_k - x_i)b = \sqrt{p_1}\sqrt{p_2}(y_k - y_i) \tag{18}$$

or,

$$b = \frac{(y_k - y_i)}{(x_k - x_i)}, \text{ weight } p_1p_2(x_k - x_i)^2 \tag{19}$$

The number of equations of type 19 (i. e., the number of possible solutions for b from the four observation equations) will be the same as the number of combinations of four things two at a time, or $\frac{4 \times 3}{1 \times 2}$. If all these solutions be made, there will be at hand six expressions for b,

$$\begin{aligned}
b &= \frac{y_4 - y_1}{x_4 - x_1}, \text{ weight } p_1p_4(x_4 - x_1)^2 \\
b &= \frac{y_4 - y_2}{x_4 - x_2}, \text{ weight } p_2p_4(x_4 - x_2)^2 \\
b &= \frac{y_4 - y_3}{x_4 - x_3}, \text{ weight } p_5p_4(x_4 - x_3)^2 \\
b &= \frac{y_3 - y_1}{x_3 - x_1}, \text{ weight } p_1p_3(x_3 - x_1)^2 \\
b &= \frac{y_3 - y_2}{x_3 - x_2}, \text{ weight } p_2p_3(x_3 - x_2)^2 \\
b &= \frac{y_2 - y_1}{x_2 - x_1}, \text{ weight } p_1p_2(x_2 - x_1)^2
\end{aligned} \tag{20}$$

If all these values be considered as possible, and hence probable, values for the proper value of b, then the most probable value can be assumed to be the weighted mean of them all, i. e.,

$$b=\frac{\Sigma p_i p_k(x_k-x_i)(y_k-y_i)}{\Sigma p_i p_k(x_k-x_i)^2} \tag{21}$$

and it is to be shown that this is identical with the least-square solution,

$$b=\frac{\Sigma p\Sigma pxy-\Sigma px\Sigma py}{\Sigma p\Sigma px^2-(\Sigma px)^2} \tag{22}$$

The denominator of 21 becomes, when expanded,

$$\Sigma p_i p_k(x_k-x_i)^2=\Sigma p_i p_k x_k^2+\Sigma p_i p_k x_i^2-2\Sigma p_i p_2 x_i x_k$$

Now $k>i$ and can take any value between 2 and 4, inclusive. i can take any value between 1 and 3, inclusive.

Suppose $k=4$.

The equation $a+bx_4=y_4$ can be combined with the remaining three equations to obtain a separate solution for b. If $k=3$, there will be two equations that can be combined with $a+bx_3=y_3$ to obtain a separate solution for b; and if $k=2$, there will be left one equation that can be combined with $a+bx_2=y_2$ to obtain a separate solution for b.

x_4 will, therefore, appear as x_k in the summations of 21 three times, x_3 will appear as x_k twice, and x_2 will appear as x_k once.

Similar reasoning will show that x_1 will appear as x_i three times, x_2 will appear as x_i twice, and x_3 will appear as x_i once.

$$\therefore \Sigma p_i p_k x_k^2+\Sigma\, p_i p_k x_i^2-2\Sigma p_i p_k x_i x_k$$

$$=\left\{\begin{array}{l} p_1p_4x_4^2 \\ +p_2p_4x_4^2 \\ +p_3p_4x_4^2 \\ +p_1p_3x_3^2 \\ +p_2p_3x_3^2 \\ +p_1p_2x_2^2 \end{array}\right\}+\left\{\begin{array}{l} p_1p_4x_1^2 \\ +p_2p_4x_2^2 \\ +p_3p_4x_3^2 \\ +p_1p_3x_1^2 \\ +p_2p_3x_2^2 \\ +p_1p_2x_1^2 \end{array}\right\}-2\left\{\begin{array}{l} p_1p_4x_1x_4 \\ +p_2p_4x_2x_4 \\ +p_3p_4x_3x_4 \\ +p_1p_3x_1x_3 \\ +p_2p_3x_2x_3 \\ +p_1p_2x_1x_2 \end{array}\right\}$$

