A DETERMINATION OF THE INTERNATIONAL AMPERE IN ABSOLUTE MEASURE

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CONTENTS

I. INTRODUCTION

1. RELATION OF ABSOLUTE TO INTERNATIONAL UNITS

The three fundamental units in electrical measurements are the ohm, the ampere, and the volt. By means of Ohm's law any one can be fixed from the other two. Hence, at the London conference of 1908 it was agreed to define the international ohm in terms of the resistance of a specified column of mercury and the international ampere in terms of the amount of silver deposited per second in the silver voltameter, while the international volt is to be derived from these two independent units, being defined as the electromotive force which will cause an international ampere to flow through an international ohm.

The important distinction between the *ohm*, which is equal to 10^9 cgs units in the electromagnetic system, and the *international ohm*, which is the resistance of a certain column of mercury (and similarly for the ampere, volt, watt, etc.), was clearly drawn by the London conference. Heretofore, there has often been confusion in this respect, and the international ohm has sometimes been defined both as 10^9 cgs units and as the resistance of a certain column of mercury (and similarly for the other units).

The international ohm, ampere, volt, coulomb, and watt, therefore, all depend upon the two concrete standards, the mercury ohm and the silver voltameter, and not upon absolute measurements of resistance and current. However, it is very important that we know the value of the small difference between the ohm and the international ohm and between the ampere and the international ampere, in order that we may, by applying the proper corrections, obtain absolute units of power, energy, etc., from the international electrical units. The *watt* is a rate of work equal to 10^7 ergs per second; the *international watt* is the rate of work when an international ampere flows through an international ohm. If we know accurately the correction to be applied to convert international ohms into ohms and international amperes into amperes, we can obtain, with very high precision, the rate of work in an electric circuit in *watts*, or the total work in joules or ergs. Owing to the close interrelation of all physical phenomena and the increasing accuracy demanded in physical measurements, it has become much more necessary to be able to do this now than formerly.

Inasmuch as precise electrical measurements are usually made in terms of standard resistances and standard cells, it would simplify the problem if a voltage instead of a current were directly determined in absolute measure. The direct determination of the volt with precision in absolute measure, however, presents very formidable difficulties and has never been made. It was largely for this reason that the ampere was chosen as the second independent unit instead of the volt. Many absolute measurements of the ampere and of the ohm have, however, been made since Gauss and Weber first showed in 1832 that a system of absolute units is possible. These determinations, having been made at different times and at different places, can be compared only by comparing the various concrete standards adopted at the different times. As there can be no concrete standard ampere, two expedients have been adopted for preserving the results of the absolute measurement of current. That most favored in the past is to determine the mass of silver deposited per second in a silver voltameter by a current of one ampere; the other, which in many cases in the past served as intermediary between the absolute determination and the silver voltameter measurement, is to determine the electromotive·force of a certain standard cell in terms of a standard of resistance and the absolute ampere determined by the balance.

Until quite recently, the constancy and reproducibility of these concrete representations of the results scarcely surpassed 1 in 10 000, and the construction of the absolute instruments was such as to give an accuracy inferior to this. Indeed, previous to 1907 many thought that an accuracy of 1 in 10 000 was the ideal to be aimed at, and that its achievement would be a triumph.

In that year, however, appeared the report of the work on the absolute measurement of current by means of the new current weigher of the National Physical Laboratory.[1] This work, claiming an accuracy (exclusive of errors in the value of the acceleration of gravity) of 1 in 100 000, marks the beginning of a new epoch in the history of the absolute measurement of electrical quantities. No work of this kind begun after the publication of the above can be considered satisfactory unless it attempts to attain an accuracy considerably surpassing 1 in 10 000.

The extreme accuracy claimed for the work done at the National Physical Laboratory exceeds the reproducibility of the silver voltameter when used according to any specifications as yet officially adopted. It is somewhat better than the reproducibility of the standard cell, or of the international ohm from its specifications in terms of the resistance of a column of mercury. It thus gives promise of an early realization of the conditions under which our electrical standards can be determined in terms of the absolute units, with an accuracy equaling that with which we can trust our concrete standards, unless, indeed, as we hope may be the case, the constancy and reproducibility of our concrete electrical standards shall be appreciably improved in the near future.

2. RECENT ABSOLUTE MEASUREMENTS OF CURRENT

The results of the various absolute measurements of current made prior to the appearance of the paper of which we have been speaking, having been discussed in that and other papers and belonging to a distinctly different epoch, need not be considered here. This leaves but five determinations to be considered, all of which were begun prior to 1907.

(*a*) *Ayrton, Mather, and Smith*[2].—In the measurements made at the National Physical Laboratory, a beautifully constructed

[1] Ayrton, Mather, and Smith: Phil. Trans., 207A, pp. 463–544; 1908.
[2] See note 1.

current weigher, with single layer cylindrical coils wound upon marble forms, was used. The balance was a double instrument, symmetrical with reference to the plane through the central knife edge and normal to the beam. The entire instrument was carried by a massive stand of phosphor bronze, which had satisfactorily undergone tests for magnetic impurities. When in adjustment, each moving coil was situated so that its end planes coincided with the mean planes of two fixed coils of the same length, wound upon opposite ends of the same hollow marble cylinder. The axes of the moving coils were vertical, and coincided with the axes of the fixed coil. The constant of such a balance is determined from direct measurements of the linear dimensions of the coils. These measurements can be made for single-layer coils with much greater accuracy than is possible for multiple-layer coils. The accuracy of measurement required, however, to give the desired precision in the result is extraordinary, even for the most favorable case of single-layer coils. From data given in the paper, however, it appears probable that sufficient accuracy has been secured. All weighings were made within the first 30 minutes after putting on the current, because when the current was kept on for a longer period the behavior of the balance became erratic, on account of the heating due to the current. Working in this way, under rather rapidly changing temperature conditions, it would appear that the dimensions of the coils, and so the constant of the balance, might be subject to some uncertainty. The detection of a minute quantity of magnetic material in a large mass of nonmagnetic metal, such as the stand on which the balance and coils rest, is difficult; and one can not avoid thinking that the presence of this stand may possibly be a source of very slight error. A direct indication of the existence or nonexistence of such an error can be obtained by measuring the difference in the forces exerted upon either moving coil by the two fixed coils in which it is hung; then interchanging these fixed coils (turning the cylinder end for end) and repeating. If the stand is without effect, these two values will be the same. We have found such a procedure of great value in the work to be described in this paper.

This most elaborate and excellent piece of work gave for the value of the Weston normal cell, at 17° C, in terms of the international ohm, and the absolute ampere, the value 1.018 30, which corresponds to the value 1.018 19 at 20°.00 C. Somewhat more recent but as yet unpublished measurements, kindly communicated by the Director of the National Physical Laboratory, gave 1.018 18 as the value at 20°.00 C.

(*b*) *Janet, Laporte, and Jouaust*[3]—The next article upon this subject that appeared described the work done at the Laboratoire Central d'Électricité. The authors used a current balance of the Rayleigh type. The distance between the fixed coils was greater than that which would give the maximum force upon the moving coil, and hence the force varied less rapidly in the region occupied by the moving coil. This permitted a less exact placing of the moving coil, but requires that the distance between the mean planes of the fixed coils be determined with greater accuracy; an error in this case of 0.01 mm in this distance produces an error of 5 parts in 100 000. The radii of the coils were determined from the measurement of the length of the wire wound on them. As stated in a note appended to the article, and as verified since (Bull. Soc. Int. d'Élec.; 1910), the international ohm as used at the Laboratoire Central is 1 in 10 000 smaller than that used in England, Germany, and America. Further, in calculating the constant of their instrument, the authors assumed that the axial breadth of a coil was given by the total breadth of the channel in which it was wound; but that the radial depth was given by the distance from the *axis* of the wire in the bottom layer to the *axis* of the wire in the top layer. This assumed radial depth is evidently too small (p. 373). In a recent paper[4] the authors have given the result of a complete recomputation of the constant of their balance using the correct sectional dimensions. They now find for the value of the Weston normal cell at 20°.00 C., taking into account the difference between the French ohm and those of the other countries, the value 1.01836.

(*c*) *A. Guillet*—The next paper in order appears in the same volume as the preceding (pp. 535–561), and describes work carried out at the Sorbonne. This work was done by the method

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[3] P. Janet, F. Laporte, and R. Jouaust: Bull. de. la Soc. Internat. des Électriciens (2), 8, pp. 459–522; 1908.

[4] Comptes rendus, 153, p. 718, Oct. 16, 1911.

suggested by Prof. G. Lippmann in 1906. Here again the instrument is of the attracted-coil type, but the multiple-layer fixed coils are placed very close together and the multiple-layer moving coils are but slightly smaller than the fixed coils. The constant of the instrument is not determined from its dimensions, but the mutual inductance of the fixed coils with respect to the moving coil, for various positions of the latter, is determined by direct comparison with absolute standards of mutual inductance. From these observations, an empirical equation, connecting the mutual inductance with the position of the moving coil, is determined; whence, by differentiation, we get the force for any given position of the coil. The difficulty in carrying out the method in practice is due to the rapid variation in the mutual inductance with the displacement of the moving coil. From the data given, it appears that the relative positions of the moving coil can not be determined with an accuracy exceeding about 1 micron. By using the mean of several readings carried out to 0.1μ, Guillet finds for the factor a in his notation the value $1.660\,91 \times 10^7$. If these values for the relative positions of the coil be rounded off to the nearest micron (which involves an increase of 0.4μ in the first and third values and no change in the second value) we find for the constant a the value $1.660\,58 \times 10^7$. Since a is proportional to the force, we see that this slight change will produce a change of about 1 in 10 000 in the current. The balance being double and symmetrical, there are two groupings of the currents that may be used; the constant was independently determined for each of these groupings, and thus two independent determinations were obtained. As only a single value was given for each grouping, it is impossible to determine the reproducibility of the observations. The value found for the Weston normal cell, at 20°00 C, after applying the correction of 1 in 10 000 for the difference in the French ohm, was 1.018 12.

(*d*) *Pellat.*—In the same volume (pp. 573–633) there is still a third paper on this subject. In this paper Prof. Pellat gives a thorough discussion of the construction, the computation of the constant, and the result obtained by him with a radically different type of instrument. This consists of a long multiple-layer solenoid, placed with its axis horizontal, in the interior of which is placed a small single-layer coil with its axis vertical. The center of the two coils coincide. The small coil is supported on a knife-edge, normal to the axes of the two coils. The torque which the current in the long solenoid exerts upon the small coil is balanced by means of weights applied to the end of an arm attached to the small coil. Thus it belongs to the type of instrument usually denoted by the term electrodynamometer, although the torque which balances the effect of the current is due to gravity instead of being produced by the torsion of a wire. The result obtained for the electromotive force of the Weston normal cell at 20°.00 C, corrected for the difference in the ohm, is 1.018 31 ± 0.000 15.

In all of the papers so far considered it is implied, if not explicitly stated, that the constant of the instrument has a zero temperature coefficient if all parts are constructed of the same material. Consequently little or no attention has been paid to the temperature of the instrument. That the coefficient is zero under these conditions is true, *provided that all parts of the instrument are at the same temperature.* But in practice this condition is never fulfilled; the coils are heated by the current, and in general they are not all heated to the same extent, and in the last case considered the balance arm, from which the weight is suspended, is undoubtedly at a different temperature from the coils. In the absence of the necessary data it is impossible to form an estimate of the errors that may thus be introduced; but in order to obtain the highest accuracy of which an instrument is capable it is very necessary to study carefully the temperatures of its various parts.

(*e*) *Haga.*—The most recent work on the subject is that by Prof. Haga, carried out at the University of Groningen.[5] He has measured the current in absolute units by means of a tangent galvanometer, and has thus based it upon the calculated constant of the galvanometer and the absolute determination of the horizontal component of the earth's magnetic field. Owing to continual variations in the latter, the method is exceedingly difficult, but Prof. Haga appears to have brought the work to a very satisfactory conclusion. He finds for the Weston normal cell in terms of the international ohm and the absolute ampere, as given

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[5] Prof. H. Haga and J. Boerema: Konink. Akad. Wetensch. Amsterdam Proc., p. 587; 1910.

by his instruments, the value 1.0183_6 at 17°.00 C, or 1.0182_5 at 20°.00 C.

Collecting these values, we have

N. P. L.	1.018 18
L. C. E.	1.018 36
Guillet	1.018 12
Pellat	1.018 31
Haga	1.018 25
Mean	**1.018 24 semiabsolute volts.**[6]

Considering all the circumstances, this agreement is very striking and leaves no doubt that the value 1.0182 is correct to at least 1 in 10 000.

3. TYPES OF INSTRUMENTS AVAILABLE

In all absolute measurements of electric current a force or torque produced by the current is balanced against another force or torque, which in turn is determined in dynamical units. In choosing an instrument to be used in the absolute measurement of current the first thing to be considered is the nature of the auxiliary force or torque that is to be employed. There are three available methods: (*a*) Using a known magnetic field, (*b*) using an elastic deformation of some body, (*c*) using the gravitational attraction of the earth. The only magnetic field for which the value of the force or torque produced can be at all readily determined in dynamical units is that due to the earth; and, owing to the continual variations in the strength of this field, an accurate determination of its strength at any given time and place involves many difficulties and scarcely gives promise of an accuracy superior to 1 in 10 000. Though the recent determination by Haga, using this method (with a tangent galvanometer), gave excellent results, it would appear that any method based upon the direct measurement of a given magnetic field would involve many difficulties when extreme accuracy is desired.

Measurements based upon the second method involve a direct determination of the force of restitution of the deformed body at the time and under the exact conditions under which it was used

[6] See footnote to p. *362*.

in the electrical measurements, and in addition are subject to some uncertainty introduced by the elastic after effect shown by all material bodies. While these difficulties are not insuperable, Dr. Guthe[7] and others have shown that they are quite formidable.

In measurements based upon the third method the evaluation of the force or torque can be reduced to the direct comparison of two masses and the determination of the acceleration of gravity. The first can be readily done with extreme precision; the second involves many difficulties, but these are largely offset by the facts that we have every reason to believe that in any region, geologically stable, the value of the acceleration of gravity remains unaltered over long epochs and that relative values of the acceleration of gravity at any two places can be readily determined with high precision. The latter fact enables the results obtained by this method at different times and places to be rendered strictly comparable with one another; the former renders the results obtained capable of correction at any future time when the absolute value of the acceleration of gravity shall have been determined to a higher accuracy than that with which it is now known.

For these reasons it appeared desirable that the work should be based on the third or gravitational method. Furthermore, it appeared on the whole desirable that the construction should be such that a force rather than a torque be measured. Accordingly a current weigher, or current balance, was constructed.

4. THE RAYLEIGH BALANCE

The particular type decided upon was that used by Lord Rayleigh in 1884, and since employed by Janet, Laporte, and Jouaust, at Paris. It consists essentially of a pair of multiple layer fixed coils placed coaxially, with their planes horizontal, and at such a distance apart that the vertical force which they exert for a given current upon a smaller coil placed coaxially with the fixed coils, and midway between them, either is a maximum or varies at a minimum rate for small vertical displacements. The smaller coil is suspended from one pan of a balance, being counterpoised by equal weights in the other pan. The force exerted upon it is counterpoised by additional weights placed on one pan or the other,

[7] This Bulletin, 2, p. 33; 1906.

according to the direction of the force. Preferably, however, the weight is equivalent to twice the force, which is the change produced by reversing the current in the fixed coils. The earth's magnetic field causes a torque which will not affect the measured force; local disturbances of the magnetic field would be eliminated by taking the mean of the two forces observed for currents in direct and in reversed directions through the fixed coils, the direction in the moving coil being the same in both cases. The most striking feature of the balance, theoretically, is the fact that the principal constant need not be determined by direct measurements upon the coils, but may be determined by an electrical method. This eliminates the difficulties incident to the direct measurement of the mean diameter of a multiple layer coil, and enables the constant to be determined with extreme precision and to be redetermined at any time.[8]

Rayleigh used a single balance, consisting of one pair of fixed coils and one moving coil. Janet, Laporte, and Jouaust used a double current balance, with two pairs of fixed coils and two moving coils, one suspended from each pan of a balance. There are both advantages and disadvantages in a double balance, and after careful consideration we decided to use a single balance. The force is doubled by a double balance, or for the same force one can use coils of smaller cross section. But, as some of our coils were much larger than those of Janet, Laporte, and Jouaust, it would have required a beam at least 75 cm long instead of 30 cm, as used by us. The balance would have been much more complicated, and harder to manipulate and adjust accurately, and we feel now, as we did four years ago when considering the question, that for work of the highest precision the simplification gained by using a single balance of this type more than overbalances the disadvantage of having a larger cross section of the fixed coils for the same force.

The coils of such a balance must have a large number of turns of wire in order to give a force large enough to be measured with precision. Lord Rayleigh's balance, as he used it, gave a change of force of about 1 g on reversal of the current, and Janet, Laporte,

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[8] Janet, Laporte, and Jouaust, however, calculated the constant from the radii obtained by counting the turns and measuring the length of the wire as it was wound on the coils.

and Jouaust's about 4 g. The coils of our balance have been wound in such a way that with a current not exceeding 1 ampere the force (on reversal of the current) would be from 3 to 6 g. This is a large enough force to be measured with the necessary precision, if the conditions are favorable. It would have been possible to make the force 10 or 20 g, and that would have been favorable to the weighing, but there would have been a loss of accuracy in other directions. The resistance of the windings increases as the square of the number of turns for a given size and cross section of the coil. One soon reaches a limit in attempting to increase the force by increasing the number of turns. First, the heat developed should not be allowed to become excessive, as it would if the number of turns be greatly increased; second, the voltage required to overcome the resistance should not be large enough to give rise to serious electrostatic attractions due to differences of potential between the parts of the balance; and, third, the current must be maintained uniform to an extraordinary degree if very accurate measurements are to be made. This requires a special storage battery, used for no other purpose, and a ballast resistance of small temperature coefficient can be employed advantageously if the instrument resistance is small enough.

One can not use a small enough resistance, however, to make the heating effect negligible, without making the cross section of the coils too great or the force to be measured too small.

If the cross section of the winding of the coils is large, the correction for section will be relatively large, and it will be difficult to determine its value with sufficient precision. After careful study of the theory of the balance, we decided to use a square cross section of 2 by 2 cm for the large fixed coils (50 cm radius); 1.4 by 1.4 cm for the small fixed coils (40 cm radius); and 1 by 1 cm for the moving coils (20 and 25 cm radius).

To reduce the error due to the heating of the fixed coils, a system of water cooling was designed to carry away the heat as fast as generated, so that a state of equilibrium could be attained, and weighings could be made deliberately for an indefinite period with the dimensions and temperatures of the coils constant. There is, of course, a difference in temperature between the wire of the windings and the brass form on which it is wound, and hence a careful study had to be made of the effect of the load. The ratio of the radii of the fixed and moving coils was determined with water flowing through the cooling system precisely as when the balance was used in the weighings, so that the heating effect of the current was known with great accuracy and could be perfectly controlled.

The moving coil can not be water-cooled. The heat generated in it is made small by using a coil of low resistance. The number of watts varied from 0.6 to 1.2, heating the coil from 1.7 to 3.3 degrees centigrade above the surrounding temperature. Even this slight increase of temperature sets up convection currents which are affected by any changes in the temperature of the fixed coils or of the instrument case. To protect the moving coil from outside influences, and to permit the convection currents to become constant so that their lifting force on the moving coil should be sufficiently constant to be eliminated by successive weighings with the current in fixed coils reversed, the moving coil was inclosed in a water jacket, through which water at a constant temperature was passed. This gives almost ideal conditions. The fixed coils being held at a definite known temperature, and the moving coil being suspended in a constant-temperature chamber and carrying a constant current, a constant circulation of air is set up in this chamber, which removes the heat from the moving coil and exerts a constant lifting force upon the moving coil. This force is completely eliminated by the method of weighing.

In order to obtain checks upon the work and a final result of greater weight, several pairs of fixed coils have been used and several different moving coils, with two different radii both in the fixed and in the moving coils.

The details of the construction of the balance are given in Section II, the measurements of the ratio of the radii are given in Section III, the theory of the instrument and calculation of the constants are given in Section IV, and the tests of the balance and the measurement of the current are given in Section V. It is intended to give in this section only a general outline of the main features of the balance, and reasons for choosing this type of balance. Some of the advantages of the balance as we have used it may now be stated briefly, by way of a summary of what precedes.

(*a*) The only direct length measurements that have to be made are those of the sectional dimensions of the coils, and these are used only in the calculation of correction terms. The principal term in the constant of the instrument depends upon the ratios of the radii of the fixed coils to that of the moving coil, and these ratios may be obtained by an electrical method, based upon that described by Bosscha and used by Lord Rayleigh. This method is capable of giving results of extraordinary precision.

(*b*) The coils being compact and readily replaced in the instrument, a number of coils can be constructed and used interchangeably, thus giving not only great flexibility to the instrument, but also a number of independent determinations, thus affording a better indication of the magnitude of the errors in the work.

(*c*) A feature which we believe to be of great importance in a balance which is to be used at intervals throughout a long term of years, is the ease with which the ratio of the radii of the coils, and, consequently, the constant of the balance, can be redetermined from time to time, as a check upon the constancy of the coils. This has proved to be of great importance in our instrument.

(*d*) The balance lends itself readily to the water-cooling of its fixed coils, so that their temperature can be controlled and maintained constant, not only in the current weighings but also in the determinations of the ratio of the radii, permitting a more exact determination of the constant of the balance and more accurate weighings. The moving coil can also be readily inclosed in a constant-temperature chamber, so that the convection currents in the air, by means of which the heat generated in the moving coil is carried away, can become steady, and the lifting force of these air currents will be constant enough to be successfully eliminated by the method of weighing.

Other types of balances undoubtedly possess some of these features, and perhaps others of importance not possessed by this balance; but it was because the Rayleigh balance if improved in the way indicated would possess so many good qualities that we chose it in 1906 when the work was first taken up.

The results obtained (given on pp. 357–363) have more than justified our expectations, although it has required a very large amount of time and labor to carry out the work.

II. DESCRIPTION OF THE BALANCE AND OF THE ELECTRICAL CONNECTIONS

5. THE PHYSICAL BALANCE

A 2-kg precision balance by Rueprecht, with a 30-cm beam, was modified in the shop of this Bureau so as to adapt it to this work. Certain magnetic portions of the balance case and of the balance itself were replaced by brass or by phosphor bronze. Each piece of the balance, as finally constructed, except the knife-edges and the blocks holding the agate planes, was tested by a very sensitive astatic magnetometer, and all were found to be satisfactorily nonmagnetic. As used, the knife-edges were always at least 70 cm above the upper coil, and, during the latter portion of the work, they were 100 cm above it. It was proved by actual tests that a much larger mass of steel at this distance produced no appreciable effect upon the force. For further discussion of the magnetic tests of the balance, see pages 345–348.

A mirror was mounted on the beam over the central knife-edge, and the deflections of the balance were read by means of a telescope and vertical scale about two and a half meters distant. With this arrangement, 1-mm scale deflection corresponds to a difference in the force of 0.36 mg, and to a displacement of the moving coil of 0.025 mm. The time for a single swing of the balance with 1-kg load (the weight of the moving coil and suspension system) is 15 seconds.

The weights employed were of platinum, and were compared at intervals with the standards of this Bureau. They have remained constant throughout the work. By a simple device, the weights could be placed on the pan, or removed, without opening the balance case. A correction for the buoyancy of the air was, of course, applied.

The moving coil was suspended from the lower end of a tube passing through the center of the right pan, and the weighings were accomplished by the addition or the removal of weights from this pan. Consequently, neither the ratio of the lever arms nor the flexure of the beam enters into the discussion of the results.

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6. THE FIXED COILS

Three pair of fixed coils were used. All were wound bifilarly, with enamel-insulated wire, upon brass forms having sections as shown in Figs. 1 and 2. Care was taken to make the faces of the forms normal to the axis of the form. Two wires were wound side by side in 36 layers of 18 double turns in each layer in the larger coils, and 28 layers of 14 double turns in each layer in the smaller coils.

There are several advantages in a bifilar winding: (*a*) It enables the insulation resistance from one wire to the other to be measured, and any leakage from one turn to the next can be detected by a simple test. (*b*) It permits the coils to be joined in series or parallel, and so permits different currents to be used with the same heating effect. It is also convenient, in measuring the ratio of the radii, to be able to vary the number of turns, as is done effectively when the coils are joined in parallel instead of in series. (*c*) By sending the current in opposite directions in the two windings, the full heating effect may be produced without any magnetic effect. This is convenient in testing the balance, and in detecting a change in the radius of the coil with different currents.

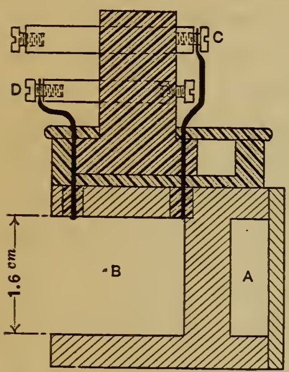

Fig. 1.—*Section of small fixed coil showing first form of terminal block*

A is the water channel. B is the channel for the wire

The enamel covering of the wire has the advantage over a silk covering in being thinner, as well as hard and unyielding, and of very uniform thickness. It can be wound as uniformly as bare wire, although it should be handled carefully to prevent injury to the insulation. When thoroughly dry, the insulation resistance is very high.

For every coil the insulation resistances between the two wires, and between each wire and the form, were frequently tested with a potential difference of 40 volts, and were found to be very good throughout the work, the insulation seldom falling below 100 megohms.

The ends of one wire are brought out through two small, axial holes bushed with ebonite and lying accurately upon the same radius of the form; the ends of the other wire of the bifilar are similarly brought out at the other extremity of the same diameter. After passing through the brass forms, the ends of the wires either are attached by spade terminals and small screws to short brass rods, supported in a radial and axial plane by ebonite posts attached to the form (Fig. 1) and forming the true terminal blocks of the coil, or they pass through an ebonite box attached to the form, as shown in Fig. 2, connections to the leads being made by drops of solder. These forms of terminals were adopted because of the facility with which they allow one terminal of the coil winding to be disconnected from the leads, and the latter short-circuited (for measuring the lead effect) without disturbing the position of the leads.

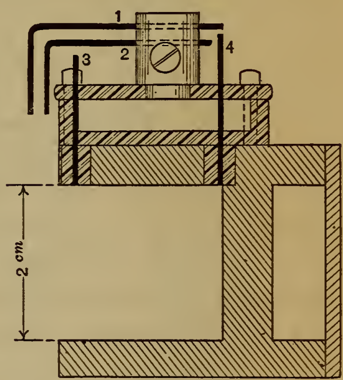

Fig. 2.—*Section of large fixed coil showing second form of terminal block*

Connections are made as desired by drops of solder

Two pair of coils, known as $S1$, $S2$, $L1$, and $L2$, were wound in 1907 upon forms of cast brass. A specimen of the brass, cast as a bar, was tested by means of an induction balance, and the forms were tested by a magnetometer, and were pronounced good. Later, by means of the extremely sensitive instrument described below, these coils were found to be slightly magnetic. In order to determine whether or not this might be an appreciable source of error it was deemed advisable to build a pair of coils having forms that were still less magnetic, and accordingly coils $L3$ and $L4$ were built. The forms for these coils were built up entirely of rolled brass, riveted together, and soft soldered. These forms are somewhat better than $S1$, the best of the old coils, and only about one-eightieth as magnetic as $L1$, the worst of the old coils.

The most satisfactory means found for testing the magnetic properties of conductors, especially of various portions of large masses, is a delicately suspended astatic magnetic system, so arranged that the body under test can be brought very close to one pole. The instrument used had needles about 6 mm long mounted on opposite ends of a slender glass rod 5 cm long. The whole was suspended by a silk fiber 10 cm long. The suspension was completely inclosed and the lower needle hung near the bottom of a glass tube 1 cm in diameter. By suitably leveling the instrument, one pole of the lower needle can be placed very near the wall of the tube so that the test object can be brought within a few millimeters of the pole. The deflections were read by means of a mirror, telescope, and scale.

With such an instrument, and with a scale distance of 2 meters, deflections of 5 cm were obtained when a tube containing ferrous sulphate was presented to the needle. This sensibility is ample for our purpose, but it could undoubtedly be increased without very great trouble. Though we have tested many kinds of brass, we have never found a specimen that did not produce a slight deflection, nor one that could not be permanently magnetized by subjecting it to a strong magnetic field. We have found rolled brass to be the most uniformly good magnetically, though excellent cast brass can be obtained, as is shown by *S*1.

In order to improve the insulation of the coils, the wire channels of the forms were lined with paper attached to the metal with thin shellac; in the case of *L*3 and *L*4 thin paper, soaked in hot paraffin, was ironed down to the bottoms of the channels. As each layer of wire was wound it was covered with a strip of onionskin paper (0.05 mm thick). Owing to the fact that the coils were not sealed air tight, the paper absorbed moisture to a degree depending on the average humidity of the atmosphere, and swelled slightly in consequence. This caused very slight but appreciable changes in the mean radii of the coils. Consequently, in the summer of 1910 the paper covering the outer layer of wire was saturated with paraffin melted in with a clean hot soldering copper, the paraffin being well melted to the sides of the form. Then a strip of muslin, well soaked in a hot mixture of beeswax and Venice turpentine, was wrapped around the coil and melted to the underlying paraffin; over the whole was wrapped a strip of binder's cloth soaked in hot paraffin and melted to the muslin and form. This sealed the coils very effectually against the absorption of moisture from the air.

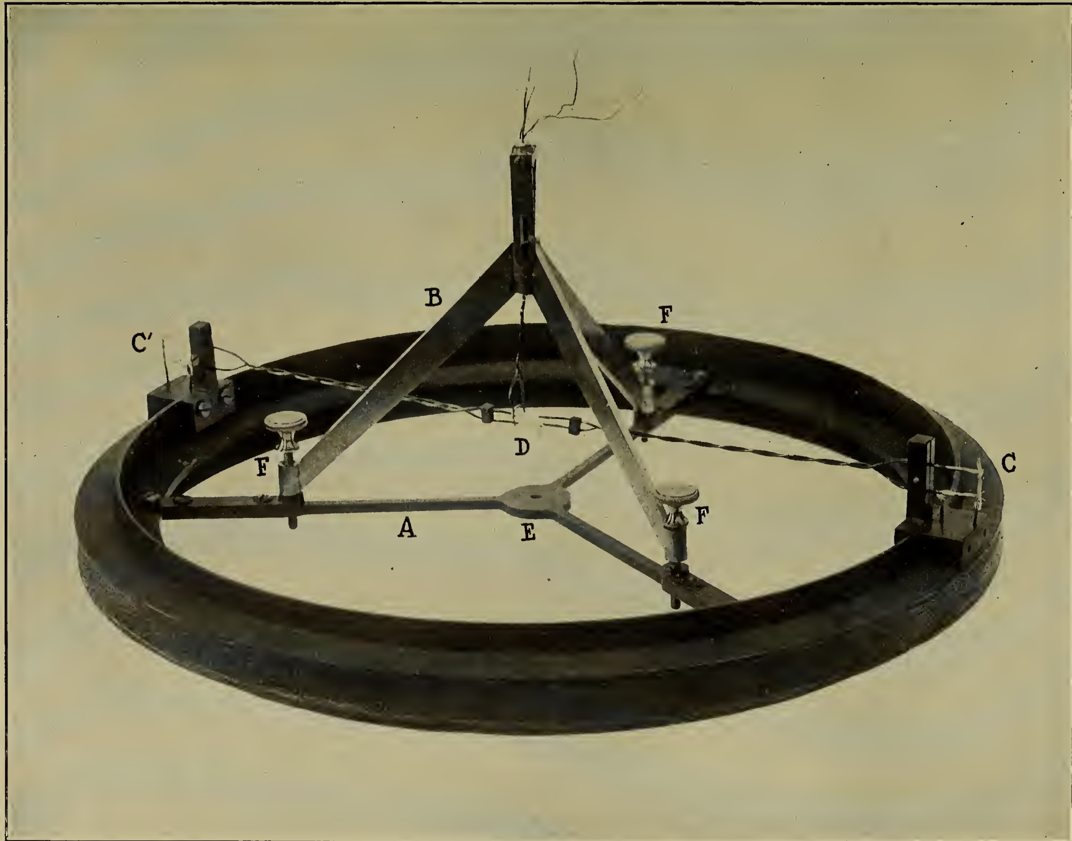

Fig. 3.—*Photograph of a moving coil, showing the leads, the star, and the tripod*

The width of the channels after lining was designed to take an integral number of turns of the bifilar winding. Owing, however, to slight variations in the thickness of the insulated wire, the fit was better in some places than in others. In all cases the wire was transferred from one layer to the next at points previously determined upon.

The wire was approximately 0.5 mm in diameter, and was wound under a tension of about 1 kilogram. The diameters of the forms and of each layer of wire were measured as the coils were wound, so as to obtain a very approximate measure of the mean diameter of each coil, and to keep a check on the uniformity of the winding. These measurements are given in the appendix, page 385, but were not used in the calculation of the constants of the balance because they can not compare in accuracy with the measurements by the electrical method. In winding the earlier coils ($S1$, $S2$, $L1$, $L2$) an integral number of turns of each wire was placed in each layer. This necessitated bringing up the wire from one layer to the next, always at the extremities of the same diameter, and gradually the winding became slightly elliptical with the long axis along this diameter. Consequently, when winding $L3$ and $L4$, the wire was brought up from one layer to the next at a point one thirty-sixth of a revolution short of the point at which it was brought up the layer before. Thus, the coil is kept circular and each winding has one turn less than by the old method.

7. THE MOVING COILS

Four moving coils have been built at various times during the progress of the work. They are all wound bifilar, as were the fixed coils, of enamel insulated wire upon brass forms finished dead black and having a section as shown in Fig. 4. The black finish and the winglike projections facilitate the dissipation of heat. The two windings of the moving coils were generally joined in parallel during the weighings, but were used in series in some cases to obtain a check upon the work. The method of connecting the windings to the leads is shown in Figs. 3 and 4. Here also the leads may be short-circuited by drops of solder, and their effect in situ determined without any current flowing through the windings.

The three coils, $M1$, $M2$, and $M3$, are wound upon forms of cast brass, the winding, the treatment of the terminals, and the sealing being done in essentially the manner already described for the fixed coils. The fourth coil, $M4$, was wound upon a form cast of rolled brass. Its windings differed from the others in that the windings in each layer were not uniformly distributed, but in adjacent layers were crowded toward opposite sides of the channel; hot paraffin was painted on and into each layer.

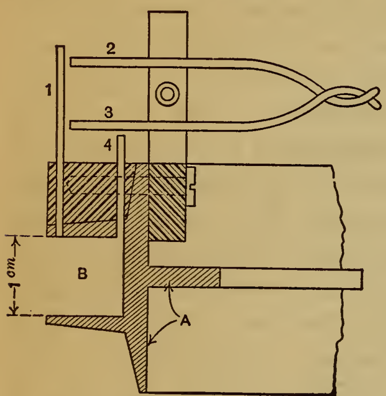

Fig. 4.—*Section of moving coil showing third form of teminal block*

Connections are made as desired by drops of solder. *A* are flanges to secure stiffness. *B* is channel for the wire

Over each layer a strip of onion-skin paper (0.03 mm thick) soaked in paraffin was pressed down closely with a hot copper; over the top layer the coil was sealed like the others. Thus the windings of this coil are embedded in a solid block of paraffin, and most effectively protected from atmospheric action. As shown in Section III, the failure to distribute the wires uniformly across the channel was a source of uncertainty not recognized at the time.

These coils were tested by the magnetometer in the same manner as were the fixed coils, and, excepting $M1$, were found to be very good. $M1$ was much worse than $L1$, and so was not used in the later work.

The insulation was frequently tested as for the fixed coils, and always found to be very high.

The results of the direct measurements of the coils will be found in the appendix, page 386.

8. THE COOLING SYSTEM

As shown in Figs. 1 and 2, each form on which a fixed coil is wound has back of the slot in which the wire is wound a channel

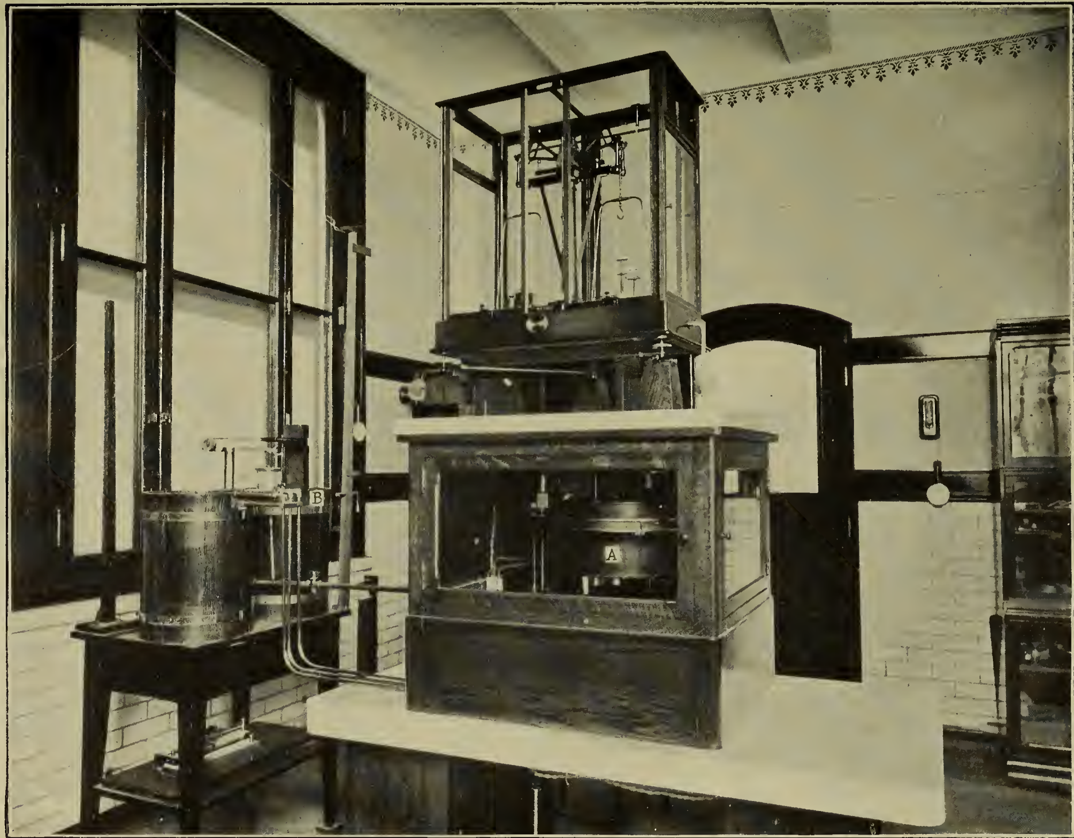

Fig. 5.—*Photograph of the current balance as used in the final measurements*

The moving coil is suspended from the pan of the balance. A is the water jacket surrounding the moving coil, B is the tank containing the thermostat which controls the temperature of the water through which water can be passed to remove the heat generated by the current and, therefore, to control the temperature of the coils.

During the weighings, the moving coil is surrounded by a cylindrical copper jacket, double-walled on the sides, completely closed at the bottom, and covered by a lid having a hole about a centimeter in diameter in its center, through which passes the tube from which the moving coil is suspended. The space between the two cylindrical walls is filled with circulating water, which carries away the heat generated by the current in the moving coil, and maintains the jacket at a constant temperature so that a temperature equilibrium can be attained without undue delay. Such an equilibrium is exceedingly important, for, unless the convection currents set up by the heating of the moving coil are maintained constant, the weighings will be erratic. By the use of such a jacket we have succeeded in obtaining weighings of such concordance that the mean departure of the individual results from the mean of a group is only about 1 per cent of the total force exerted by the convection air currents.

Water from a cylindrical tank, 25 cm in diameter by 40 cm deep, is forced, by means of an electrically driven turbine pump, through three pipes to the water channels in the forms of the two fixed coils, and to the space between the walls of the water jacket. After passing through the channels and the jacket, respectively, the water, now warmed a few degrees by the heat generated by the current, is conveyed by three pipes, each provided with a valve, to a small trough at the top of a second tank similar to the first. From this trough it passes through a pipe immersed in iced water contained in the second tank. During its passage through this pipe the water is cooled to a temperature somewhat lower than the water in the first tank into which it then enters. This overcooling of the water is compensated for by electrical heating, thermostatically controlled, thus giving a supply of water at a constant temperature. Since the two coils and the jacket are all supplied in parallel, the two coils may be maintained at very nearly the same temperature.

The water tanks and connections may be seen in Fig. 5.

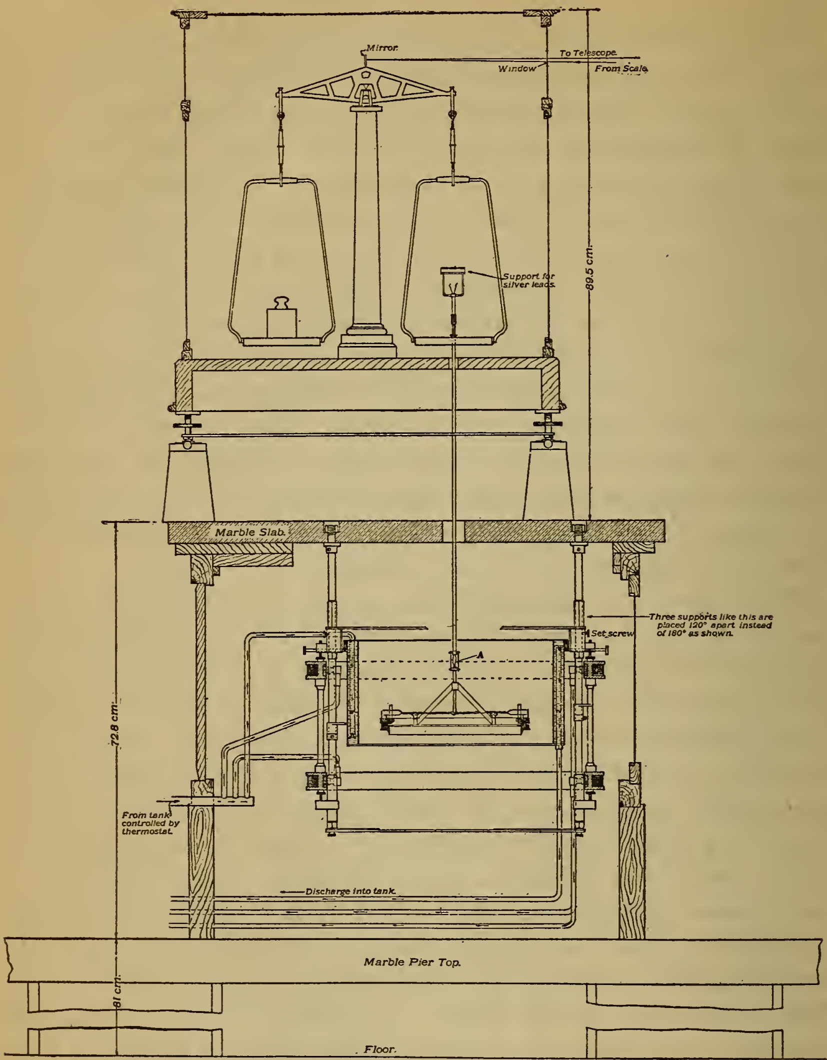

Fig. 6.—*Section of the assembled balance*

The fixed coils are suspended from the marble slab. The water circulation through the jacket surrounding the moving coil and through the channels of the fixed coils is shown

9. THE ASSEMBLED CURRENT BALANCE

The assembled balance is shown in Figs. 5 and 6. As may be seen, three brass rods suspend the fixed coils from the marble top of the case which surrounds them. This case has no magnetic material in its construction, and all the material used in the supports, pipes, etc., was tested with the magnetometer and found to be good. The balance rests on oak pyramids fastened to the marble top of the coil case.

The method of supporting the large fixed coils is shown in Fig. 7. *C* is one of the rods bolted to the marble top of the case; it carries two collars, *M* and *K*, to which are attached arms carrying the leveling screws *F* and *F′*, between which the coils and their distance piece *B* are clamped. A light brass rod, *J*, which fits rather snugly into holes drilled through the coils, passes through the coils and their distance piece and into holes of the proper size in the ends of the leveling screws. The ends of the distance piece *B* are faced accurately parallel to one another. By proper manipulation of the three pairs of leveling screws it is easy to level the coils and to clamp them rigidly together.