$$=\left\{\begin{array}{l} p_4x_4^2[p_1+p_2+p_3\quad\quad]+p_4p_4x_4^2-p_4p_4x_4^2 \\ +p_3x_3^2[p_1+p_2\quad\quad\quad+p_4]+p_3p_3x_3^2-p_3p_3x_3^2 \\ +p_2x_2^2[p_1\quad\quad+p_3+p_4]+p_2p_2x_2^2-p_2p_2x_2^2 \\ +p_1x_1^2[\quad\quad+p_2+p_3+p_4]+p_1p_1x_1^2-p_1p_1x_1^2 \end{array}\right\}$$

$$-\left\{\begin{array}{l} p_1x_1[\quad\quad+p_2x_2+p_3x_3+p_4x_4] \\ +p_2x_2[p_1x_1\quad\quad\quad+p_3x_3+p_4x_4] \\ +p_3x_3[p_1x_1+p_2x_2\quad\quad\quad+p_4x_4] \\ +p_4x_4[p_1x_1+p_2x_2+p_3x_3\quad\quad] \end{array}\right\}$$

$$
\begin{aligned}
&= p_4x_4^2\Sigma p + p_3x_3^2\Sigma p + p_2x_2^2\Sigma p + p_1x_1^2\Sigma p \\
&\quad - [p_1x_1\Sigma px + p_2x_2\Sigma px + p_3x_3\Sigma px + p_4x_4\Sigma px] \\
&= \Sigma p[p_4x_4^2 + p_3x_3^2 + p_2x_2^2 + p_1x_1^2] \\
&\quad - \Sigma px[p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4] \\
&= \Sigma p\ \Sigma px^2 - \Sigma px\ \Sigma px \\
&= \Sigma p\ \Sigma px^2 - (\Sigma px)^2
\end{aligned}
$$

which is identical with the denominator of 22. The numerator of 21 becomes, when expanded,

$$
\Sigma p_ip_kx_ky_k + \Sigma p_ip_kx_iy_i - \Sigma p_ip_kx_ky_i - \Sigma p_ip_kx_iy_k
$$

$$
= \left\{\begin{array}{r} p_1p_4x_4y_4 \\ +p_2p_4x_4y_4 \\ +p_3p_4x_4y_4 \\ +p_1p_3x_3y_3 \\ +p_2p_3x_3y_3 \\ +p_1p_2x_2y_2 \end{array}\right\} + \left\{\begin{array}{r} p_1p_4x_1y_1 \\ +p_2p_4x_2y_2 \\ +p_3p_4x_3y_3 \\ +p_1p_3x_1y_1 \\ +p_2p_3x_3y_3 \\ +p_1p_2x_2y_2 \end{array}\right\}
$$

$$
- \left\{\begin{array}{r} p_1p_4x_4y_1 \\ +p_2p_4x_4y_2 \\ +p_3p_4x_4y_3 \\ +p_1p_3x_3y_1 \\ +p_2p_3x_3y_2 \\ +p_1p_2x_2y_1 \end{array}\right\} - \left\{\begin{array}{l} p_1p_4x_1y_4 \\ p_2p_4x_2y_4 \\ p_3p_4x_3y_4 \\ p_1p_3x_1y_3 \\ p_2p_3x_2y_3 \\ p_1p_2x_1y_2 \end{array}\right\}
$$

$$
= \left\{\begin{array}{l} p_4x_4y_4(p_1 + p_2 + p_3 \qquad\quad) + p_4p_4x_4y_4 - p_4p_4x_4y_4 \\ +p_3x_3y_3(p_1 + p_2 \qquad\quad + p_4) + p_3p_3x_3y_3 - p_3p_3x_3y_3 \\ +p_2x_2y_2(p_1 \qquad\quad + p_3 + p_4) + p_2p_2x_2y_2 - p_2p_2x_2y_2 \\ +p_1x_1y_1(\qquad + p_2 + p_3 + p_4) + p_1p_1x_1y_1 - p_1p_1x_1y_1 \end{array}\right\}
$$