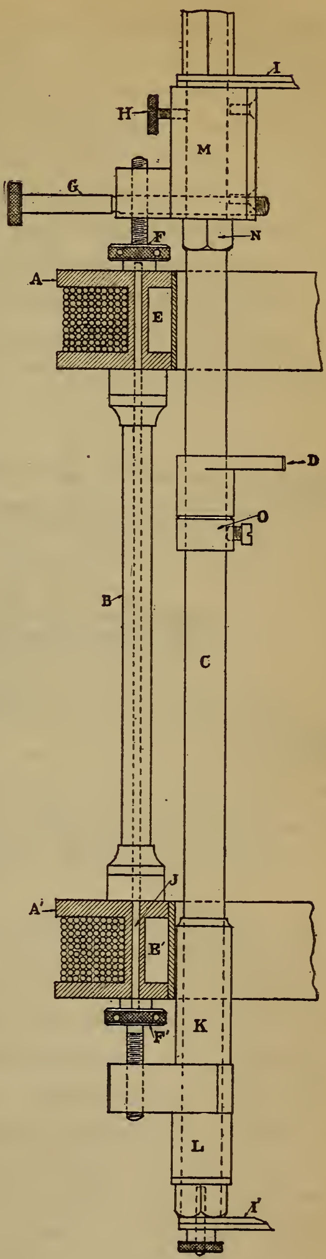

Fig. 7.—*Method of supporting fixed coils*

C is a stout brass rod bolted to the marble top of the coil case, *A*, *A′* are sections of the fixed coils, *B* is a distance piece separating the coils, *F*, *F′* are leveling screws, and *J* is a slender rod threading the coils and distance piece, and ending in cavities in the leveling screws

The holes through the coil forms having been accurately spaced when the forms were made, the coils will be very nearly coaxial when they are thus clamped. An electrical method for testing the accuracy of this adjustment is described in Section V. Here it will suffice to say that in no case was this adjustment found to be sufficiently in error to cause an error of 5 parts in a million in the force. The collar *M* rests upon a shoulder *N*, and when adjusted it can be clamped to the rod *C* by means of the set screw *H*. *D* is a flange attached to a collar fitting *C* and resting upon an adjustable collar *O*. It can be turned out under the upper coil so as to support it while the lower coil is being put in place. By using

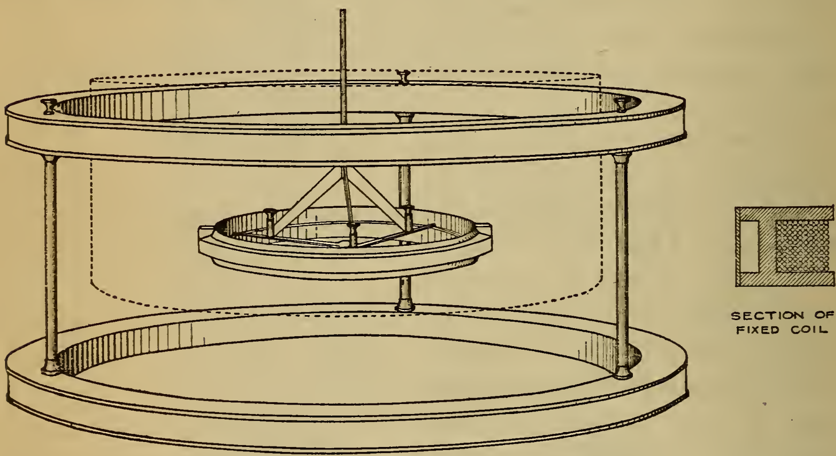

Fig. 7a.—*Perspective drawing of the coils. The dotted line shows the position of the water-jacket*

collars (*L*) of different lengths, distance pieces (*B*) of various lengths may be used. The three rods (*C*) are connected to one another both above and below the coils by means of light triangles of brass, *I*, *I′*, so that the entire mounting is very rigid.

The mounting for the smaller fixed coils differs from that just described merely in having the lower portions of the rods *C* offset with reference to the upper so that they can go inside the coils and still be attached to the marble at the same places.

To the inner flange (Fig. 3) of the moving coil is attached a light three-armed star *A*, having a 3-mm hole at its center, and to this star is attached, by means of three long leveling screws, a tripod *B* carrying a short section of thin-walled 5-mm brass tubing. By means of a collar and a pair of brass links this piece of tubing may be attached to a collar on a piece of similar tubing, which passes through the center of the right pan of the balance and is supported by a nut which rests on the top of the pan. By means of the three screws attaching the tripod to the star the height of the moving coil can be adjusted and the coil leveled. When in position and leveled, the axis of the coil passes through the tube at the top of the tripod. By undoing the links it is very easy to replace one moving coil by another.

Since the fixed coils are rigidly attached to the coil case, the lateral adjustment of the moving coil can be accomplished only by moving the balance. In order to be able to make changes of known amount in this adjustment, the plates, on which the front leveling screws of the balance rest, are provided with screw motion (Fig. 5), the left plate in a direction parallel to the length of the case and the right plate in a direction perpendicular to the latter. In order to eliminate any trouble from the springing of the leveling screws, they are connected to one another near their lower ends by means of collars and a rigid frame of brass.

The water jacket used with the large fixed coils is supported by the rods carrying the fixed coils; a smaller one used with *S*1 and *S*2 is supported on oak blocks resting on the pier top.

Connections between the brass pipes of the water-circulating system, and the coils and the water jacket, are made by means of glass and rubber tubing.

The coil case rests on a marble slab 152 by 76 by 7.5 cm, supported by two heavy oak piers resting on the concrete floor. In the earlier part of the work piers of white enameled brick set in Portland cement were used; but as the brick, and especially the sand used in the cement, were later found to be slightly magnetic, it was deemed advisable in the subsequent work to replace these piers by wooden ones. With the coils situated as shown in Figs. 5 and 6, this change produced a perceptible but very slight change in the force. The iron in the floor construction has been found to produce no effect upon the measurement of current. (See p. 347.)

10. THE ELECTRICAL CONNECTIONS

The two windings of each fixed coil are connected in parallel by means of a pair of closely twisted enamel-insulated wires, passing halfway around the circumference of the coil; a similar pair of twisted leads runs from the nearer terminals to binding posts set in the left wall of the coil case and connected with the commutator on the outside of the case. These posts and their connections to the commutator are carefully insulated with ebonite. Where the twisted leads expand into loops to connect with the coil terminals, the planes of the loops are carefully adjusted so as to lie as nearly as possible in a radial-axial plane. The outstanding effect of these loops is determined experimentally by measuring the force exerted upon the moving coil when the leads are short-circuited (without changing their positions) and a rather heavy current is passed through them, with no current passing through the fixed coils and with the normal working current flowing in the moving coil. The effect is always very small, but the proper correction has been applied to the observations.

Twisted leads pass from the terminals of the moving coil along a diameter of the coil to the axis, where they are soldered to twisted leads passing along the axis of the coil through the suspending tube into the balance case above the right pan of the balance. Here they are connected to the terminals of two sets of silver leads, one above the other, each consisting of 25 wires 0.02 mm in diameter and 8 cm long. The terminals at the other ends of these leads are connected by means of twisted leads to the commutator on the end of the coil case.

Where the leads pass through the brass tube, they are wrapped with a strip of paraffined paper so as to protect them from abrasion by the tube. They have soldered joints at the point *A*, Fig. 6, so that the coil and the lower section of the leads may be removed without disturbing the upper section. Wherever loops occur, they are placed as nearly as may be in radial-axial planes, and the outstanding effect is determined and allowed for by a method analogous to that described for the fixed coils. In order to be able to measure the resistance of the moving coil when carrying the working current, a pair of fine twisted leads runs up the tube with the current leads from the axial junction of the latter, and is connected with the outside terminals by means of a single pair of fine silver wires.

The terminals of the two sets of silver leads are insulated from one another by a block of ivory; the outer pair of terminals is carried by a stand which rests upon the floor of the balance case and can be moved so as to adjust the slackness of the wires. These leads affect the sensibility of the balance only slightly. The sensibility given on page 283 is with the leads in position.

The mercury commutator attached to the end of the coil case

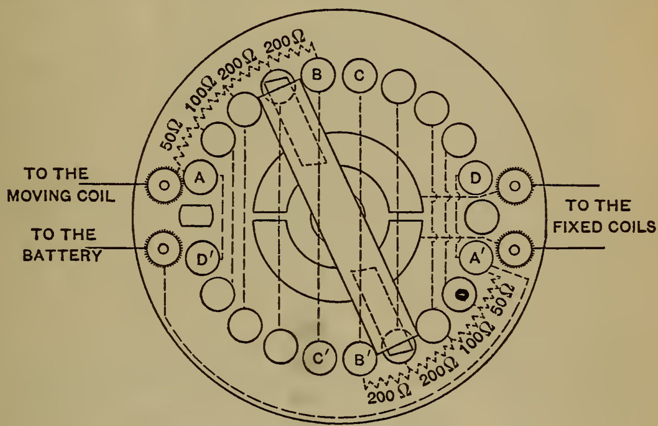

Fig. 8.—*Circular reversing switch*

As the handle is rotated from *AA′* to *DD′*, the current is gradually reduced, reversed through the fixed coils, and then increased to its original value

enables the current in any coil to be reversed without changing the direction of the current in any other coil. From suitable terminals of this commutator, leads run to the circular reversing switch shown in Fig. 8. By means of this switch, which is placed on the end of the coil case where it can be readily reached by the observer at the balance, the current can be rapidly and smoothly reduced to a very low value, reversed through the fixed coils only, and then increased to its normal value. A reversal of this nature eliminates the danger of damaging the coils by a high induced electromotive force, and greatly facilitates the manipulation of the balance.

From the commutator, twisted leads with heavy rubber insulation run to the floor, and then through a conduit, one going directly to one terminal of a 25-ampere 120-volt storage battery, and the other to the standard resistance, regulating resistances, small fuse block, and switch, to the other terminal of the battery. The complete set-up is shown in Fig. 9. The resistances and potentiometer for measuring the current in terms of the international ohm and the Weston normal cell are on the table placed parallel to the pier, with its nearer end about 1.5 meters west of the axis of the coils. The working standard of electromotive force was given by four Weston normal cells kept in a thermostatically regulated kerosene bath on the small table in front. These cells and the working standard of resistance—an Otto Wolff 1-ohm coil—were frequently compared with the standards of this Bureau, and have been found to remain very constant.

The potentiometer was frequently calibrated and has been found to remain very constant. Its coils have been dipped in hot paraffin to prevent changes in resistance due to changes in atmospheric humidity. During the first portion of the work, the potentiometer was used to measure directly the voltage at the terminals of the 1-ohm coil; during the latter portion it has been customary to measure the small difference between this voltage and that of the standard cell, the two being connected in series and in opposition. This increases the sensibility and minimizes the errors introduced by slight variations in the potentiometer current.

By using as a source of current a storage battery of large capacity, which is entirely disconnected from all other circuits, and by waiting until temperature equilibrium of the coils of the balance and the ballast resistance has been attained, it has been possible to obtain a very constant current. Normally, it is possible to control the current so that its oscillations will amount to not more than one or two millionths of an ampere.

In order to facilitate the balance work, the current has always been adjusted so as to give a force of 1.5 or of 3.0 g, giving a change of force on reversal of the current of 3.0 or 6.0 g, respectively. A single platinum weight, of 3 or of 6 g, can therefore be used in each case in weighing the force.

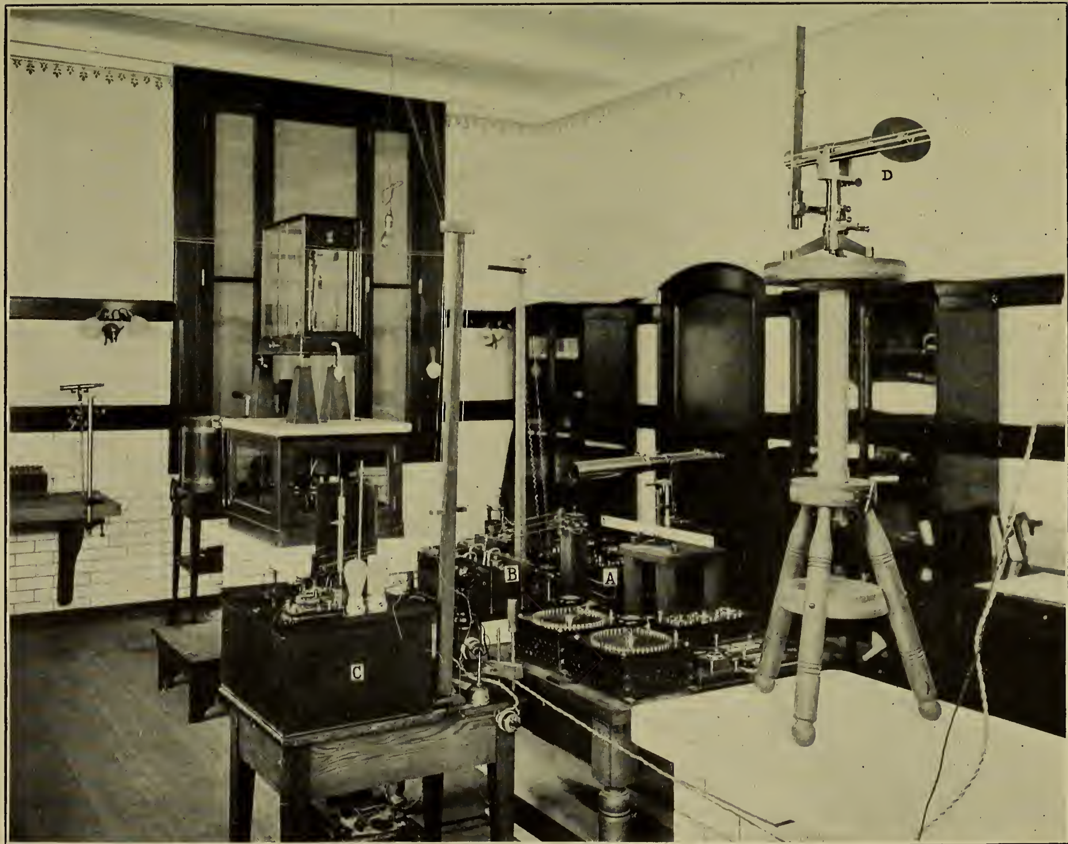

Fig. 9.—*Photograph of the auxiliary apparatus and of the current balance as mounted for the preliminary work*

A is a Wolff potentiometer, at B are standard resistances, C is oil tank containing the standard cells and thermostat, D is telescope and scale used in reading the rest points of the balance

III. THE RATIO OF THE RADII

11. DEVELOPMENT OF THE METHOD

Since the mutual force between given currents in two *coaxial circular circuits* varies in such a way as to become a maximum for a certain value of the distance between the planes of the circuits, the distance for maximum force, as well as the magnitude of this maximum force, must be a function solely of the radii of the coils. (That the force has a maximum is at once evident from the fact that it is zero both when the planes of the coils coincide and when they are at an infinite distance apart, but is finite for intermediate positions.) Furthermore, knowing that the dimensions of the square of a current in electromagnetic units are the same as those of a force, and that the force between two circuits such as we are considering is equal to the product of the two currents into some function of the radii of the circuits and of their distance apart, it is evident that this maximum force, of which we have spoken, is a function solely of the *ratio* of the radii of the two circuits.

Excepting for correction terms depending upon their finite section, the same is true for coaxial circular *coils*. Hence, the determination of the constant of the balance, in which the coils are so spaced as to exert their maximum mutual force, depends solely upon a knowledge of the sectional dimensions of the coils and upon the ratios of the mean radii of the fixed coils to the mean radius of the moving coil.

The value of this ratio can be obtained from the direct measurement of the mean radii of the coils, but such measurements for multiple-layer coils are very difficult to obtain and are at best of a relatively low order of accuracy.

A far better and more accurate method is to measure directly the ratio of the galvanometer constants of the two coils. From this ratio and a knowledge of their sectional dimensions we obtain at once the ratio of the two mean radii, the quantity upon which the constant of the balance depends.

In taking account of the sectional dimensions of the coils in the computation of the constant of the balance, it is, however, simpler—at least, the steps appear more tangible—if we deal with the radii directly rather than with their ratios. In order to obtain numbers corresponding to the mean radii, the absolute values of these radii affecting only the correction terms necessitated by the finite sectional areas of the coils, it is allowable to assume *any* approximate value for the mean radius of one coil, and from this and the measured ratios to deduce the corresponding radii for the other coils.

In the present work we have assumed that the mean radius, corrected for the temperature, of the coil designated as $S1$ is, and has always remained, exactly that calculated from the direct measurements made upon the various layers of the coil as it was wound.

Methods—In 1854 Bosscha[9] described a method for the determination of the ratio of the galvanometer constants of two coils. This method, which was employed by Lord Rayleigh to obtain the radius of the moving coil of his balance in 1884, consists in placing the two coils concentric and with their planes in the magnetic meridian; in connecting the coils, with suitable resistance in series with each, in parallel and in such a way that when a current is passed through this compound circuit the magnetic fields at the center of the coils will be opposed; and then in adjusting the resistance in one branch of the compound circuit until the torque exerted upon a small magnetic needle suspended at the center of the coils is zero. When this condition is attained, the ratio of the current in the two circuits, and consequently the inverse ratio of the resistances of the two circuits, will be equal to the inverse ratio of the two galvanometer constants (except for a correction term depending upon the length of the magnetic needle employed). Hence, excepting for this correction term, the ratio of the galvanometer constants is equal to the ratio of the two resistances, which can be measured.

The method as described is ideally simple and easy of application where extreme accuracy is not needed. When, however, an accuracy of 1 in a million, or even of 1 in 100 000 is desired, the heating of the coils by the measuring current introduces almost insuperable difficulties.

[9] Pogg. Ann., 93, p. 402; 1854.

In order to minimize these difficulties, Lord Rayleigh so arranged the coils and standard resistances that, after the balancing of the magnetic fields, the removal of a single link sufficed to convert the resistances, of which we desire the ratio, into the adjacent arms of a Wheatstone bridge, the opposite arms of which were composed of standard coils. Thus, it was possible to pass rapidly from one measurement to the other, and so to reduce the change of temperature to a relatively small amount. He was thus able to obtain an accuracy that was ample at that period.

In order to increase the accuracy still further, the link used by Rayleigh was omitted (thus keeping the coils in the Wheatstone net during the process of balancing the magnetometer), and a simultaneous balance of both the bridge and the magnetometer was obtained. Then the ratio of the resistances in the arms containing the coils will, at the instant of balance, be exactly that of the ratio of the other two arms of the bridge. These, which we shall call the ratio arms, must be of low resistance, and must have small temperature coefficients, as they carry the full currents passing through the coils. They must further be capable of fine adjustment in order to obtain an exact magnetometer balance.

These conditions can scarcely be fulfilled by coils designed for precision resistance measurements, so arrangement was made for quickly transferring these ratio arms to a second bridge in which they can be measured against precision resistances.

This method, though workable and yielding results of higher accuracy than those previously used, was abandoned early in the work for three reasons: (*a*) It is slow, and therefore variations in the earth's field are a grave source of error; (*b*) it is difficult to avoid slight trouble due to the heating of the ratio arms; and, most important of all, (*c*) the current being alternately off and on, the coils to be compared never attain a stationary temperature condition, and hence an exceedingly accurate interpretation of the results is not possible.

In recent times the facilities for the measurement of current have improved to such an extent that it is now almost as easy to measure two currents by means of potentiometers and to determine their ratio as it is to measure the ratio of. two resistances. This very obvious change in the details of the method involves simultaneous observations by three observers, but is capable of the most extreme accuracy. This may be called the *potentiometer method.*

Another modification of the details of the method renders it even more simple than as used by Bosscha, but is applicable only to coils having very nearly the same galvanometer constant. Two such coils are connected in series, and the coil with the larger galvanometer constant, together with an added resistance, is shunted so as to obtain a zero field at the center of the coils. If the galvanometer constants of the coils differ by a small amount it is easy to adjust the resistance in series with the shunted coil so that the shunt is large enough to give the desired sensibility of balance. Neither the shunt nor the shunted resistance need be known accurately, since a very small part of the total current flows through the shunt.

This method may be called the *shunt method;* it requires but a single observer. By using the shunt as a volt box it is easy to determine under working conditions the resistance of the circuit shunted in terms of a standard coil through which the total current passes. Since the galvanometer constants are assumed to be very nearly equal, very little current will flow through the shunt and, consequently, it will not be appreciably heated.

Under certain conditions a combination of these two methods may be desirable. This may be called the *combination method.* It is described in a later portion of the paper.

In order to obtain the best results it is necessary to make the observations at a time when the earth's magnetic field is as steady as possible. This is usually at night.

12. THE POTENTIOMETER METHOD

The connections for the potentiometer method are shown in Figure 10. M and F are the two coils; they are actually coplanar and concentric, though, for simplicity, shown otherwise. R_1 and R_2 are two standard resistances from which leads run to the potentiometer for measuring the currents. From that resistance which is in series with the fixed coil, auxiliary leads run to a second potentiometer. By means of this potentiometer and a continuously variable resistance in circuit with the fixed coil the current can be held constant at any desired value. C_2 is a commutator by means of which the two resistances may be interchanged with reference to the coils M and F; r, r_1, and r_2 are adjusting resistances. L is a self-inductance. C_1 is a pair of commutators for reversing the current through the coils. Since it is impossible to so construct C_1 that it will make and break both circuits at identically the same time, it is connected with the switch S so that they can all be thrown at one operation, and

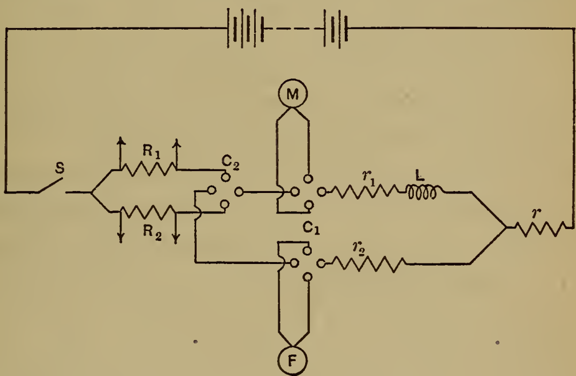

Fig. 10.—*Connections for the potentiometer method of measuring the ratio of the radii*

so that S is opened before C_1 and closed after C_1, thus avoiding severe deflections of the magnetometer.

In the present case the coils M and F are wound on metal forms, and, consequently, owing to eddy currents, the time constant of the field at the center of each, when alone, is quite large. But as adjusted for the ratio of the radii work, the field inside M is very small, being largely neutralized by the field due to F; that inside F, but outside M, is much greater than it would be if M were absent. Hence, as thus adjusted, the eddy currents in the form of M will be small, while those in the form of F will be about as strong as if M were absent. Hence, the time constant of the field of F will be much greater than that of the field of M. This will cause a deflection of the magnetometer whenever the current is stopped or started. Furthermore, when the coils are connected for the two-potentiometer method there is an interval between the opening of S and that of C_1, during which the circuit of F is closed through M so that the induced electromotive force in the former gives rise to a current through both F and M in such directions that the two fields thus produced at the center are in the same direction. This adds to the effect just mentioned, and causes a very violent deflection of the magnetometer whenever the current is made or broken during the use of this method. To reduce this deflection, and so increase the speed of the work, a suitable *large* inductance L is introduced in the moving-coil circuit. It is placed at a distance from the magnetometer so as to reduce its effect upon the needle, it being desirable to have the zero of the magnetometer on closed circuit the same as on open circuit. However, by the procedure adopted in this work, in no case can the field of the inductance produce any effect upon the observed ratio of the galvanometer constants.

The method of procedure was as follows: The resistances r, r_1, and r_2 were adjusted so as to make the two fields approximately equal and of such a strength as to give the desired magnetometer sensibility (a change in reading of 1 mm on reversal corresponding to a difference in the two fields of about 3 or 4 in a million). The current was then left on continuously, with water of the proper temperature circulating through the fixed coil, for about an hour before observations were begun. Then the resistances of the coils were measured, the thermometers attached to the coils were read, and while one observer held the current through the fixed coil constant and at the desired value, a second observer, with a potentiometer which could be connected to either R_1 or R_2, connected his potentiometer across the standard resistance in series with the fixed coil, and adjusted the potentiometer current until he obtained the nearest possible balance with some even setting of the potentiometer dials; the lack of exact balance was measured by the galvanometer deflection and allowed for in the reduction. He then threw his potentiometer across the standard resistance in the moving-coil circuit, adjusted this current approximately to its correct value, and then allowed it to drift slowly toward his potentiometer balance, while a third observer damped and read the magnetometer. The third observer gave a signal at the instant of reading the magnetometer, and the second observer noted the galvanometer deflection at that instant. The first observer thoughout this time held the current through the fixed coil at a constant value. The switches C_1 are now thrown so as to reverse the currents through both coils, and the operation is repeated. After taking several such pairs of observations, the second observer again puts his potentiometer across the standard resistance in the fixed-coil circuit, and observes the deflection when the dials are set as at first, and the first observer's indicating apparatus is balanced as at the start. The very slight difference between this and the deflection observed at the start is due to the relative change in the two potentiometer currents, and is allowed for in the reduction. The thermometers are then read, C_2 is reversed so as to interchange the standard resistances with reference to the balance coils, and the operation is repeated. Taking the mean of these two sets of observations eliminates the values of the two standard coils, and hence makes it unnecessary to know their values accurately. Two such sets, each of five pairs of magnetometer readings, takes from 15 to 20 minutes. The resistances of the coils are then measured.

By this method of procedure, it is evident that the only stray field that can affect our results is that due to the leads from C_1 to the coils. The effect of these is always very small, and was always measured by noting the magnetometer deflection produced by reversing, through the leads alone, short-circuited at the coil terminals, a current of 4 or 5 amperes. Of course, one terminal of each winding was entirely disconnected from the leads, so that there was no possibility of any of the current passing through the coil itself.

The relative values of the standard resistances R_1 and R_2 can be eliminated by interchanging them only if they are of the same denomination. In some cases standard resistances of different denominations would have to be used for the measurement of the ratio of the galvanometer constants; in which cases the relative values of R_1 and R_2 would have to be known with high precision. The necessity of using standard resistances of different denominations arises from the fact that the lower setting on the potentiometer must be great enough for its value to be determined to the required degree of accuracy, and the higher one must, of course, not exceed the range of the instrument.

13. THE SHUNT METHOD

In the present work *M3* was so constructed that when its windings are in series it has very nearly the same galvanometer constant as *S1* or *S2*, when their windings are in parallel. Consequently, the ratio of its constant to that of either *S1* or *S2* could be determined by the shunt method. In fact, both methods were used, and checked very closely.

The connections used for the shunt method are shown in Fig. 11. As in the previous method C_1 and S were thrown simultaneously

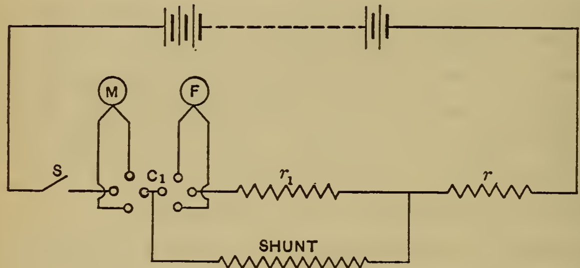

Fig. 11.—*Connections for the shunt method*

and in such a way that S broke circuit before either of the commutators C_1, and closed circuit after C_1.

After the temperature equilibrium had been reached, the shunt was adjusted for approximate balance, and the magnetometer was read; then the currents through the coils were reversed, and the magnetometer was read again, and so on. About nine such pairs of observations can be made in five minutes.

14. THE COMBINATION METHOD

If the constants differ so much that in order to obtain the accuracy required in the value of the ratio it is necessary to know r_1 (Fig. 11) to an inconveniently high degree of accuracy, we can use the same schematic arrangement, but by means of a potentiometer and of standard coils in r_1 and in r we can measure directly the currents in the two coils, as in the potentiometer method; or, if the constants are quite different we can incorporate the standard coils with r_1, and with the shunt, so that we can measure the currents in these branches. This is an especially desirable arrangement if the constant of F is twice that of M.

In these combination methods, it is desirable that the observer who holds the current constant should have his potentiometer so connected as to control the *total* current. The second observer, with his potentiometer across a standard coil in r_1, will then observe practically no drift, and so can read his deflection at the desired instant with a maximum of accuracy; he then throws his potentiometer across the other standard coil and measures that current. If the constant of F is very nearly equal to twice that of M, then these two standard coils should have the same value and the two potentiometer settings will be nearly the same; everything will be suitable for a well-balanced measurement. The main objection to this method is the purely practical one of facility in manipulation. As every change in r_1 or in the shunt affects the current through M, the balancings of the two potentiometers have to proceed simultaneously. This makes the attainment of a balance distinctly slower than in the potentiometer method, where the two circuits are almost independent of one another.

All of these methods have been used, but as no case existed in which the ordinary potentiometer method could not be satisfactorily employed, we have in the final work used that and the shunt method only.

15. ADJUSTMENTS OF COILS AND NEEDLE

As shown in Fig. 12, the coils were clamped concentric with one another by means of three brass blocks fastened to a marble slab. The slab is supported vertically on a wooden stand having leveling screws. All portions of the mounting were tested by the astatic magnetometer and were found to be satisfactory. The magnetometer needle was of tungsten steel, about 1.9 mm long, 1 mm wide, and not over 0.2 mm thick, fastened to a slender glass rod, to the lower end of which was attached a small plane mirror. By means of a quartz fiber, about 5 cm long, the whole was suspended in a brass tube, 17 mm in diameter, which was attached to the

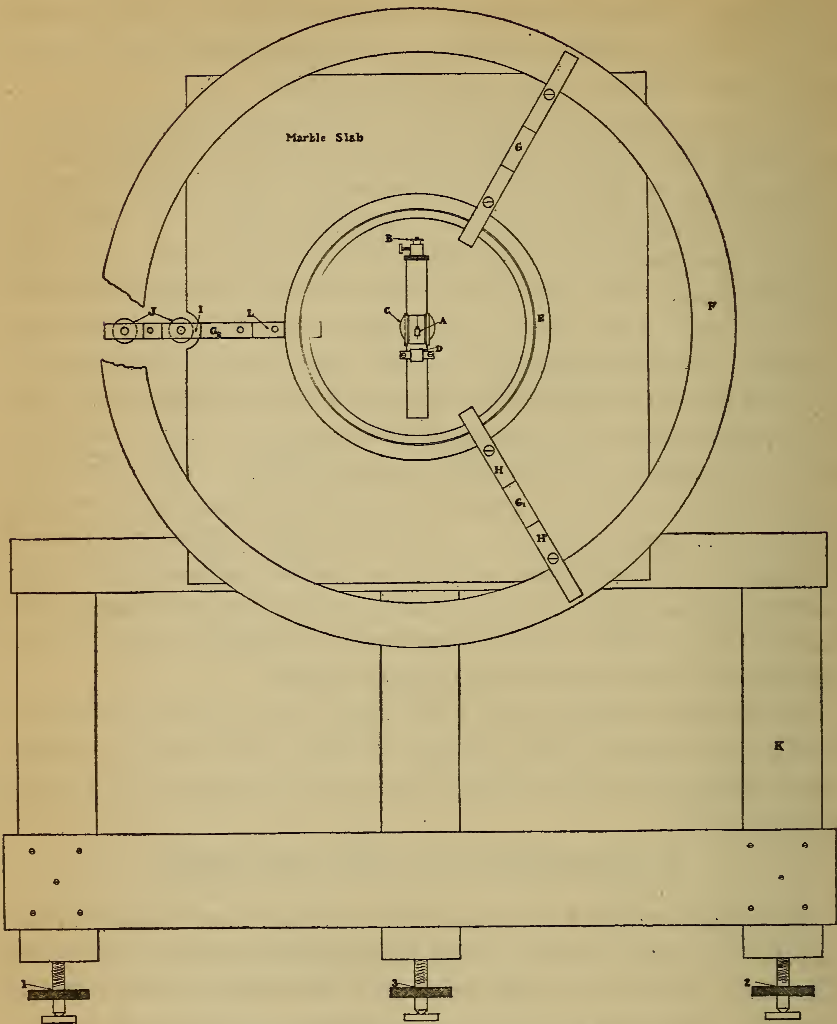

Fig. 12.—*Mounting for the determination of the ratio of the radii*

A is suspended magnet and mirror, *B* is screw for adjusting height of magnet, *C* is hole in marble slab with cross-hairs for preliminary adjustment of magnet, *D* is fixed mirror for determining position of stand, E is 20-cm moving coil, *F* is 50-cm fixed coil, *G*, G_1, G_2 are brass blocks attached to the marble slab, *H*, *H'* are clamps, *I* is slot for 40-cm fixed coil, *J* are micrometers for rotating fixed coils around a vertical axis, *L* is removable block for admitting 25-cm moving coil marble so that the needle hung near the axis of the coils. By means of a screw *B* (having a graduated head) the vertical position of the needle could be adjusted over a range of several millimeters. The needle could be displaced horizontally parallel to the planes of the coils by means of the leveling screws 1 and 2. By means of a graduated series of brass liners, each coil could be independently adjusted so that its plane passed through the needle. By means of the micrometer screws *J* with threads of 0.5-mm pitch, the large coil could be delicately adjusted by rotation about a vertical axis so as to place its plane parallel to the plane of the small coil; and by rotating the entire stand the planes of the coils could be set parallel to the needle.

Now it is evident that if the mean planes of the two coils exactly coincide, then the ratio of the moments exerted on the needle for given currents in the two coils will be independent of the angular position of the needle with reference to the coils, except for a very small second-order correction term, depending upon the variation in the correction for the length of the needle.

If the needle lies along the bisector of the angle between the planes of the two coils, then the ratio of the moments will be independent of this angle, except for the same small correction term.

But if neither of these conditions is fulfilled, then that coil which is the more nearly parallel to the needle will exert a relatively greater moment, and for two reasons. First, the poles of the needle lie in a field which, relative to the other coil, is stronger; and, second, the direction of this field is more nearly normal to the length of the needle than is that of the other coil. The first of these effects is very small, as it concerns the second-order correction term spoken of above. The second effect results from the fact that the moment which either coil exerts upon the needle is proportional to the cosine of the angle between the magnetic axis of the needle and the plane of the coil, so that the ratio of the two moments is proportional to the ratio of these two cosines. For example, suppose the needle makes an angle of $2°$ with the plane of one coil and $2° \, 1'$ with the plane of the other—that is, the planes of the coils are inclined at an angle of $1'$—then the value of the ratio of the galvanometer constants, calculated from the currents on the assumption that the adjustments are perfect, will be in error by a factor equal to the ratio of cos 2° to cos 2° 1′—that is, it will be in error by 1 in 100 000.

With a needle but 2 mm long, and a fiber which must be stout in order to withstand the rather rough treatment to which it is necessarily subjected, it is evident that the adjustment of the needle to within 2° by any process of inspection is out of the question. The same is true with respect to the adjustment of the coils to within a single minute of arc. But, by making use of this very error due to maladjustment, it is a comparatively simple process to set *both* coils and needle in parallel planes. We have adopted the following procedure:

The coils are adjusted so that their planes are vertical and approximately coincide in the magnetic meridian. The planes of the coils are made vertical with sufficient precision by means of a plumb line. The vertical height of the needle is adjusted so that the latter lies on a level with the centers of the coils, and the fiber is free from torsion. The large coil is rotated about a vertical axis through an angle of a few minutes by means of the adjusting screw. Then, by means of an auxiliary coil, suitably placed, and a constant battery, the needle is deflected from its normal position by about 24°, and the apparent ratio of the galvanometer constants is measured with the needle in this deflected position. The current through the auxiliary coil is now reversed so as to deflect the needle by about the same amount in the other direction, and the ratio is again measured. Suppose the angle between the coils is 3′; then, when the needle was deflected (with reference to the small coil) in the same direction as the large coil, the moment exerted by the latter relative to that exerted by the small coil was approximately 39 parts in 100 000 too great; when deflected in the other direction, it was too small by the same amount, provided that the undeflected position of the needle bisected the 3′ angle between the coils and the deflections were equal in both directions. Otherwise one of these quantities will be increased, and the other decreased, by slightly different amounts. If we plot these observed ratios of the galvanometer constants as ordinates against angular positions of the needle as abscissæ, the straight line connecting them will pass very nearly through the point representing the correct ratio at that abscissa which corresponds to the position of the needle when it bisects the angle between the two coils.

Now rotate the fixed coil in the opposite direction, and repeat the measurements, and (if the coil has been turned beyond the position where they are coplanar) we obtain a line sloping in the other direction. The abscissa of the intersection of these two lines is very nearly that setting of the needle which makes it parallel to the moving coil. It is exactly that setting if the lines intersect at their middle points. Then the stand carrying the coils and magnetometer is rotated in the proper direction through an angle, as indicated by the reflection of the scale from a mirror fixed to the magnetometer tube, that is equal to the ascertained angle between the coils and the needle. The observations are then repeated in order to test the accuracy of the setting. Thus, the needle is placed in the plane of the coils to within the accuracy with which we can adjust the stand. In practice, 1 cm on the scale was considered sufficiently accurate; at a scale distance of 3 m this corresponds to an angle of about 6′. With an error of this magnitude in the position of the needle, an angle of 2′ between the planes of the coils will give an error of 1 in a million in the result. Hence, it is necessary to assure ourselves that the coils are parallel to within this limit.

From what has been said above, it is evident that it is impossible for the apparent ratio with the needle deflected +25° to be the same as that with the needle deflected —25°, unless the coils are parallel to one another. Hence, all that is necessary in order to set the coils parallel to one another is to rotate the large coil until this condition is fulfilled. The amount and sense of this rotation are indicated by the slopes of the two lines on the plot. Using a 25° deflection, and 1 in 100 000 as an accuracy rather easily obtained without refined reductions, we see that we can detect an angle of 2.4×10^{-5} radians, or about 5″ between the planes of the coils. This, for the mounting of the small fixed coils, corresponds to an advance of the adjusting screw of only 7 microns. Occasionally an accuracy of this amount was accidentally attained, but usually the accuracy of setting was about one-fifth of this, or 25 seconds. With an error of 6′ in the setting of the needle, and 25″ between the coils, the observed ratio of the galvanometer constants will be in error by 2 in ten million. This is about the error that this maladjustment may introduce into our latest determinations of the ratio of the radii.

In the earlier work the importance of this adjustment was not recognized, and as a consequence those measurements may, from this cause, be in error by over 1 in 100 000.

Besides the orientation of the coils, four other adjustments must be made, viz, the vertical adjustment of the needle, the adjustment of each coil so that its mean plane passes through the needle, and the lateral adjustment of the needle in the plane of the coils. In each case measurements of the ratio were made for various adjustments of the type under investigation and the results were plotted. The theoretical curve, representing the variation of the ratio with variations in this type of adjustment and calculated from the known dimensions of the apparatus, was fitted as well as possible to the plotted results, and the particular adjustment was made to correspond to the vertex of the curve. Any slight failure in realizing this exact adjustment, at the time the observations were taken, was allowed for in the final precise reductions.

The adjustment of the angle between the planes of the coils was always repeated after all other adjustments had been made.

16. CORRECTION TERMS

In addition to the corrections already considered, there are three others of prime importance.

(a) *Needle correction.*—This depends upon the distance between the effective poles of the needle. For large magnets this has been found to be about five-sixths the total length of the magnet. This value was used in the earlier portion of the work. Later it was noticed that the amount of the correction can be determined directly by a comparison of the apparent ratio of the constants when the needle is deflected with that found when the needle is not deflected. A series of observations has been made with this object in view, and the corrections to the observed ratios as determined experimentally are given in Table I. The effect of the length of the needle is such as to increase the measured ratio of the radii of the larger to the smaller coil, and the correction, therefore, has the negative sign.

TABLE I

Correction for the Length of the Needle (Needle Length = 2.0 mm)

Radii of coils		Parts per million
20 cm	10 cm	−56.3
25	10	−63.0
25	12.5	−35.9

These corrections correspond to a polar distance of 2.00 mm; five-sixths of the total length of the needle is 1.93 mm, but it is evident that the vertical width of the needle must increase its effective length; hence the discrepancy between the observed polar distance and that calculated from data obtained with much larger magnets is but 3 per cent. We are not acquainted with any previous attempt to measure the polar length of such small magnets.

(*b*) *Correction for the temperatures of the coils.*—Since it was found that the temperature indicated by a thermometer attached to the form of a fixed coil varied appreciably as the position of the thermometer was changed, and as it was obviously impossible to read a thermometer attached to the moving coil when the latter was surrounded by the water jacket, it was deemed best in the latest work to derive all temperatures from the resistances of the coils. This involves a knowledge of the resistances of the coils at some known temperature and of the temperature coefficient of the resistance of the coil. The first is easily obtained, and the latter can be found by a preliminary measurement. But, as it is necessary to determine experimentally the coefficient of expansion of the coils, it was considered simpler and better to assume a value of the resistance temperature coefficient that is approximately right, and to determine the coefficients of expansion in terms of the temperatures calculated on the basis of this coefficient. Then any error in the assumed resistance coefficient will be exactly compensated for by the observed coefficient of expansion. We have taken as our temperature of reference 22° C, and as the assumed resistance temperature coefficient per 1° C the value 3.9×10^{-3} of the resistance at 22°.0 C. We have called the temperatures thus determined the "electrical temperatures" of the coils.

The values of the coefficients of expansion of the coils have been determined from measurements of the ratio of the galvanometer constants for various temperatures of one coil, the other changing in temperature but little. The temperatures of the fixed coils were regulated by the temperature of the water circulating through them; those of the moving coils by the temperature of the room. It was found experimentally that the temperature coefficients as thus determined are independent of the actual temperature and of the rapidity of the flow of the water within the limits of accuracy with which we are concerned. The resistance of a coil at any time was determined by potentiometer measurements of the current through it and of the drop of potential across its terminals. The values of the coefficients of expansion of the coils are given in Table VII, page 322.

(*c*) *Correction for the load.*—It is evident that the wire when carrying a current will be at a higher temperature than the form on which it is wound, and this excess will be proportional to the load—that is, to the number of watts expended upon the coil. But the radius of the coil is determined by neither of these temperatures, but by some temperature between them. Hence the radius of a coil will not be determined solely by its electrical temperature, but after the latter correction is applied there will be still another one needed to take account of the load. The amount of the latter correction will depend upon the heat insulation of the windings from the form and may be expected to vary greatly from coil to coil.

To determine this correction, it is sufficient to measure the ratio of the galvanometer constants with various loads on one coil and a constant load on the other. To realize such a condition, we have made use of the property of conjugate conductors in a

Wheatstone bridge. The leads from the reversing switch, C_1, of Fig. 10 (p. 301), instead of going directly to the terminals of F, with its two windings in parallel, run, as shown in Fig. 13, to the opposite corners of a Wheatstone net. Two arms of this net are composed of the two windings A and B of the coil F, the others of resistances r_a and r_b. R_a and R_b are two standard 1-ohm coils

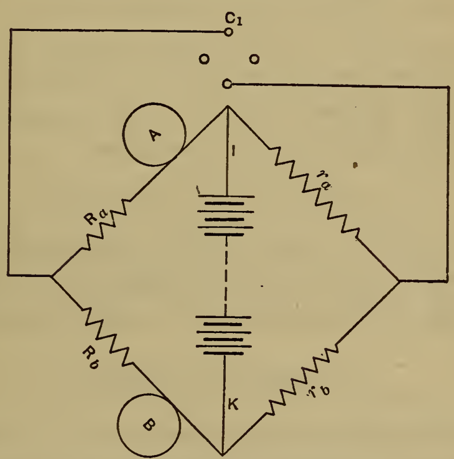

Fig 13.—*Bridge used for electrically loading a coil in the determination of the correction for the load*

by means of which the current through each winding can be measured. The windings are so connected that the current from C_1 passes through them in the same direction—that is, their magnetic fields add together. By adjusting r_a and r_b, the bridge is balanced, and then any electromotive force desired may be placed across IK without affecting the current from C_1, which measures the galvanometer constant. Furthermore, the constants of the two windings of any coil are practically identical, and, as would be expected, the current from the IK battery flowing through them in opposite directions has been found to produce no effect on the magnetometer. Also, since measurements of the constants are made by reversing C_1, IK remaining unchanged, the only possible effect the current from IK could produce upon the magnetometer would be a permanent displacement of the zero. Furthermore, any slight lack of balance of the bridge (called, in general, the heating bridge) will produce no error, for the resulting current through the bridge will add itself with proper sign to the measuring current and after passing C_1 will be measured with the latter. Hence the only point requiring especial care is that the *IK* battery shall be well insulated from the battery furnishing the current for the measurement of the ratio. Thus we are enabled to study the effect of the load upon each coil. It has been found that the apparent radius of a coil after being corrected to a constant electrical temperature decreases linearly as the load increases. Hence the radius of the coil is given by the expression:

$$A = A_0 \{1 + \tau (t - 22) - \lambda w\}$$

where A is the radius at the electrical temperature t, and with the load of w watts, and A_0 is the radius at 22°.00 C and with no load. The values of the coefficients τ and λ are given in Table VII, page 322.

17. SOURCES OF ERROR

The sources of error to be considered are (1) maladjustment, (2) effect of leads, (3) changes in the size of the coil, (4) error in the measurement of the sectional dimensions, (5) magnetization of the forms.

The first two have already been considered. The third may arise from either of two causes; either from the strains incident to winding the coil, or from changes in the humidity. The latter may be obviated by sealing the coils, and the former by heating the coil (by means of a current) for several hours to a temperature appreciably higher than any at which it will afterwards be used.

The magnitude of the fourth source may be estimated from a consideration of the equation expressing the relation between the galvanometer constant of a coil and its sectional dimensions, viz:

$$A_0 = \frac{2\pi n}{G_0}\left\{1 - \frac{1}{2}\left(\frac{\alpha_0}{A_0}\right)^2 + \frac{1}{3}\left(\frac{\rho_0}{A_0}\right)^2 + \frac{3}{8}\left(\frac{\alpha_0}{A_0}\right)^4 + \frac{1}{5}\left(\frac{\rho_0}{A_0}\right)^4 - \frac{\alpha_0^2\rho_0^2}{A_0^4} + \cdots\right\}$$

where A_0 is the mean radius of the coil, n is the number of turns of wire, G_0 is the galvanometer constant, $2\,\alpha_0$ is the axial breadth of the windings, $2\,\rho_0$ is the radial depth of the windings.