$$
- \left\{\begin{array}{l} p_4x_4(p_1y_1 + p_2y_2 + p_3y_3 \qquad\quad) \\ +p_3x_3(p_1y_1 + p_2y_2 \qquad\quad + p_4y_4) \\ +p_2x_2(p_1y_1 \qquad\quad + p_3y_3 + p_4y_4) \\ +p_1x_1(\qquad + p_2y_2 + p_3y_3 + p_4y_4) \end{array}\right\}
$$

$$
\begin{aligned}
&= p_4x_4y_4\Sigma p + p_3x_3y_3\Sigma p + p_2x_2y_2\Sigma p + p_1x_1y_1\Sigma p \\
&\quad - [p_4x_4\Sigma py + p_3x_3\Sigma py + p_2x_2\Sigma py + p_1x_1\Sigma py \\
&= \Sigma p[p_4x_4y_4 + p_3x_3y_3 + p_2x_2y_2 + p_1x_1y_1] \\
&\quad - \Sigma py[p_4x_4 + p_3x_3 + p_2x_2 + p_1x_1] \\
&= \Sigma p\ \Sigma pxy - \Sigma py\ \Sigma px
\end{aligned}
$$

which is identical with the numerator of 22.

The above analysis, though limited to four observations, will apply equally well to any number of observations, odd or even. Equations 21 and 22 are, therefore, identities, and hence the two-point method may be made to give results identical with the least-square method by making all possible separate solutions for c_2 furnished by the data and assigning weights to them according to 16 before taking the mean.

Substitute in 16,

$$\frac{1}{\lambda_1\theta} \text{ for } m_1, \frac{1}{\lambda_2\theta} \text{ for } m_2,$$

there results

$$E_i^2E_k^2(m_k - m_i)^2 = \frac{E_i^2E_k^2}{\theta}\left(\frac{\lambda_i - \lambda_k}{\lambda_i\lambda_k}\right)^2 = p$$

and since relative weights are all that are required the constant θ may be dropped,

$$\therefore p \propto E_i^2E_k^2\left(\frac{\lambda_i - \lambda_k}{\lambda_i\lambda_k}\right)^2 \tag{23}$$

The p in equation 23 approaches zero when λ_k is near to λ_i, as may be the case when both points are near or on the same side of the maximum ordinate, and when E_i and E_k are small.

It will be noted that in making the individual computations for c_2 from equations 10 and 11 that an assumed approximate value of c_2 is necessary in the exponential term. If, after applying the above system of weights, the mean does not agree with the assumed value, another solution will be necessary in which the computed value may be used as the approximate value in the exponential term. In no case should more than two of these processes be necessary.

A numerical illustration of the above theory is briefly represented in the accompanying Table 4. Six pairs of observations, E_i, λ_i, E_k, λ_k, were selected at different parts of the distribution and values of c_2 computed from them according to equations 10 and 11.

Table 4 is prepared to show (1) there is considerable variation in the weights of the values of c_2; (2) that values of c_2 differing widely from the mean are likely to be of small weight; (3) that the simple arithmetic mean of a limited number of computations is likely to differ considerably from the weighted mean (the variation shown in the computations of Table 4 is probably an extreme one. Any other arrangement may show a better agreement with the

value given by least squares); (4) that for a limited number of computations the weighted mean approximates better the most probable value, i. e., the value given by a least-square solution.

TABLE 4

List of Values of c_2 Given by the "Two-Point" Method. Computed From One Arrangement of the Observations of Table 1 Using Each Observation Once

λ_i	λ_k	E_i	E_k	c_2	p	$p_x c_2$
0.9338	5.001	3.12	9.10	14 478	6.117	88 555.7
1.526	3.569	23.95	19.20	14 308	297.556	4 257 437.3
1.054	4.638	6.75	11.02	14 358	29.740	427 007.1
3.760	5.329	17.45	7.70	14 487	1.107	16 036.3
1.197	4.446	11.80	12.10	14 360	75.963	1 090 829.7
1.357	4.031	18.03	15.05	14 337	175.958	2 522 709.8
			$\frac{\Sigma c_2}{6}=14\ 388\pm21$		$\Sigma p=586.440$	$8\ 402\ 575.9=\Sigma pc_2$
					$\frac{\Sigma pc_2}{\Sigma p}=14\ 328\pm10$	