If, owing to errors in measurement, we assume the values $\alpha=\alpha_0+\delta\alpha$, $\rho=\rho_0+\delta\rho$ instead of α_0 and ρ_0 we shall find from the observed galvanometer constant not A_0 but A. Expanding the above expression by Maclaurin's theorem we find

$$\frac{A-A_0}{A_0}=-\left(\frac{\alpha_0}{A_0}\right)^2\frac{\delta\alpha}{\alpha_0}+\frac{2}{3}\left(\frac{\rho_0}{A_0}\right)^2\frac{\delta\rho}{\rho_0}$$

In the case of the coils used in this work $\frac{\alpha_0}{A_0}$ and $\frac{\rho_0}{A_0}$ were approximately either 0.04 or 0.05, so that the per cent error in A is about 0.002 times the per cent error in either dimension of the section. Hence, an error of 0.01 mm in the width or in the depth of the windings of a coil with a section 1 cm square will produce an error of about 2 in a million in the radius. Excepting the moving coils, all coils have a larger section than this, and so the effect of an error of this magnitude in the measurements will produce for them a smaller error in A.

As pointed out by one of us,[10] and as shown at length in the appendix to this paper, page 375, the sectional dimensions of the windings of any coil are to be understood as determined by the expression ns where n is the number of wires in the direction of the dimension considered and s is the distance between the axes of consecutive wires.

Hence, in the above expression, $2\rho_0=\frac{n}{n-1}(r_0-r_i)$, where n is the total number of layers, r_0 is the radius to the axis of the wire in the outer layer, and r_i is the radius to the axis of the wire in the bottom layer. These quantities can be readily measured with considerable accuracy, probably to within 0.01 mm. For details, see page 390 in the appendix.

As already stated, the coils were wound with enamel-covered copper wire, which winds more smoothly and uniformly than silk-covered wire, the finished coils presenting a very beautiful appearance. However, adjacent spires are not in actual contact throughout, so that the distance between the axes of adjacent wires slightly exceeds the diameter of the wire

[10] This Bulletin, 2, pp. 77, 413; 1906, and 3, p. 235; 1907.

(about 0.01 mm to 0.03 mm). This will produce no error, provided the spacing is uniform and half as wide a space occurs on the average between the end turns and the sides of the channel; $2\alpha_0$ is then equal to the breadth of the channel.

After calculating the magnitude of the maximum error that could occur due to this cause, we think it improbable that an error greater than 3 parts in a million exists in any coil, except moving coil $M4$.

Moving coil $M4$ was wound in such a way that in each layer the wires were crowded toward one side of the channel, adjacent layers being displaced in opposite directions. The wires were maintained in the position which they occupied on the completion of the layer, by the paraffin which was thoroughly melted into the layer. Owing to very slight, almost microscopic, kinks in the wire, and to its tendency to slide into the spaces between the wires in the underlying layer, it is practically impossible to pack the wires so closely that the actual width of the space in which the n wires lie is equal to nd, where d is the diameter of the wire. Hence, the value $2\alpha_0$ for this coil lies between the values nd and the width of the channel. Now it is easy to show that the galvanometer constant of a coil having layers of width nd, and so wound that each layer is displaced in its plane by an amount h, with reference to the adjacent layers, in the manner in which $M4$ is wound, is approximately equal to that of a coil of breath $\sqrt{n^2d^2+3h^2}$. For $M4$ the breadth of the channel is 10.877 mm, $nd=10.50$ mm, $h=0.377$ mm. Hence, $\sqrt{n^2d^2+3h^2}=10.520$ mm. This differs from the breadth of the channel by 0.357 mm, which, by what precedes, corresponds to a difference of about 63 in a million in the radius as calculated from the observed galvanometer constant.

This considerable range, within which the radius may lie, makes the results obtained by this coil of no value so far as the absolute measurement of current is concerned, but as it is satisfactory in every other respect, and is especially well protected from the effects of atmospheric humidity, observations with it will serve as a test of the constancy of the other coils.

In the computation of the constants for this coil, we have assumed, as for the others, that the effective axial breadth is equal to the breadth of the channel. As shown above, this is too great in the present case and the effect will be to yield too high a value for the standard cell.

As regards the fifth source of error, that due to the magnetization of the forms, it is difficult to obtain an estimate of its magnitude. It is evidently very small, for otherwise the variations in the results obtained in the measurements of current should have some relation to the magnetic properties of the coils. No such relation is evident, and we conclude that even the most magnetic of the fixed coils causes no appreciable error on this account.

18. RESULTS

In order to exhibit the method of reduction adopted, and to give an idea of the reproducibility of the observations, a few of the reductions and results of the most recent determinations are given in Tables II and III.

In Table II are given the determinations of the ratio of the galvanometer constants of $M3$ and $S1$ for various temperatures, and loads on the latter. These particular observations were obtained by the shunt method, and the values of the ratios of the current in $M3$ to that in $S1$, as given in column 6, have been corrected for the lack of balance of the magnetometer. For the potentiometer method, the numbers in this column would be the mean of two ratios, each corrected for the errors of the potentiometer; otherwise, the table would be the same.

TABLE II

Ratio of Galvanometer Constants, S1 to M3 (Determined by the Shunt Method) [11]

Date	Temperature		Load in watts		Observed ratio	Temperature correction		Load correction		Corrected ratio	Δ
	M	F	M	F		M	F	M	F		
Nov. 28	20°95	20°17	0.285	0.928	1.004604_1	+20.7	−31.1	+2.3	− 0.4	1.004596	3
	21.61	23.46	0.285	0.928	4559_5	+ 7.7	+24.8	+2.3	− 0.4	94	1
	22.99	20.79	1.200	3.860	4627_0	−19.5	−20.6	+9.8	− 1.8	95	2
	22.98	20.76	1.194	3.872	4626_8	−19.3	−21.1	+9.8	− 1.8	94	1
Nov. 29	21.06	23.78	0.294	21.25	4551_6	+18.5	+30.2	+2.4	−10.0	93	0
	21.06	23.78	0.290	21.15	4548_1	+18.5	+30.2	+2.4	− 9.9	89	4
	21.15	22.76	0.288	15.20	4568_7	+16.8	+12.9	+2.4	− 7.1	94	1
	....	....	....	....	4567_7	....	....	....	....	93	0
	21.12	22.75	0.286	15.19	4566_3	+17.3	+12.7	+2.3	− 7.1	92	1
	21.05	20.94	0.286	4.56	4594_4	+18.7	−18.0	+2.3	− 2.1	95	2
	....	....	....	....	4596_6	+17.5	−18.0	+2.3	− 2.1	96	3
	....	....	....	....	4593_5	+16.0	−18.1	+2.3	− 2.1	92	1
	....	....	....	....	4595_3	+14.5	−18.1	+2.3	− 2.1	92	1
	....	....	....	....	4595_7	+13.7	−18.2	+2.3	− 2.1	91	2
	21.35	20.93	0.286	4.57	4596_6	+12.8	−18.2	+2.3	− 2.1	91	2
	21.39	20.32	0.286	0.929	4609_2	+12.0	−28.6	+2.3	− 0.4	94	1
Dec. 2	21.65	20.34	0.282	0.918	4612_2	+ 6.9	−28.2	+2.3	− 0.4	93	0
	21.65	20.34	0.282	0.918	4610_9	+ 6.9	−28.2	+2.3	− 0.4	92	1
Mean	....	....	....	....	....	....	....	....	....	1.004593	1.4

[11] In the last column are given the differences from the mean value in parts in a million.

From these observations the temperature coefficient and the load correction for *S*1 are determined; these quantities for *M*3 are similarly determined from values not given in this table. By the use of these coefficients we deduce from the observed values the values that would have been found had both coils been at 22°00 C and without load—were such a measurement possible. These corrected values are given in the column headed "Corrected ratio," and their agreement is an indication of the accuracy of the measurement; their deviations from the mean average only 1.4 in 1 000 000. The mean of the values in this column after being corrected for the length of the needle, for maladjustment as determined by preliminary observations, and for the effect of the leads as determined just before removing the coils, gives us the most probable value of the ratio of the galvanometer constants of these two coils.

In Table III are given these ratios for the various combinations studied.

TABLE III

Corrected Observed Ratios of Galvanometer Constants

Fixed	M2	M3	M4
S1	....	1.004652	....
S2	....	1.005101	....
L1	2.247584	1.325326	2.273009
L2	2.247672	1.325371	2.273088
L3	2.247122	1.325054	2.272534
L4	2.246676	1.324790	2.272087

In order to obtain a better insight into the reliability of this work, and to average out the accidental errors as much as possible, it appeared desirable to deduce from these values as many nominally identical ratios as possible, and to use the means of these nominally identical ratios in the computation of the radii. For example, by dividing the numbers in column 2 by those in column 3, we obtain four independent values of the ratio of the galvanometer constant of $M3$ to that of $M2$; similarly, we can get four values of the ratio of the constants of $M3$ to $M4$. By a similar treatment of the rows, we obtain sets of three nominally equal ratios. Thus, we find the values given in Tables IV and V.

TABLE IV

Ratios of the Galvanometer Constants of M3 to those of M2 and of M4 as obtained from observations using coils L1, L2, L3, L4

	L1	L2	L3	L4	Mean
$\frac{M3}{M2}$	1.695872_6	1.695881_3	1.695872_0	1.695873_3	1.695874_8
$\frac{M3}{M4}$	1.715056_6	1.715058_0	1.715050_2	1.715054_6	1.715054_8

TABLE V

Ratios of the Galvanometer Constants of L1, L2, and L3 to L4, obtained from observations with M2, M3, and M4

	M2	M3	M4	Mean
$\frac{L1}{L4}$	1.000404_1	1.000404_6	1.000405_8	1.000404_8
$\frac{L2}{L4}$	1.000443_3	1.000438_6	1.000440_6	1.000440_8
$\frac{L3}{L4}$	1.000198_5	1.000199_3	1.000196_7	1.000198_2

From these tables it appears unlikely that the means are in error by more than about 2 in 1 000 000. This accuracy could not have been obtained had the work not been done at night, and only during steady magnetic periods. Frequently the work was suspended during times of magnetic disturbance.

In order to obtain the radii from these values it is necessary to know the radius of one coil and to know the value of

$$\frac{AG}{2\pi}=n\left\{1-\frac{1}{2}\left(\frac{\alpha}{A}\right)^2+\frac{1}{3}\left(\frac{\rho}{A}\right)^2+\frac{3}{8}\left(\frac{\alpha}{A}\right)^4+\frac{1}{5}\left(\frac{\rho}{A}\right)^4-\frac{\alpha^2\rho^2}{A^4}+\ldots\right\}$$

for every coil.

As already remarked, no knowledge of the absolute value of the radius is required except in the calculation of the corrections for the sectional dimensions of the coils, and, consequently, it is allowable to assume any approximate value for the radius of the coil of reference. We have accordingly assumed that the mean radius of S1 at 22°.00 C, and with no load, is exactly equal to 19.97510 cm, the value calculated from the direct measurement of the coil.

The data for the computation of *AG* is given in the following table.

TABLE VI

Correction to Galvanometer Constant

	Coil	Turns	Axial breadth 2α	Radial depth 2ρ	Radius A	$\frac{AG}{2\pi n}$
			cm	cm	cm	
Moving	M1	2×72	0.985_0	0.908_0	9.961	0.9994682
	M2	2×72	0.956_4	0.954_8	12.499	0.9997535
	M3	2×98	0.996_7	1.029_4	10.030	0.9996406
	M4	2×71	1.087_7	1.073_7	12.460	0.9996648
Fixed	S1	2×392	1.580	1.528	19.97	0.9997045
	S2	2×392	1.579	1.522	19.96	0.9997013
	L1	2×648	2.027	2.035	25.03	0.9997299
	L2	2×648	2.027	2.054	25.03	0.9997402
	L3	2×647	1.969	1.943	25.00	0.9997270
	L4	2×647	1.965	1.925	25.00	0.9997209

The correction factor given in the last column would be unity for a coil of zero section.

From these values and the assumed radius of $S1$ we can at once obtain from the first two ratios given in Table III the value of the radii of $M3$ and of $S2$. From this value of $M3$ and the mean ratios of Table IV, we get the radii of $M2$ and of $M4$. From the radii of the three moving coils and the last line of Table III we obtain three values for the radius of $L4$; and from the mean of the latter and the mean ratios of Table V we get the radii of $L1$, $L2$, and $L3$. The three values found for $L4$ were 25.002 47, 25.002 47, and 25.002 48 cm.

These are the most probable values that can be obtained from our observations. These radii, as well as the coefficients of expansion and the load corrections, are given in Table VII.

The radius in cm of a coil at the electrical temperature $t°$ and with a load of w watts is related to the quantities given in this table, as shown (p. 314) by equation

$$A = A_o\{1 + \tau(t - 22.0) - \lambda w\}$$

In the earlier work the importance of the exact parallelism of the coils and the needle was not realized, and the coils were not sealed until the summer of 1910. Hence, the earlier measurements of the radii are subject to uncertainties. However, the

TABLE VII

Radii and Coefficients

Coil	Radius A_0	Temperature coefficient τ	Load coefficient λ
	cm		
M2	12.50248	20.3×10^{-6}	2.9×10^{-6}
M3	10.03337	19.7×10^{-6}	8.2×10^{-6}
M4	12.46716	19.7×10^{-6}	10.7×10^{-6}
S1	19.97510	17.0×10^{-6}	0.47×10^{-6}
S2	19.96611	17.6×10^{-6}	0.93×10^{-6}
L1	25.03121	17.8×10^{-6}	0.61×10^{-6}
L2	25.03056	....	0.75×10^{-6}
L3	24.99767	18.9×10^{-6}	0.81×10^{-6}
L4	25.00247	18.5×10^{-6}	0.84×10^{-6}

measurements were all made in the autumn or winter, when the humidity in the laboratory is low and not subject to great variations, so that the discrepancies between the various measurements should be a minimum. Merely in order to show the variations observed under such conditions, these older values are given in the following table:

TABLE VIII

Comparison of Radii Determinations

Coils	Mechanical [12]	Electrical			
		Oct., 1908	Feb., 1909	Jan., 1910	Nov., 1910, to Jan., 1911
	cm	cm	cm	cm	
M1	9.9684	9.96389	9.96374	9.96362	....
M2	12.4989	....	12.50290	12.50250	12.50248
M3	10.0308	10.03342	10.03347	10.03330	10.03337
M4	12.4713	....	....	....	12.46716
S1	19.9751	19.97510	19.97510	19.97510	19.97510
S2	19.9658	19.96636	19.96612	19.96601	19.96611
L1	25.0216	....	25.03109	25.03019	25.03121
L2	25.0282	....	25.02971	25.02976	25.03056
L3	25.0030	....	....	[13] 24.99851	24.99767
L4	25.0021	....	....	[13] 25.00362	25.00247

[12] These measurements were not used in the computation of the constants.
[13] These measurements were made Mar. 29, 1910.

The radius of $S1$ is assumed to remain constant, and the table shows that in spite of the unfavorable conditions just mentioned the greatest range relative to $S1$ is only about 5 in 100 000.

IV. THEORY OF THE BALANCE AND COMPUTATION OF CONSTANTS

19. THEORY OF THE RAYLEIGH BALANCE

The mutual energy of two coils being equal to the mutual inductance of the coils multiplied by the product of the two currents, it is evident that the force, in any direction, which one coil exerts upon the other is equal to the product of the currents multiplied by the rate of change of the mutual inductance, as the second coil moves in the direction specified.

Hence the expression for the mutual force per unit current in each may be obtained by differentiating any of the expressions which may be derived for the mutual inductance.

The derivation of all of these expressions includes two distinct steps, namely, the formation of an expression for the mutual inductance of linear circuits, and the modification of this expression, necessitated by the finite section of the coil.

One of the simplest methods that has been suggested to correct for the sectional dimensions of the coil is the method, described by Lyle [14] in 1902, which makes use of an "equivalent radius." Lyle showed that so far as its external field is concerned, a circular coil of square section is equivalent to a linear circular circuit of radius a_e, where $a_e = a\left(\frac{1+4\alpha^2}{24a^2}\right) = a + \frac{\alpha^2}{6a}$ and a is the mean radius of the coil having a section $2\alpha \times 2\alpha$. This approximation neglects quantities of the order of the fourth and higher powers of the ratios of 2α to the radius of the coil, and of 2α to the distance of the point considered from the plane of the coil. The quantity we have denoted by a_e is called the "equivalent radius" of the square coil.

Similarly it is shown that if the axial breadth of the coil is 2α, its radial depth is 2ρ, and its mean radius a, then if 2α is

[14] Phil. Mag., 3, p. 310; 1902; this Bulletin, 2, pp. 374–378; 1906.

greater than 2ρ the coil is equivalent to two linear circular circuits, each of radius $a_e = a\left(1 + \frac{4\rho^2}{24a^2}\right) = a + \frac{\rho^2}{6a}$, the planes of these circles being a distance 2β apart and symmetrically placed with reference to the mean plane of the coil. The value of β is given by the expression

$$\beta^2 = \frac{4\alpha^2 - 4\rho^2}{12} = \tfrac{1}{3}(\alpha^2 - \rho^2)$$

If 2α is less than 2ρ, the coil is equivalent to two coplanar circular circuits lying in the mean plane of the coil and having radii

$$a'_e = a + \frac{\alpha^2}{6a} + \delta$$

$$a''_e = a + \frac{\alpha^2}{6a} - \delta$$

where

$$\delta^2 = \frac{4\rho^2 - 4\alpha^2}{12} = \tfrac{1}{3}(\rho^2 - \alpha^2)$$

Should α and ρ be so great that this order of approximation is not sufficient, then the coil can be subdivided until the sections of the subdivisions are sufficiently small to give the desired accuracy, and each section be replaced by its equivalent circles. The total current turns in the actual coil must, of course, be equally divided among the various circles into which the coil is thus resolved.

In this manner the computation is reduced to the case of linear circular circuits.

Probably the simplest expression for the mutual inductance of two coaxial circular filaments is its spherical harmonic expansion given by Maxwell.[15] When this is expanded in terms of the linear quantities involved, and differentiated with reference to the distance between the coils, it yields the following expression

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[15] Elec. and Mag., §§ 696–699.

for the mutual force of attraction between the circles per unit current in each.[16]

$$F=\frac{\pi^2A^2a^2}{C^4}\left\{1.2.3\frac{B}{C}+2.3.4\frac{(B^2-\frac{1}{4}A^2)b}{C^3}+3.4.5\frac{B(B^2-\frac{3}{4}A^2)(b^2-\frac{1}{4}a^2)}{C^5}\right.$$
$$+4.5.6\frac{(B^4-\frac{3}{2}B^2A^2+\frac{1}{8}A^4)(b^2-\frac{3}{4}a^2)b}{C^7}$$
$$\left.+5.6.7\frac{B(B^4-\frac{5}{2}B^2A^2+\frac{5}{8}A^4)(b^4-{}^3b^2a^2+\frac{1}{8}a^4)}{C^9}+\ldots\right\}$$

where A and a are the radii of the circles, A is assumed to be the larger, B and b are the distances of the centers of these respective circles from some point on their common axis, $C^2=A^2+B^2$.

Owing to the slow convergence of this series, it is impracticable to use it in the computation of the force when high accuracy is desired. However, this series is of great value in the study of the relation of the force to the various linear quantities involved, and in this connection has been carefully studied by Lord Rayleigh in the paper referred to, in which will be found some of the conclusions given below.

From this expansion it is at once evident that F is of zero dimensions in length; that is, it depends solely upon the ratios of the linear quantities involved. Hence, increasing all the dimensions in the same proportion will leave F unchanged.

Again for very small values of b (in the limit for $b=0$), the derivative of F with respect to b becomes

$$\left(\frac{dF}{db}\right)_{b=0}=\frac{\pi^2A^2a^2}{C^4}\left\{2.3.4\frac{B^2-\frac{1}{4}A^2}{C^3}-3.5.6\frac{a^2(B^4-\frac{3}{2}B^2A^2+\frac{1}{8}A^4)}{C^7}+\cdots\right\}$$

This will equal zero, and hence F will be a maximum, for a value of B nearly equal to $0.5A$ (that it is really a maximum can be seen from a consideration of the third term in the expansion for F). For a second approximation give B this value where it occurs in the second term in the above expression, equate to zero, and we find for the new value

$$B^2=\frac{1}{4}A^2\left\{1-\frac{9}{5}\cdot\frac{a^2}{A^2}\right\}$$
$$B=\frac{1}{2}A\left\{1-\frac{9}{10}\cdot\frac{a^2}{A^2}\right\}$$

[16] Maxwell, Elec. and Mag., §699. Rayleigh, Phil. Trans., 175, pp. 411–460; 1884; Scientific Papers, 2 pp. 278–332.

That is, if the distance between the planes of the circles be equal to

$$\tfrac{1}{2}A\left\{1-\tfrac{9}{10}\cdot\frac{a^2}{A^2}\right\}$$

the force between them will be very nearly a maximum for variations in B, and consequently will vary but slowly with changes in the value of the latter; thus, the exact measurement of B becomes of relatively slight importance. This is greatly to be desired in practice, as B is the distance between the mean planes of the coils and, consequently, can not be measured directly with high precision.

It is also evident that when B is such that F is a true maximum, then b will enter into the expression only as the square, cube, and higher powers. Of these the square term will be much the most important.

If there are two coaxial circles of radius A, and the origin is taken midway between them, then, from symmetry, the odd powers of b will be absent from the expression which represents the sum of the forces exerted upon the small circle by the two large ones (the currents being such that these forces are in the same direction). This force will be a maximum when the distance from the small circle to each of the large ones is given by the expression found above, and variations of the force from this maximum will be an even function of the displacement (b) of the moving coil from this position. For small displacements, the change in the force is proportional to the square of the displacement.

If the currents in the two large circles are so directed that the forces which they exert on the small circle are opposed, we obtain the resultant force by subtracting the expressions of F for the two circles, and thus find that the difference in the forces is an *odd* function of the displacement of the small circle. If the spacing is that which gives the maximum force, the first term in the series for the difference in the forces involves the cube of the displacement of the small circle. Hence, for small values of this displacement, the difference is nearly independent of the displacement. If the spacing is different from that assumed, the difference in the forces is nearly a linear function of the displacement.

It will also be noticed that when the two large circles are so spaced that

$$B^2 = \tfrac{3}{4}A^2$$

the square term in the expression for the sum of the forces vanishes, and therefore the expression for the total force is reduced to a constant term, and those involving the fourth and higher powers of the displacement of the small circle. Hence, for this adjustment small displacements of the smaller circle produce a minimum effect upon the force.

For this reason this adjustment has been used in some cases, but in our opinion the advantage gained in this respect is, in practice, more than outweighed by the fact that, when not working at the position of maximum force, the distance between the mean planes of the fixed coils must be measured with considerable accuracy. Consequently the coils have been spaced in our work so as to obtain the maximum force.

Any of the other series formulas for the mutual inductance of coaxial circular circuits may be used with more or less facility in the computation of the force, but the only exact expression for the force that has been given is the one in terms of elliptic integrals which was given by Maxwell,[17] namely,

$$F = \frac{\pi B \sin \gamma}{\sqrt{Aa}} \left\{ 2F_\gamma - (1 + \sec^2\gamma) E_\gamma \right\}$$

where A and a are the radii of the two coaxial circles, B is the distance between their planes, F_γ and E_γ are the two complete elliptic integrals of argument γ, and

$$\sin \gamma = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + B^2}}$$

20. METHOD EMPLOYED IN THE CALCULATION OF THE CONSTANTS

Each coil has been replaced by the eight equivalent circles obtained by quartering the coil and replacing each quarter by its two equivalent circles (as defined by Lyle) in the manner already described. The force between the various circles was calculated by means of the elliptic integral expression just given.

[17] Electricity and Magnetism, 2, § 701.

This computation is greatly facilitated by the tabulation of log [sin γ $\{2F_\gamma-(1+\sec^2\gamma)E_\gamma\}$] for values of γ progressing by equal steps. Such a table has been constructed by Lord Rayleigh[18] using seven-place logarithms. Owing, however, to the accumulation of small errors, many of the values in that table are in error by several units in the seventh place of the logarithms. In a few cases it amounts to 10 units, giving an error of about 3 in a million in the result. While this is a small amount, it appeared desirable, especially in the study of the effect of the subdivision of the coils and in the calculations for the determination of the distance for maximum force, to attain a higher accuracy in the computation; consequently, the table has been recalculated, using Legendre's tables of elliptic integrals and Vega's ten-place table of logarithms. It is given in the appendix to this paper. We believe that the table contains no errors as great as 1 unit in the last place. In order to attain an accuracy of 1 in 1 000 000 in the force, it is necessary to know γ to about 0°.000 008, but a unit in the seventh place of the logarithm of the sine of γ corresponds to about three times this change in angle; hence, in all final computations we have calculated the sine of γ either by means of Vega's ten-place table or by a calculating machine. In both cases ten-place tables were used in passing from sin γ to γ.

21. DISTANCE FOR MAXIMUM FORCE

Before proceeding to the computation of the constants of the balance it is necessary to know the distance between the coils which will give the maximum force, the latter being the quantity directly measured. As seen above, an approximate value of this distance is given by the relation

$$B'_m=\tfrac{1}{2}A\left\{1-\tfrac{9}{10}\cdot\frac{a^2}{A^2}\right\}$$

In order to obtain a more accurate value for B_m, and at the same time to obtain an accurate expression for the variation of the force with the distance for values differing from B_m by several

[18] Phil. Trans., 175, pp. 411–460; 1884; Scientific Papers, 2, p. 327.

millimeters, calculations of the force were made for various values of B lying on each side of the approximate value B'_m given by the above expression. From these values the distance B_m for maximum force was determined, and the coefficients in the equations representing (1) the variation of the force with the distance $(x-B_m)$, (2) the variations in the magnitude of the maximum force with variations in A, the radius of the moving coil remaining unchanged, and (3) the variations in the distance for maximum force with variations in A, a being constant as before.

In this computation two assumptions have been made for the sake of simplicity. First, that the coils are equivalent in their action to coils of square cross section, and of the same mean radii and same sectional area as the actual coils; second, that any such square coil produces the same effect as its "equivalent" circular current, as defined by Lyle. These assumptions are equivalent to the single assumption that each coil may be regarded as a linear circular current lying in the mean axial plane of the coil, and having a radius (A_e) defined by the equation

$$A_e = A + \frac{\alpha \rho}{6A}$$

where 2α and 2ρ are the sectional dimensions of the coil, and A is its mean radius.

These computations have been performed for each moving coil, and the largest and the smallest of the large fixed coils, and for $M3$ and each of the small fixed coils. The agreement of the coefficients in the variation formula furnishes a good check on the work. The values of B_m for the intermediate values of the large fixed coils were determined by interpolation.

TABLE IX

Equivalent Radii and Distances for Maximum Force

Coil	A_e	Coil	A_e	Coils	B_m	Coils	B_m	Coils	B_m
M2	12.50552	L1	25.03808	M2, L1	9.6092	M3, S2	7.6344	M4, L1	9.6256
M3	10.03763	L2	25.03749	M2, L2	9.6089	M3, L1	10.6673	M4, L2	9.6253
M4	12.47106	L3	25.00405	M2, L3	9.5882	M3, L2	10.6669	M4, L3	9.6047
S1	19.98014	L4	25.00877	M2, L4	9.5912	M3, L3	10.6476	M4, L4	9.6077
S2	19.97113			M3, S1	7.6400	M3, L4	10.6502		

The equivalent radii A_e used, and the values found for the distances for maximum force B_m, are given in Table IX.

The values of the coefficients in the following variation formulæ are given in Table X:

$$F = F_m\{1 - \gamma(x - B_m)^2 + \delta(x - B_m)^3\}$$

$$\frac{\Delta F}{F} = -\epsilon\frac{\Delta A}{A}, \qquad a \text{ constant}$$

$$\frac{\Delta B_m}{B_m} = \eta\frac{\Delta A}{A}, \qquad a \text{ constant}$$

TABLE X

Variation Coefficients

Radii	1000γ	1000δ	ϵ	η
50 cm, 25 cm	7.523 cm^{-2}	0.566 cm^{-3}	2.555	1.62
50 cm, 20 cm	6.433 cm^{-2}	0.407 cm^{-3}	2.317	1.35
40 cm, 20 cm	11.874 cm^{-2}	1.101 cm^{-3}	2.555	1.62

The value of B_m can also be obtained from the fact that it must be such a distance that each coil will exert the same force upon each of the two flat annular current sheets, which form the upper and the lower boundaries, respectively, of the other coil. The necessary computation may be readily made by replacing one coil by its equivalent turn, and each of the two current sheets by their two equivalent turns, as defined by Lyle. This has been done in one case, as a check upon the other work, and yielded a result differing but 2 microns from the value as found by the other method.

22. COMPUTATION OF FORCE

Having found the value for the distance for maximum force, and having found by a preliminary computation that ample accuracy is obtained by assuming that the fixed coil is equivalent to the eight turns obtained by quartering it and replacing each quarter by its two equivalent turns,[19] and that the moving coil is likewise equivalent to the eight turns obtained by replacing each quarter by two equivalent turns,[20] the computation was made in the following two ways:

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[19] Although the sections of the coils were nearly square, they departed too much from a square to permit replacing each quarter by a single turn.

[20] It would probably have been sufficiently accurate in most cases to replace the moving coil by two turns without quartering.

(1) Instead of making 64 computations (for the force of each of the 8 equivalent turns of the fixed coil upon the 8 turns of the moving coil) for each pair of coils used, it is simpler to replace each coil by a single equivalent turn on the assumption that it has a square section and calculate the force F_1 for each combination of coils used. Then with any moving coil replaced by a single turn as before, we may calculate for each fixed coil used with the moving coil, the force F_f obtained by replacing the fixed coil by 8 turns. The percentage difference between F_1 and F_f for any coil—that is, the correction for its sectional dimensions—will be the same for every moving coil having approximately the same radius as that used in the computation of F_f. Hence, this correction need not be redetermined for other moving coils of the same size.

Having thus determined the sectional correction for each fixed coil for each group of moving coils with which it is to be used, we in a similar manner determine the sectional corrections for each moving coil. Calling F_m the force obtained when the fixed coil is replaced by a single turn and the moving coil is quartered, we have for the corrected force, both coils subdivided, the expression

$$F = F_1(1 + \delta_f + \delta_m)$$

where $\delta_f = \frac{F_f - F_1}{F_1}$, and is determined by calculation with the special fixed coil under discussion but with any moving coil having approximately the same radius as that concerned in F; and $\delta_m = \frac{F_m - F_1}{F_1}$ and is determined for the special moving coil concerned in F, but for any fixed coil of approximately the same radius as that used in F.

(2) Or, after calculating the value of F_1 as defined above, we may proceed by the method just described to the determination of the effect of the area of section for a coil of approximately the same radius and of strictly square cross section of approximately the same sectional area as the actual coils. This effect will be very small, and over a considerable range will be independent of the radius of the coil. Then, by means of the variation formula given below (and derived in the appendix, p. 376, where this method

is considered more in detail), the correction for the departure of the actual section from an exact square may be determined and applied for each coil. This method greatly lightens the labor of computation, but was not used in the present work except as a check, because of a failure to realize earlier the range of application and the accuracy of the method. A thorough test of the method has been made, and in the future it can be used with all confidence.

Using the values of the mean radii as given in Table VII, and the sectional dimensions as given in Table VI, and the distances for maximum force as just determined, we find the values (F_1) for the forces in dynes per C. G. S. current turn, as given in Table XI. In the column headed "I for 1.5 g" in this table are given the currents in absolute amperes which, when passed in series through the given pairs of coils, will give in each case a force corresponding to 1.5 g at a place where the acceleration of gravity is 980.091 cm per second per second, the windings of each coil being in parallel and the radii the same as if the coils were at 22°.00 C and carried no load

TABLE XI

Constants of the Balance

Coils	F_1	$\delta_f \times 10^{+6}$	$\delta_m \times 10^{+6}$	F	I (in absolute amperes) for 1.5 g
M2, L3	5.355834	− 48.1	− 3.4	5.355558	0.767641_0
M2, L4	5.353251	− 80.9	− 3.4	5.352799	0.767838_7
M3, S1	5.417298	−136.8	+100.2	5.417100	0.840503_1
M3, S2	5.423552	−149.9	+100.2	5.423282	0.840023_9
M3, L1	3.137781	+ 16.6	+ 69.6	3.138051	0.858911_1
M3, L2	3.137953	+ 50.7	+ 69.6	3.138331	0.858872_8
M3, L3	3.147665	− 44.5	+ 69.6	3.147744	0.858250_4
M3, L4	3.146292	− 70.9	+ 69.6	3.146282	0.858448_9
M4, L3	5.318310	− 48.1	− 28.0	5.317906	0.775759_7
M4, L4	5.315754	− 80.9	− 28.0	5.315175	0.775959_2

23. EFFECT OF ERRORS IN SECTIONAL DIMENSIONS

In the section dealing with the determination of the ratio of the radii, we have considered how an error in the sectional dimensions affects the radius as computed from the observed ratio of the galvanometer constants. This per cent error in the radius multiplied by the factor ϵ (Table X) gives the per cent error, thus introduced into the force.

But having assumed values for the radii, the sectional dimensions again enter in the computation of the force, and so an error in them produces a second error in the force. The most direct estimation of this error is obtained by a consideration of the spherical harmonic expansion of the force between two coils of finite section. This expression may be obtained from that for two linear circular circuits either by direct integration, or by the use of Taylor's theorem as was done by Maxwell (Elec. and Mag., Vol. II, §700), and is of the form

$$F_1 = F_0 + \lambda'_1 \frac{\rho_1^2}{A^2} + \lambda'_2 \frac{\alpha_1^2 + \alpha_2^2}{a^2} + \lambda'_3 \frac{\rho_2^2}{a^2}$$

Whence, by Maclaurin's theorem, the value F of the force given by $\rho = \rho_1 + \delta\rho_1$, etc., is given by the expression

$$\frac{\Delta F}{F} = \frac{F - F_1}{F_1} = \lambda_1 \frac{\rho_1^2}{A^2} \cdot \frac{\delta\rho_1}{\rho_1} + \lambda_2 \frac{\alpha_1^2}{a^2} \cdot \frac{\delta\alpha_1}{\alpha_1} + \lambda_3 \frac{\rho_2^2}{a^2} \cdot \frac{\delta\rho_2}{\rho_2} + \lambda_2 \frac{\alpha_2^2}{a^2} \cdot \frac{\delta\alpha_2}{\alpha_2}$$

While it is impracticable to calculate the coefficients with sufficient accuracy to enable us to determine F_1 from F_0 and a knowledge of the sectional dimensions of the coils, they can be readily determined with ample accuracy for use in the variational equation just given. In the sectional dimensions, the subscript *1* refers to the fixed coil and the subscript *2* to the movable one. The expressions for the coefficients in terms of the radii of the coils and of their distance apart are given in the appendix, page 378, and when evaluated yield the numbers given in Table XII.

TABLE XII

Variation of Force with Sectional Dimensions (Radii Constant)

$\frac{A}{a}$	λ_1	λ_2	λ_3
2	+2.523	−0.8478	+1.704
2.5	+1.954	−0.4361	+1.201

Denoting the errors thus introduced by the letter C, and those introduced by the error in the determination of the radius from the galvanometer constant by the letter G, we find that an error of 0.01 mm in the axial breadth 2α or in the radial depth 2ρ produces the errors in the force given in Table XIII.

TABLE XIII

Effect of Errors in the Sectional Dimensions

Radii		Dimensions of square section		δF in parts per million for a variation of 0.001 cm in the radial depth or axial breadth											
				Fixed coil						Moving coil					
				For variation in depth			For variation in breadth			For variation in depth			For variation in breadth		
Fixed	Moving	Fixed	Moving	G	C	Sum	G	C	Sum	G	C	Sum	G	C	Sum
cm	cm	cm	cm												
20	10	1.58	1.0	−1.68	+2.50	+0.82	+2.53	−3.35	−0.82	+4.26	+4.26	+8.52	−6.39	−2.12	−8.51
25	12.5	2.00	1.0	−1.36	+2.02	+0.66	+2.04	−2.71	−0.67	+2.72	+2.73	+5.45	−4.09	−1.36	−5.45
25	10	2.00	1.0	−1.23	+1.56	+0.33	+1.85	−2.18	−0.33	+3.86	+3.02	+6.88	−5.79	−1.09	−6.88

From Table XIII it appears that while the two errors nearly balance one another in the case of the fixed coils, they add together in the case of the moving coils, and even for the small assumed error of 0.01 mm in 2α or 2ρ may amount to over 8 in a million in the force, or 4 in a million in the current. This is about the magnitude of the uncertainty in the present work, due to errors in measuring the cross sections of the coils. If the cross sections were larger, this error would be increased, and hence the cross section should be kept as small as practicable.

V. THE MEASUREMENT OF CURRENT

24. GENERAL DESCRIPTION

As already stated, the current which passed through the coils of the current balance also passed through a standard resistance, and, by means of a calibrated potentiometer, the drop of potential between the terminals of this resistance was measured in terms of a Weston normal cell.

Knowing the electromotive force of this cell in international volts, as defined by that value for the mean Weston normal cell (1.0183, at 20°00), which was adopted by the international committee to take effect after January 1, 1911, and knowing the resistance in international ohms, the value of the current in international amperes (1911) is known. The term "international ampere (1911)" is used in this paper to denote that current which, when passed through a resistance of 1 international ohm, will produce between the extremities of the latter a potential difference of 1 international volt, as defined by the value adopted for the mean Weston normal cell.

This current, measured in *international amperes*, as just described, is measured in *absolute amperes* by means of the current balance. To find the ratio of these two numerical values of the same current is the object of this investigation.

The same result may be stated in a different form by saying that the quotient of this ratio into the electromotive force of the mean Weston normal cell at 20°00 C (by definition 1.0183 international volts) is equal to the electromotive force of this mean cell in terms of the absolute ampere and the international ohm. The unit of electromotive force as thus expressed depends both upon the absolute and the international system, and consequently is called, in this paper, a "semiabsolute volt." In this investigation we have found that the electromotive force of the mean Weston normal cell at 20°00 C is 1.018 22 semiabsolute volts.

If the silver voltameter were as convenient an instrument as the standard cell and were defined by complete official specifications, a better method would be to state the result in terms of the mass of silver deposited in a silver voltameter per second by the absolute ampere, as Rayleigh did thirty years ago. But the measurement of current with high precision by the silver voltameter is still an investigation rather than an operation, and the complete specifications are as yet undefined; consequently, for precise measurements in practice, the standard cell and resistance are always employed. We have, however, in section 38 given our final result in terms of the electrochemical equivalent of silver as found by a large number of observations with two types of voltameters at the Bureau of Standards.

In the actual work the potentiometer was connected to measure directly the drop of potential across the standard resistance, or to measure the small difference between this drop in potential and the electromotive force of the cell. In the first case a sensibility of about 8 in 1 000 000 for 1 mm, and in the second of about 3 in 1 000 000 for 1 mm deflection, was secured. The scale distance was 270 cm. By means of a continuously variable resistance, the deflection could be kept, as a rule, within a quarter of a millimeter during the time necessary for the determination of the rest point of the balance, and frequently it could be kept apparently zero for this length of time. Hence, there is no appreciable error in the work due to variation of the current during measurement.

25. MANIPULATION OF THE BALANCE

In the manipulation of the balance three points, in addition to the control of the current, have to be considered.

First.—Slight changes in the zero of the balance may be produced by arresting it. Consequently, it is better to change the weight and to reverse the current without arresting the balance. By means of a suitable arrangement for manipulating the weight without opening the balance case, the circular reversing switch already described, and an air jet (produced by squeezing a rubber bulb) for deflecting the balance beam, it is possible with practice to make the reversal of the current and the change in weight simultaneous, so that the balance receives no bump or jar in the process. This procedure has been followed throughout this work.

Second.—Though the change in the weight does not jar the balance, it does usually start the pan and the coil to swinging slightly. This vibration is stopped by gently touching with a camel's-hair brush the tube by which the coil is suspended from the pan. This is done in the space between the balance case and the coil case.

Third.—Owing to slight changes in temperature or other causes, the zero of the balance is usually continually drifting. Also, the earth's field exerts a slight force upon the current in the moving coil. Both of these effects may be eliminated at the same time if a series of weighings are made with the weight alternately on and off, the current in the moving coil being always in the same direction and that in the fixed coils being alternately direct and reverse. As stated in an earlier portion of this paper, the current is always adjusted to such a value that these simultaneous changes in the weight and the direction of current produce but a slight change in the rest point of the balance.

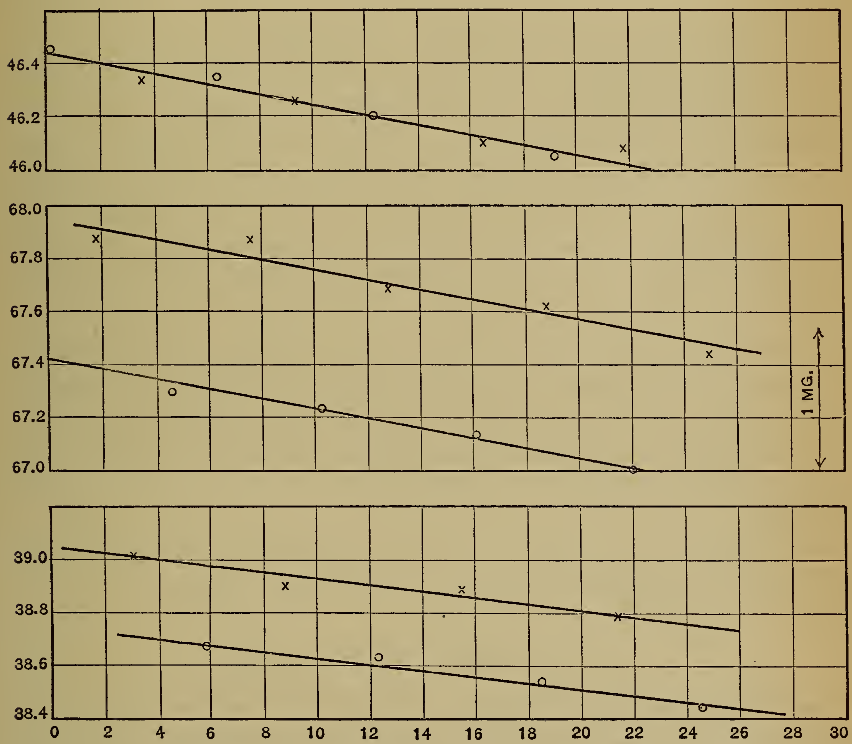

Fig. 14.—*Plots of three typical runs on the current balance. The abscissae are times in minutes; the ordinates are doubled scale readings in centimeters*

By this procedure the effect of the earth's field is reduced to a very slight permanent displacement of the zero of the balance. The difference between the change in the weight and twice the force exerted by the fixed coils on the moving coil is directly given by the difference in the rest points of the balance.

To eliminate the drift, the time at which each rest point is taken is noted, and the rest points are plotted as functions of the time. Under favorable conditions they will lie upon two parallel straight lines, generally slightly inclined to the time axis. The distance between these lines measured perpendicularly to the time axis is proportional to the amount by which twice the force due to the current differs from the gravitational force exerted upon the weight used. A set of four pairs of observations can be obtained in about 30 minutes. Three such "runs" are shown in Fig. 14.

For two reasons this graphical treatment of the observations is considered superior to the mere numerical averaging, as is usual in the determination of the rest point of a balance. First, the observations can not be taken at strictly equal intervals of time; and, second, irregular behavior due to imperfect current control, swinging of the coil, variations in the air currents, change in the rate of drift, etc., is detected much more readily when the results are plotted; and single erratic points can be readily given reduced weight.

As already stated, the deflection of the balance beam is read by a mirror and telescope and scale, at a scale distance of 250 cm. Each resting point is determined from at least five turning points. The amplitude of oscillation is usually such that the displacement of the moving coil from its position of rest does not exceed about 0.2 mm.

The weights used were of platinum, and the correction for the buoyancy of the air was taken as 56 parts per million.

26. ADJUSTMENT OF THE COILS

Having carefully leveled and clamped the fixed coils in position, as already described, the moving coil is adjusted to approximately the proper height by means of the nut screwing on the suspending tube and resting upon the pan of the balance. Then, by means of the long screws by which the coil is connected to the suspending tripod, it is leveled and adjusted (by direct measurement), as near as may be, midway between the fixed coils. These adjustments are tested by means of a surface gauge resting on the surface of the lower coil, and moved from point to point. While testing this adjustment, the balance pan should be hanging freely on its knife-edge, and the beam should be just locked.

Then a star, which just fits inside the forms of the fixed coils and which has a hole in its center through which passes a conically pointed brass rod, is placed in the lower coil. The star is held in place by flanges resting on the surface of the coil, and when so placed the rod in its center coincides with the axis of the coil. While the balance pans are freely suspended from their knife-edges, this rod is pushed up until its tip passes into the 3-mm hole in the center of the star by which the moving coil is suspended. If the tip of the rod is not central to the hole, the balance is properly shifted until the observer is no longer able to decide which side of the hole is first touched by the conical tip as the rod is raised. When this condition is attained, the center of the moving coil must lie very nearly upon the axis of the lower coil.