The above table is computed from an arrangement of the observations using them all once. There are 11 different arrangements of the observations two at a time in each of which every observation appears only once. Thus there are 66 possible solutions for c_2 from the data of Table 1 by the two-point method. If any part of the remainder of these solutions be made and the results added to those in the table before taking the mean, the final value of c_2 will, no doubt, be altered to some extent. In case the experimenter considers the mean of a limited number of computations satisfactory for his purpose, they should be arranged so that each observation enters the computations the same number of times, and thus prevent any one point from unduly affecting the results. That is, for $2k$ observations, k, or $2k$, or $3k$, etc., to $(n-1)k$ computations will be necessary. The same number of computations will be necessary for $2k+1$ observations, and in each set of k computations one of the observations can not appear. This does not mean that as a method of approximation the two-point method is a failure in this case, for the experimenter may use his discretion and drop in order from each set that observation that can be combined with others of the group to give relatively small weights to the value of c_2.

The above system of weighting is only theoretically correct when applied to all possible solutions for c_2 from the data. However, the system applied to a limited number tends to make the mean stable and approach the value given by least squares. By a judi-

cious selection of combinations it is possible to give a comparatively large weight to a mean value of c_2 with one or two sets of weighted computations. The pairs of observations should be selected so as to keep the difference of the λ's large and at the same time keep the products of the E's large. Any set of k computations in which each observation is used once need not give the same result as any other set, nor need the weights of the results of the two sets be the same. Thus it is seen that the value of a result obtained from a limited number of computations depends, first, upon the arrangement of the observations and, second, upon the number of separate computations for c_2.

To a skilled computer the labor involved in a least-square computation is by no means prohibitive of its application, and whether it is to be preferred to the two-point method to make a solution for the most probable value of the constants depends to some extent upon the factor of personal skill. In using the two-point method for n observations there will be necessary $\frac{n(n-1)}{2}$ individual computations for c_2 and the same number of computations for the weights. Thus the method will become impractical for a large number of observations. In fitting a curve to a few observations, say six or seven, the complete solution can perhaps be most readily accomplished with the two-point method by anyone. For a larger number of observations the advantages of the two-point method are less. The fact that the least-square computation gives both constants may or may not be of practical value. If c_1 is desired, another extended computation to determine c_1 will be necessary after determining c_2 by the two-point method.

The possibilities of the equal-ordinate method depend upon the computer's skill as a draftsman, for in fixing the curve he tacitly predetermines a value of c_2. In a high-grade set of data one skillful with a pen may be able to fit a curve so closely by eye to the least-square position that the results of the individual calculations for c_2 will not differ appreciably among themselves, and any system of weights will therefore have no appreciable effect. An appreciable difference among individual solutions by this method signifies that the curve is not properly located. If for a given position of the curve an appreciable variation is not obtained in a reasonable number of solutions selected from place to place on the distribution, the curve can be considered properly located and the unweighted mean of the solutions should approximate well the least-square solution.

In the data used in this paper the gain in precision of the least-square method over the weighted two-point method using a limited number of computations is slight for the particular arrangement of the observations in Table 4. However, the element of chance enters to a considerable degree in the worth of the results of Table 4. It must be remembered that there are 10 other arrangements of the observations using them all once. Had any one of these happened to be selected for the illustration, the results might or might not have agreed so well both in the value of the constant and the probable error. In the same way the element of chance is a large factor in the disagreement of the simple mean with the least-square value. Individual computations for c_2 by the two-point method can be arranged to present a practical significance to a physicist which the least-square method does not directly possess, but consideration of these is outside the scope of this paper.[12]

SUMMARY

A least-square reduction of a set of observations of the Planck radiation-equation type has been effected by taking logarithms of both sides and assigning proper weights to the transformed equations. The method of assigning these weights has been given in a general form that can be adapted to any scheme of transformation.

How to combine and weight separate computations for c_2 by the two-point method so as to yield a value that agrees with the least-square value has been shown. A numerical demonstration is given that the proposed system of weights applied to an arbitrarily selected set of two-point calculations leads to a result which approximates the least-square value better than the simple mean of the set.

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[12] J. H. Dellinger, this Bulletin, 13, p. 541, et seq.; 1916.