Since, when properly adjusted, the force for a given current is a minimum for lateral and a maximum for vertical displacements of the moving coil, it is very easy to determine the proper adjustment from a study of the variations in the force as the coil is slightly displaced in different directions. This was always done.

For each of the three mutually perpendicular adjustments weighings were made with the moving coil in at least three positions, of which at least two were approximately equidistant from and on opposite sides of the position corresponding to the maximum (or minimum) force. These forces reduced, if necessary, to the same current, were plotted, on a suitable scale, in terms of the position of the coil; the theoretical curve, connecting the force with the displacement of the moving coil, was plotted on the same scale and fitted to the observations, as well as possible. The position, thus determined, of the vertex of the curve is the proper setting for the moving coil. Four of these curves are shown in Figs. 15 and 16. The two curves in Fig. 15 are drawn to the same scale; *A* applies to the small moving and the small fixed coils, *B* to the same moving coil and the large fixed coils. In Fig. 16, *C* is drawn to the same scale as *A* and *B*, and applies to large moving coil and the large fixed coils. For *D* the scale of the abscissas is reduced by the factor $\sqrt{2}$; otherwise *D* is the same as *C*, but

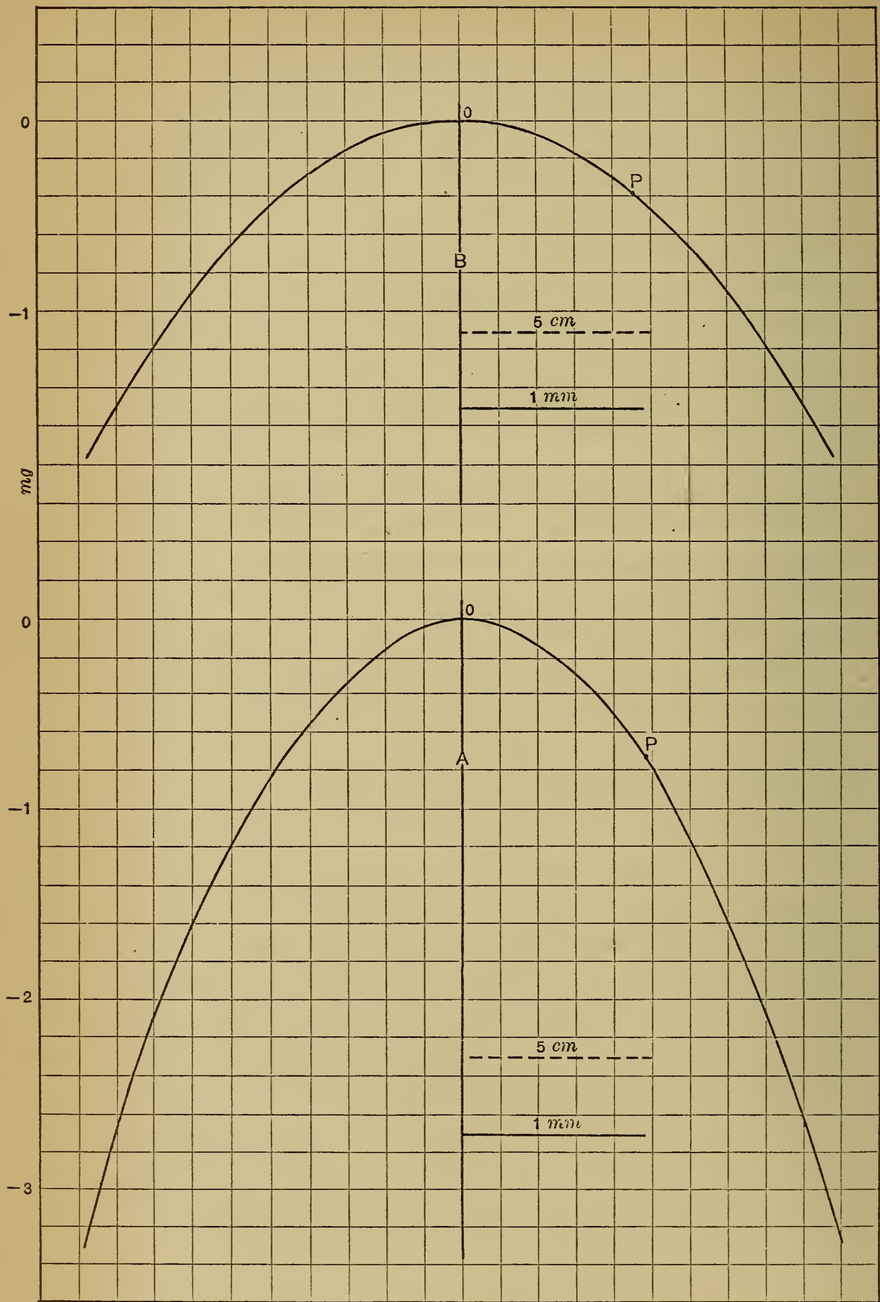

Fig 15.—*Vertical displacement curves*

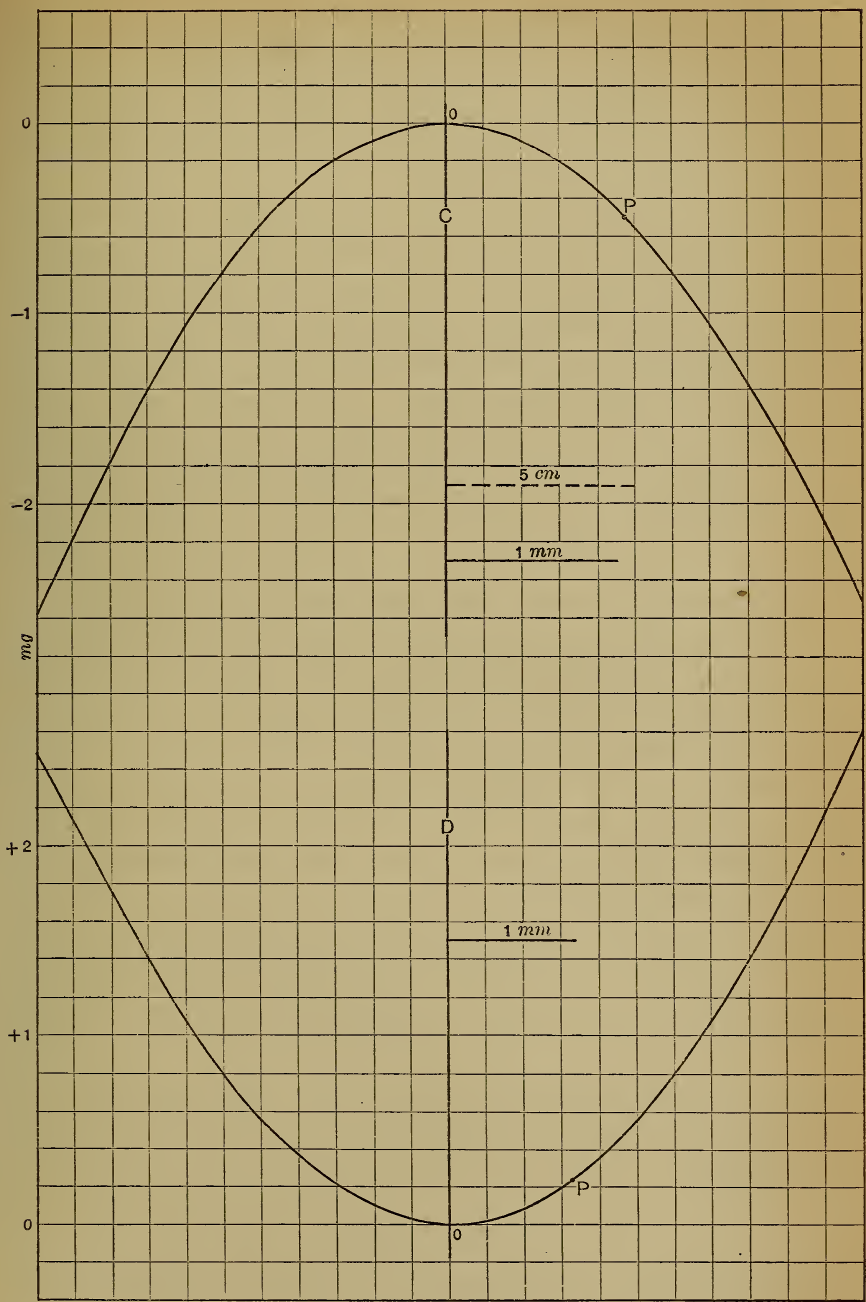

Fig. 16.—*Vertical and horizontal displacement curves*

inverted.[21] This change of scale is made so that the same theoretical curve can be used for the observations on lateral as on vertical displacements. The point P gives the force when the moving coil is displaced 1 mm from the maximum. This corresponds to a change of about 5 cm on the scale, which is used to read the rest points of the balance.

As there appeared to be no reasonable chance of the balance being accidentally displaced horizontally after it was once adjusted and the screws locked, and as numerous observations during the preliminary work had failed to disclose any such shift, this horizontal adjustment was in the final observations usually made but once for each combination of coils. On the other hand, a displacement of the reading telescope might occur, and this would change the reading for the correct vertical position of the coil. Consequently, the correct vertical position of the coil was determined daily; seldom, however, was any accidental change observed.

27. INSULATION OF THE COILS AND ELECTROSTATIC EFFECT

Each time the coils were adjusted, their insulation resistances were measured just before proceeding to the electrical adjustment of the moving coil. To do this each coil, taken one at a time, had one terminal of each winding disconnected from the leads. Then the insulation resistance between the two windings was tested by a direct deflection method, using a well insulated portable battery of 40 volts. The sensibility was such that 1 mm deflection corresponded to 25 000 megohms.

When the circuit connecting the balance with the apparatus on the table was opened by disconnecting one coil from the commu-

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[21] From considerations of symmetry and the fact that the force satisfies Laplace's equation, we have the expression

$$F = F_0 + k\,\delta^2 P_2(\theta) + \textit{terms in fourth and higher powers of } \delta$$

connecting the force F_0 acting on the moving coil when symmetrically placed with reference to the two fixed coils and the force F when the coil is displaced from this position by the amount δ, along a line inclined at an angle θ with respect to the common axis of the fixed coils. $P_2(\theta)$ is the second zonal harmonic. Hence, denoting the change in F for an axial displacement by dF_a and the change for a radial displacement by dF_r, we have (since $P_2(\theta) = 1$ when $\theta = 0°$ and is equal to $-\frac{1}{2}$ for $\theta = 90°$)

$$dF_a = k\delta^2_a$$

$$dF_r = -\tfrac{1}{2}k\delta^2_r = -k\left(\frac{\delta_r}{\sqrt{2}}\right)^2$$

which shows that, in order to use the same curve (only inverted) for both the axial and the radial displacements, the scale of abscissas must be changed by the factor $\sqrt{2}$.

tator, its resistance was never low enough to give a deflection exceeding 3 mm for 40 volts.

The large coils showed considerable absorption, the deflection on closing the circuit (after the initial charge due to the electrostatic capacity) being large, this deflection rapidly decreased, and after a minute seldom exceeded 10 cm. In the later measurements it but once amounted to so much as 25 cm; that is, the insulation resistance was but once so low as 100 megohms. The capacity and the leakage between the windings and the form on which they were wound were somewhat greater than those between the windings, but seldom did the insulation resistance fall appreciably below 100 megohms. The capacity effect for the moving coils was always much smaller than for the fixed coils, and the insulation resistance was considerably higher. It thus appears that the insulation of the coils and connections throughout was so high that there was no error due to leakage.

To obviate any electrostatic force upon the moving coil, its windings were at all times directly connected to the water jacket by means of a wire running from the commutator. Thus the water jacket and all of the surrounding framework were brought to the potential of the moving coil. A number of tests showed that no appreciable current flowed through this wire.

28. TEMPERATURES

In the earlier work, the measurements of the temperatures of the coils were somewhat uncertain; the temperatures of the fixed coils were determined by thermometers attached to the forms, and that of the moving coil was inferred from the temperature of the surrounding water jacket, a preliminary experiment being made to determine the difference between them under working conditions.

In the later work—that is, after the summer of 1910—all temperatures are based upon the electrical resistances of the coils as described in the section on the ratio of the radii. The resistances at a known temperature near 22°0 C, and with no load, were measured when the temperature conditions were steady, usually in the morning. It was done by a potentiometer method, using a low current (0.1 ampere) and keeping it on for as short a time as possible. In the case of the fixed coils, the measurements were made at the proper terminals of the mercury commutator attached to the end of the coil case. For the moving coil, a pair of potential leads extends down the suspending tube and connects with the vertical current leads, where the latter join the horizontal leads extending across the coil.

These measurements, together with the temperatures of the coils as given by the thermometers attached to the fixed coils, and inserted in the water jacket surrounding the moving coil, enable us to calculate the electrical temperature of the coils (as defined in the section on the ratio of the radii) from any future measurement of the resistance, provided the lead connections have not been changed in the interval. The resistance with no load was measured on at least two days for each combination of coils used; these observations were always quite concordant.

No weighings were taken until the working current had been on the circuit, and water at the desired temperature had been circulating through the system for an hour or an hour and a half. By this time temperature equilibrium has been about established, the drift of the balance has become slight, and the resistances of the coils have become nearly constant.

During the weighings, the resistances of the coils were measured several times a day, and the thermometers in the oil baths containing the standard resistance and the cells, those attached to the fixed coils and those in the water jacket, and the coil case, were read after nearly every run. The latter observations were for the purpose of checking the constancy of the conditions, and were not used in the actual reduction of the results.

The temperature of the oil in the cell bath, and of the water circulating through the coils and water jacket, were thermostatically controlled. Both were well stirred by propellers driven by electric motors.

The oil in the bath containing the standard resistance was stirred by a propeller pump driven by a small electric motor. The excess in temperature of the wire of the resistance above that of the oil was determined experimentally to be 0°.56 C per watt under working conditions, and the proper correction has been applied.

29. SUM AND DIFFERENCE OF FORCES

By means of the mercury commutator attached to the end of the coil case, we can readily connect the fixed coils either so that the forces which they exert upon the moving coil are in the same direction, and thus can measure their sum; or by reversing either of the fixed coils we can oppose the forces and so measure their difference. By measuring both the sum and the difference we can readily determine the force exerted by each of the two coils.

The measurement of the difference in the forces, combined with the effect of interchanging the upper and lower coils, gives a check on the accuracy in the computation of the two constants, and a splendid check on the presence of any unsymmetrical disturbance due to fixed magnetic masses. Such a disturbance will cause the force exerted by the lower coil to be too small or too great always by the same amount.

As we shall see in a later section, measurements of the difference in the forces of the two coils also furnish valuable information as to the magnitude and effect of certain maladjustments.

30. MAGNETIC TESTS BY BALANCE

As regards their effect upon the measurement of an electric current by the balance, magnetic bodies other than the coils themselves may be divided into two classes.

The first class includes those attached to the swinging portions of the balance, beam, pointer, pans, etc. These produce their effect solely in virtue of the forces which the fixed coils exert upon them.

The second class includes all other bodies; what we may call the fixed masses. Their effect arises from the forces which they exert upon the moving coil, and upon bodies of the first class. When observations are taken in the manner we have described, the effect of these bodies upon the final result depends solely upon that portion of their magnetization, which is reversed when the current through the fixed coils is reversed.

The effect of bodies of the first class, in so far as it is independent of the current in the moving coil, can be determined by observing the deflection of the balance when a current passes through the fixed coils only. It is, in fact, determined along with the leads of the moving coil, and is separated from the latter by making two sets of observations, differing only in the direction of the current through the leads of the moving coil. This was always done and the proper correction, though seldom amounting to 0.01 mg, has always been applied. It seems scarcely probable that the value of this correction can be appreciably changed by the presence of a current in the moving coil.

As we have seen, measurements upon the difference in the forces exerted by the two fixed coils will serve to detect the presence of bodies of the second class, except in so far as their effect is symmetrical with reference to the two fixed coils. Indeed, the presence of unsymmetrically distributed fixed magnetic masses (especially above and below the coils) affects the difference in the forces to a much greater extent than it does the sum; for, in the latter case, the axial fields of the fixed coils are opposed, and so the bodies are but slightly magnetized, while in the former the fields are added and, consequently, the bodies are much more strongly magnetized. This magnification of the disturbance increases the value of this method for detecting the presence of unsuspected magnetic material and of testing the effect of known magnetic bodies.

The chance of magnetic material being symmetrically distributed is very small, and such a possibility can be tested by taking observations with the coils in two different vertical positions, everything else being the same.

All these tests have been applied with satisfactory results.

In order to form an idea of the magnitude of the possible effect of the reenforcing in the concrete construction supporting the floor, weighings were made with a number of iron rods aggregating about 75 kg placed under the coils and at various distances below them. Both the sum and the difference in the forces were measured. The results are given in Table XIV.

These observations were made with moving coil $M3$ and the large fixed coils $L3$ and $L4$, $L3$ being the upper coil; the current used was such that the doubled force was 6 g; the distance between the mean planes of the fixed coils was about 21.3 cm.

From these observations it is evident, as would be anticipated, that the effect of iron below the coils is to decrease the force exerted by the upper coil and to increase that exerted by the lower coil. Being nearer the lower coil, the increase in the latter's force exceeds the decrease in the former's, the difference being the resultant effect upon the sum of the forces. As the distance of the iron from the coils increases the latter becomes very small. In the present case the effect of this large mass of iron upon the sum of the forces is imperceptible at 116 cm. On the other hand, at this distance it affects the difference in the forces by 0.10 mg; in this position the effect on the difference is about 10 times that on the sum.

TABLE XIV

The Increase which 75 kg of Iron at Different Distances below the Coils produces in the Double Force

Distance of iron below moving coil	Increase produced by iron			
	Sum L3+L4	Difference L3−L4	Force by L3	Force by L4
cm	mg	mg	mg	mg
116	+0.01	− 0.10	−0.05	+ 0.05
109	....	− 0.14	....	....
100	+0.12	− 0.22	−0.05	+ 0.17
86	+0.22	− 0.74	−0.26	+ 0.48
70	+1.00	− 2.79	−0.90	+ 1.90
50	+8.20	−17.76	−4.78	+12.98

The iron reenforcing in the concrete floor is about 140 cm below the moving coil. There is about 13 kg of iron per meter of each girder and 2 kg per square meter between girders. The girders are about 1.1 m apart. Containing so much less iron than we used in the test and lying at a greater distance from the coils, it would appear that the reenforcing can produce no perceptible effect upon the sum of the forces exerted by the two coils, though it may produce a slight effect upon their difference.

Weighings made with pieces of iron and steel, that were small, but distinctly larger than the knife-edges of the balance, and that were placed in various positions in the balance case, and between the latter and the coil case, proved conclusively that the magnetic susceptibility of the knife-edges was without effect upon the observed force.

In order to obtain an idea of the effect that might be produced by the slight magnetic susceptibility of the forms on which the

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fixed coils were wound, weighings were made with the fixed coil forms covered with strips of cloth to which iron filings had been stuck by means of paraffin. Though the cloth was thickly powdered, the effect on the force was very small. The best criterion, however, seems to be the agreement between the results obtained with the various combinations of coils. Although the magnetic susceptibility of all the coils is extremely slight, $L1$ has nearly 100 times the susceptibility of $L3$ and $L4$, and yet the results obtained agree very closely. This would indicate that there can be no error due to this cause.

In order to see if the wire used in winding the coils is sufficiently nonmagnetic, measurements of the difference in the forces of the two fixed coils were made both with and without a marble form wound with about 7 kg of the same wire introduced into the lower coil. There was no observable difference in the two cases.

31. CORRECTIONS TO THE OBSERVED FORCE

(*a*) *For the leads.*—The portion of the observed force which was contributed by the leads was determined experimentally in the manner already described and the proper correction was always applied.

(*b*) *For spacing and lack of coaxiality of the fixed coils.*—The constant of the balance has been computed for the moving coil in the position of maximum force. But, unless the distance between the fixed coils is exactly right, this condition can not be simultaneously satisfied for both of the fixed coils. Hence, it becomes necessary to consider the relation that exists between the maximum observed force and that which would have been observed had the distance between the fixed coils been exactly that assumed in the computation. In practice the distance between the two fixed coils was adjusted so as to make the two maxima coincide as accurately as possible. The distance apart of these two maximum points was generally not more than a few tenths of a millimeter. The correction for this displacement was determined in every case. A mathematical discussion of this subject is given in the appendix to this paper, page 380. The experimental method of determining this correction is as follows:

The two fixed coils used together were always very nearly identical. In this case, the distance for maximum force, the value of the maximum force, and the variation in the force for small displacements of the moving coil from the position of maximum are very nearly the same for both fixed coils.

Consequently, the maximum observed force is obtained when the moving coil is at C, Fig. 17, midway between the maxima A and B of the two coils respectively, for a slight displacement from this position will cause the force exerted by one fixed coil to increase by the same amount as that exerted by the other

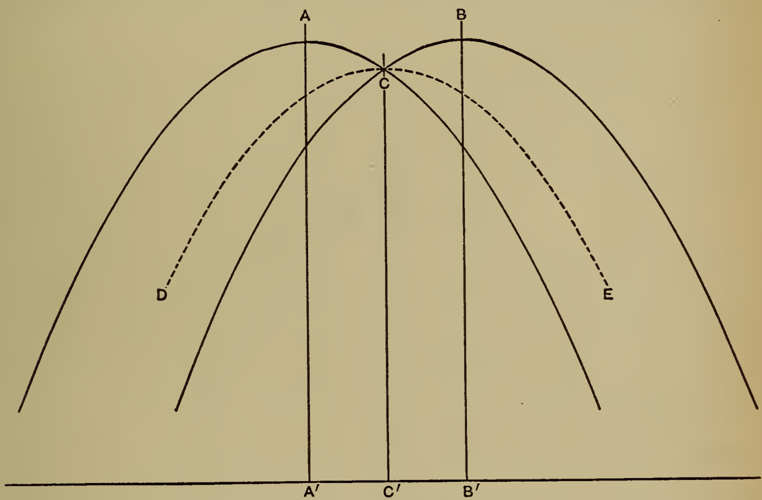

Fig. 17.—*Graphical plots of the force upon the moving coil due to each of the two fixed coils, and of one-half the sum of their forces, the distance between the fixed coils being such that the force maxima do not coincide*

will decrease. Also, for this position of the moving coil the force CC' exerted by each fixed coil will be less than the corresponding maximum forces AA' and BB' by the same amount; therefore the difference between the two forces which is observed when the moving coil is in this position is equal to the true difference between the two maximum possible forces. Also for small displacements from this position the observed difference between the two forces will vary linearly with the displacement, the rate of variation being proportional to the difference between the actual distance between the fixed coils and that corresponding to the absolute maximum of force for the given current. Hence, the slope of the line representing the difference in the forces as a function of the position of the moving coil is a criterion of the correctness of the spacing of the fixed coils.

Also, when the moving coil is in the position A, corresponding to the true maximum force of one of the fixed coils, a slight displacement will not affect the force due to this coil, and, consequently, the observed rate of variation of *both* the sum of and the difference in the two forces will be equal to the variation in the force due to the other coil, and so will be equal to one another. Also, if the displacements throughout this range are small enough to

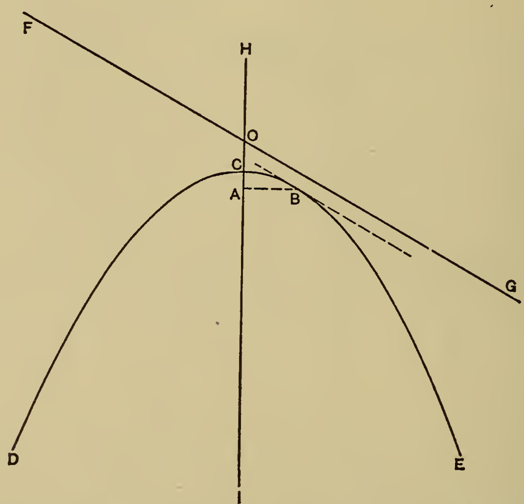

Fig. 18.—*The variation with the position of the moving coil of the observed sum and difference of the forces exerted by the two fixed coils when the distance between them is not that which gives coincidence of the maxima*

enable us with sufficient accuracy to express the variation of the force from the maximum as proportional to the square of the displacement, then it is easy to see that the sum of the true maximum forces is as much greater than the observed maximum of the sum as the latter exceeds the observed sum of the forces when the moving coil is in the position of maximum for either coil; that is, when the moving coil is in such a position that the rate of variation of the observed sum of the forces equals the rate of variation of the observed difference. Consequently, we have the following graphical method for determining the sum and the difference of the true maximum forces from the observed values. In Fig. 18 let *DCE* represent the relation between the observed sum of the forces, as ordinates, and the position of the moving coil as abscissas; *DCE* will be a parabola, *HI* its axis. Similarly and on the *same scale* let *FOG* represent the observed difference between the forces (the origin from which the difference of the forces is measured will naturally not coincide with that used for the sum of the forces). Let *B* be the point at which the tangent to the *DCE* curve is parallel to *FOG*. Draw *BA* parallel to the axis of abscissas. Then the value of the difference in the force observed at the point *O* will be the difference between the true maximum forces, the sum of the two maximum forces will exceed the force at *C* by the amount *AC*, and the distance between the fixed coils differs from that corresponding to the true maximum by twice the distance represented by *AB*.

The next approximation in which the force of each coil is represented by a cubic is considered in the appendix, page 383, where it is shown that in all cases encountered in the present work the construction just given is sufficiently exact.

A similar discussion applies to the case when the fixed coils are not coaxial. This may be investigated experimentally by displacing the moving coil in a horizontal plane. If the moving coil is coaxial with *one* of the fixed coils, the force due to the latter is a minimum. If the moving coil is slightly displaced in a horizontal plane, the force increases proportionally to the square of the displacement. If the *two fixed coils* are not coaxial, the sum of their forces will be too great even for that position of the moving coil in which the sum of the forces is a minimum. This is a parallel case to the one considered above, where the maxima do not coincide, excepting that the error is of the opposite sign. The error due to the lack of coaxiality may likewise be determined experimentally by the curve of the difference of the forces exerted by the two fixed coils. If these coils are coaxial, their difference will be constant for small displacements of the moving coils in a horizontal plane. Otherwise the difference will vary linearly with the displacement of the moving coil, the slope of the line representing the difference will be a maximum for displacements of the moving coil in the same direction in which the axes of the fixed coils are displaced relative to one another. Two experimental curves for the difference taken in a horizontal plane and in directions at right angle to each other, allow the lack of coaxiality to be measured, and the corresponding correction to be applied to the sum of the forces determined. The mechanical adjustments of the fixed coils were so good that this correction, as already stated, always amounted to less than five parts in a million in the force.

(c) *For load and temperature of coils.*—We have calculated the constants of the balances for 22°00 C and no load. As they were used at different temperatures and with different loads, it is necessary to apply temperature and load corrections.

Either of two methods may be adopted in calculating these corrections. We may correct the calculated constant for the load and the temperatures, and thus obtain the actual constant of the balance under working conditions; or we may correct the observed force for these quantities and thus obtain the force that would have been observed had the coils been of the radii assumed in the calculation of the constant. Since the temperatures of the coils differ slightly from time to time, and it is frequently desirable to study the variations in or the constancy of the force for a given current and a given balance constant, we have adopted the second method and have corrected the observed force for the temperatures and load. Knowing the coefficient of expansion of the coils, the load correction and the rate of variation of the force with the radius (Tables VII and X), this correction is readily determined.

32. THE OBSERVED DIFFERENCE IN THE FORCES

Since the study of the difference in the forces exerted upon the moving coil by the single fixed coils is of the nature of an adjustment or check measurement, it appears desirable to consider briefly at this point the results which were obtained by this study during the final set of observations. These are set forth in Table XV. Proper corrections for leads, temperatures, etc., have been applied.

TABLE XV

The Difference in the Forces (Milligrams)

Date	Coils	2×sum of forces g	Order of sub-tracting	Difference in forces: Upper coil is 2 or 3 A	Difference in forces: Upper coil is 1 or 4 B	Change on inter-chang-ing A-B	Mean $\frac{A+B}{2}$ M	Calcu-lated differ-ence C	M-C
Nov., 1910	S1, S2, M3	6	2–1	+3.45	+3.43	+0.02	....	....	....
Mar., 1911	S1, S2, M3	6	2–1	+3.45	....	....	....	....	....
Mean	....	....	....	+3.45	....	....	+3.44	+3.43	+0.01
Nov., 1910	L1, L2, M3	6	2–1	−0.04	+0.12	−0.16	....	....	....
Mar., 1911	L1, L2, M3	6	2–1	+0.11	....	....	....	....	....
Mean	....	....	....	+0.04	....	....	+0.08	+0.27	−0.19
Nov., 1910	L3, L4, M3	6	3–4	+1.26	+1.49	−0.23	....	....	....
Feb., 1911	L3, L4, M3	6	3–4	....	+1.46	....	....	....	....
Mar., 1911	L3, L4, M3	6	3–4	....	+1.61	....	....	....	....
Mean	....	....	....	....	+1.52	....	+1.39	+1.39	±0.00
Sept., 1910	L3, L4, M2	6	3–4	+1.58	+1.60	−0.02	....	....	....
Feb., 1911	L3, L4, M2	6	3–4	....	+1.57	....	....	....	....
Mean	....	....	....	....	+1.58	....	+1.58	+1.54	+0.04
Oct., 1910	L3, L4, M2	3	3–4	+0.78	+0.79	−0.01	+0.78	+0.77	+0.01
Oct., 1910	L3, L4, M4	6	3–4	+1.46	+1.60	−0.14	....	....	....
Feb., 1911	L3, L4, M4	6	3–4	....	+1.69	....	....	....	....
Mean	....	....	....	....	+1.64	....	+1.55	+1.54	+0.01
Oct., 1910	L3, L4, M4	3	3–4	....	+0.79	....	....	+0.77	....

It will be noticed that the excess of the mean difference (*M*) observed over the difference (*C*) calculated from the dimensions of the coils exceeds the experimental error in the weighing in but a single case. This is a good check on the accuracy of the radii determinations, and on the computations. Even in the case of the single exception the discrepancy is so small that were it entirely due to an error in the computed constant of the balance it could affect the measurement of the current by but 15 parts in a million.

In three cases the effect of interchanging the fixed coils lies well within the experimental error, showing the complete absence of any asymmetrical distribution of fixed magnetic masses that can affect the measurement of the current.

In the other three cases this effect, though quite measurable, is still so small that even if it affected the sum of the forces by its full amount it would in the worst case amount to but 2 in a hundred thousand in the measurement of the current. While the source of the trouble is not entirely clear, we believe, in view of the enhanced sensibility of the method of difference, that whatever its source it can affect the measurement of the current by but a small fraction of this amount. A further study of this point is to be made shortly.

33. RESULTS

The observations naturally fall into three distinct groups. The first includes those taken before November 30, 1908. These were of a strictly preliminary character, and served the purpose of discovering defects in the design and in the manipulation of the balance.

Then followed a long series of nearly 500 measurements of current made between November 30, 1908, and May 7, 1910. During this period many changes in the details of the work and of the mounting of the coils were made, and the method of procedure adopted in the final work was developed and thoroughly tested, the effects of small maladjustments were studied, and the question of the possible effect of slight magnetic impurities in the apparatus and of a permeability of the surroundings slightly greater than unity was investigated.

Unfortunately, the possible variations in the radii of the coils with the humidity (p. 286) had not yet been recognized, and the coils were not sealed. Consequently, these results are of uncertain value, and, as stated in the section dealing with the ratio of the radii, the earlier ones are also somewhat uncertain owing to the importance in that work of the parallelism of the coils (p. 307) being unrecognized at the time.

A summary of the values obtained in this interval is, however, given in Table XVI. To each value is assigned a weight equal to the total number of days on which observations were taken. While the values have a much greater range than in the later work, the mean of all is remarkably close to that obtained in the final measurements.

TABLE XVI

Results of Observations Previous to the Summer of 1910

Date	Coils	Number of weighings	Weight of observation	I_i Current in international amperes (1911)	Doubled force mg	I_A Current in absolute amperes	$\frac{I_i-I_A}{I_A}\times10^5$	Deviation
Nov. 30–Dec. 12, 1908	L2, M1	60	4	1.010370	3000.11	1.010211	16	8
Dec. 21, 1908–Jan. 5, 1909	S1, S2, M1	29	4	0.989318	6000.01	0.989215	10	2
Mar. 24–Apr. 12, 1909	L1, L2, M1	61	13	1.010370	6000.23	1.010322	5	3
Apr. 13–July 16, 1909	L1, L2, M2	102	24	0.768384	6000.29	0.768308	10	2
July 17–Aug. 3, 1909	L1, L2, M2	39	8	0.768384	6000.66	0.768331	7	1
Aug. 7, 1909	L1, L2, M3	5	1	0.858897	5999.81	0.858848	6	2
Oct. 1–5, 1909	L1, L2, M3	15	3	0.858897	5999.28	0.858811	10	2
Oct. 11–14, 1909	L1, L2, M1	23	4	1.010370	6000.69	1.010280	9	1
Oct. 19–25, 1909	L1, L2, M2	19	6	0.768384	6000.40	0.768330	7	1
Oct. 28–29, 1909	L1, L2, M3	10	2	0.858897	5999.51	0.858827	8	0
Nov. 6–11, 1909	S1, S2, M3	17	4	0.840273	5999.37	0.840224	6	2
Nov. 17–23, 1909	S1, S2, M1	16	4	0.989318	6000.79	0.989399	−8	....
Apr. 4–13, 1910	L3, L4, M3	36	7	0.858448	5999.83	0.858386	7	1
Apr. 15–16, 1910	L1, L3, M3	10	2	0.858638	5999.96	0.858581	7	1
Apr. 18–May 7, 1910	L3, L4, M2	40	10	0.767855	6000.44	0.767806	6	2
Weighted mean for the total of 482 weighings	....	....	....	....	....	....	7.4	....
Weighted mean, omitting Nov. 17–23	....	....	....	....	....	....	8.0	2

The emf of the Weston normal cell at 20°00 derived from the above results is 1.018 30 − 0.000 08 = **1.018 22** semiabsolute volts.

The last three values are especially interesting, as they indicate that the result given by $L1$, the most magnetic of the coils, does not differ appreciably from that given by $L4$, the best coil.

In reducing these results we have used the radii observations of October, 1908, for the observations between November 30, 1908, and January 5, 1909; those of February, 1909, for the observations between March 24 and August 7, 1909; and those of January, 1910, for the remainder.

The average difference of the results obtained in the 14 sets of measurements shown in Table XVI (excluding the one set of November 17–23, 1909) from the mean of the whole series is 2 parts in 100 000. If all the errors had been purely accidental, we could safely assume that in so large a number of weighings (482) these errors would be very largely eliminated, and the final result, 1.018 22 volts at 22° C for the Weston normal cell, would be considered as determined with very high accuracy.

The probable error of this weighted mean is less than 1 part in a million, and it would in that case be useless to make any further current weighings. But there was evidence in this work of certain small systematic errors which we believed could be eliminated so that a more satisfactory series of results could be obtained. Hence, after the peculiar effects, finally traced to changes in atmospheric humidity, had been overcome, it was decided to make certain changes for removing the possibility of these slight errors, and to undertake such a final series.

In May, 1910, these peculiar effects became very pronounced, and the rest of the spring and summer was devoted to an investigation of their cause (see appendix, p. 368) and to the sealing of the coils, the winding of moving coil $M4$, the replacing of the brick piers by wooden ones, and general preparation for exact measurements in the autumn and winter.

The third and final group of measurements was made between September 24, 1910, and March 25, 1911, and is composed of two parts. The first extends to November 17, 1910. Then the balance and the current connections were entirely taken apart, and the ratio of the radii of the coils, the needle correction, the temperature coefficients, and the load corrections were carefully measured. This work, with its reduction, lasted until February. The constants as thus determined were used in the reduction of all the weighings composing this group and have been given in a former section of this paper (Sec. III).

The balance was then reassembled and the remaining observations were made.

All of the results are given in considerable detail in Table XVII. As already stated, at least two additional weighings were made each day with the moving coil displaced considerably from the position of maximum force. As these observations were taken merely for the purpose of locating the position of the maximum, they are of the nature of observations for adjustment, and so are not given in the table nor considered in the determination of the value of the cell.

TABLE XVII

Third Series of Measurements of the Electromotive Force of the Weston Normal Cell in Semiabsolute Volts, Sept. 24, 1910–Mar. 25, 1911

Date	Coils			Temperatures		Double force		Current (absolute amperes)	Current int. amp. (1911)	Int.−abs.	Parts per 10^6	Excess above mean	Emf of cell (semiabsolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed	Moving	Observed	Corrected							
1910						*mg*	*mg*							
Sept. 24	L3	L4	M2	28°10	26°38	6000.08	6000.12	0.767748	0.767802	54	70	+ 6		
				28°10	26°38	.09	.13	49	02	53	69	+ 5		
Sept. 26				28°16	26°62	.10	.07	45	798	53	69	+ 5		
				28°17	26°59	.18	.17	51	95	44	57	− 7		
				28°16	26°56	.16	.15	50	97	47	61	− 3		
				28°14	26°53	.09	.08	46	97	51	66	+ 2		
Sept. 27	L4	L3	M2	28°14	26°56	.14	.07	45	99	54	70	+ 6		
				28°10	26°46	.32	.23	55	800	45	59	− 5		
				28°12	26°47	.28	.22	54	799	45	59	− 5		
				28°12	26°47	.23	.17	51	99	48	63	− 1		
Sept. 30				28°19	26°47	5999.43	5999.46	06	58	52	68	+ 4		
				28°18	26°48	6000.12	6000.14	49	96	47	61	− 3		
Oct. 1				28°16	26°37	.21	.27	58	802	44	57	− 7		
				28°16	26°37	.22	.27	58	02	44	57	− 7		
				28°16	26°37	.11	.17	51	02	51	66	+ 2		
				28°16	26°37	.09	.15	50	02	52	68	+ 4		
Mean	....	....	....	....	....	....	....	....	....	....	64	4_5	1.01823_5	+10

TABLE XVII—Continued

Date	Coils			Temperatures		Double force		Current (absolute amperes)	Current Int. amp. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of cell (semiabsolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed	Moving	Observed	Corrected							
1910						*mg*	*mg*							
Oct.[22] 6	L4	L3	M2	25°82	26°21	3000.41	3000.26	0.383887	0.383914	27	70	− 5		
				25°81	26°24	.42	.27	87	14	27	70	− 5		
				25°82	26°24	.47	.32	91	14	23	60	−15		
				25°80	26°21	.41	.26	87	14	27	70	− 5		
Oct.[22] 10	L3	L4	M2	23°02	23°42	.39	.28	88	23	35	91	+16		
				23°03	23°44	.47	.35	92	22	30	78	+ 3		
				23°03	23°47	.41	.29	89	22	33	86	+11		
Mean	....	....	....	....	....	....	....	....	....	....	75	9	1.01822_4	−1
Oct. 14	L3	L4	M4	26°26	23°94	5999.93	6000.35	0.775881	0.775914	33	43	− 3		
Oct. 15				26°24	23°98	.73	.13	67	12	45	58	+12		
				26°24	23°98	.76	.16	69	12	43	55	+ 9		
Oct. 17				23°48	21°65	.91	.22	73	09	36	46	0		
				23°49	21°68	.88	.18	70	05	35	45	− 1		
				23°49	21°68	.92	.22	73	04	31	40	− 6		
Oct. 20	L4	L3	M4	23°45	21°56	.31	5999.65	36	868	32	41	− 5		
				23°44	21°56	.98	6000.31	78	907	29	37	− 9		
Mean	....	....	....	....	....	....	....	....	....	....	46	6	[1.01825_8]	
Oct. 18	L3	L4	M4	21°84	21°03	2999.92	2999.99	0.548614	0.548631	17	31	−15		
				21°80	20°97	.54	.64	582	10	28	55	+ 5		
				21°80	20°97	3000.28	3000.38	649	69	20	36	−10		
				21°86	21°07	.24	.31	43	67	24	44	− 3		
Oct. 24	L4	L3	M4	21°73	20°68	.29	.42	53	76	23	42	− 4		
				21°74	20°70	.24	.36	48	75	27	49	+ 3		
Oct. 25				21°70	20°56	.23	.37	49	80	31	56	+10		
				21°72	20°63	.21	.33	45	76	31	56	+10		
				21°72	20°65	.27	.38	49	76	27	49	+ 3		
Mean	....	....	....	....	....	....	....	....	....	....	46	7	[1.01825_3]	
Oct. 28	L4	L3	M3	24°24	23°68	6000.51	6000.27	0.858366	0.858434	68	79	− 9		
Oct. 31				24°22	23°44	.33	.30	69	44	75	87	− 1		
				24°24	23°55	.33	.25	65	38	73	85	− 3		
				24°25	23°60	.33	.24	64	35	71	83	− 5		
Nov. 2	L3	L4	M3	24°23	23°65	.24	.13	56	37	81	94	+ 6		
				24°22	23°67	.28	.16	59	36	77	90	+ 2		
				24°22	23°67	.30	.18	60	32	72	84	− 4		
Nov. 3				28°65	27°70	.10	.02	49	32	83	97	+ 9		
				28°66	27°72	.10	.01	48	31	83	97	+ 9		
Mean	....	....	....	....	....	....	....	....	....	....	88	5	1.01821_0	−15

[22] These observations were made with the windings of the moving coil in series.

TABLE XVII—Continued

Date	Coils: Upper	Coils: Lower	Coils: Moving	Temperatures: Fixed	Temperatures: Moving	Double force: Observed	Double force: Corrected	Current (absolute amperes)	Current int. amp. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of cell (semiabsolute volts)	Excess above mean in millionths
1910						mg	mg							
Nov. 4	L2	L1	M3	24°84	23°95	5999.71	5999.79	0.858878	0.858935	57	66	+ 1		
				24°86	23°96	6000.13	6000.13	902	54	52	61	− 4		
Nov. 5				24°86	23°94	5999.35	5999.36	847	01	54	63	− 2		
				24°90	24°04	.07	.07	26	880	54	63	− 2		
Nov. 7	L1	L2	M3	25°03	23°86	.13	.19	35	90	55	64	− 1		
				25°06	23°94	.97	6000.01	93	957	64	74	+ 9		
Mean.	....	....	....	....	....	....	....	....	....	....	65	3	1.01823_4	+ 9
Nov. 11	S1	S2	M3	22°46	23°27	5998.86	5998.61	0.840166	0.840218	52	62	+ 6		
Nov. 12				22°46	23°25	.89	.70	72	20	48	57	+ 1		
				22°46	23°24	.92	.74	75	19	44	52	− 4		
Nov. 14				22°36	23°18	5999.57	5999.37	219	72	53	63	+ 7		
				22°36	23°21	6000.00	.80	49	302	53	63	+ 7		
				22°38	23°23	.05	.85	53	297	44	52	− 4		
				22°38	23°23	5999.98	.78	48	97	49	58	+ 2		
Nov. 16	S2	S1	M3	22°34	23°23	.01	5998.80	179	18	39	46	−10		
Nov. 17				22°33	23°22	6000.06	5999.85	253	303	50	60	+ 4		
				22°30	23°22	.10	.88	55	299	44	52	− 4		
				22°30	23°22	.06	.84	52	99	47	56	0		
Mean.	....	....	....	....	....	....	....	....	....	....	56	4	1.01824_3	+18
1911														
Feb. 3	L4	L3	M4	23°28	21°22	5999.64	5999.99	0.775858	0.775905	47	61	+ 7		
				23°28	21°30	.08	.39	19	862	43	55	+ 1		
				23°29	21°29	.16	.48	25	59	34	44	−10		
Feb. 4				23°28	21°14	.04	.40	20	67	47	61	+ 7		
				23°30	21°30	.60	.92	53	902	49	63	+ 9		
				23°31	21°30	.60	.92	53	898	45	58	+ 4		
				23°32	21°30	.62	.95	55	98	43	55	+ 1		
Feb. 6				23°20	20°98	.76	6000.16	74	914	40	52	− 2		
				23°25	21°09	.66	.03	66	06	40	52	− 2		
				23°28	21°16	.64	.00	64	02	38	49	− 5		
				23°30	21°22	.63	5999.98	62	01	39	50	− 4		
Feb. 7				23°28	21°15	.68	6000.04	61	03	42	54	0		
				23°29	21°22	.66	.00	58	899	41	53	− 1		
				23°30	21°29	.66	5999.98	57	95	38	49	− 5		
Feb. 8				23°28	21°20	.70	6000.04	61	902	41	53	− 1		
				23°30	21°23	.70	.04	61	01	40	52	− 2		
Mean.	....	....	....	....	....	....	....	....	....	....	54	4	$[1.01824_5]$	

TABLE XVII—Continued

Date	Coils			Temperatures		Double force		Current (absolute amperes)	Current Int. amp. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of cell (semiabsolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed.	Moving	Observed	Corrected							
1911						*mg*	*mg*							
Feb. 14	L4	L3	M2	23°24	21°73	5999.98	6000.10	0.767747	0.767801	54	70	− 3		
				23°24	21°76	.94	.05	44	00	56	73	0		
Feb. 15				23°20	21°65	.49	5999.62	16	766	50	65	− 8		
				23°20	21°67	.97	6000.11	47	804	57	74	+ 1		
Feb. 16				23°20	21°59	.94	.10	47	08	61	79	+ 6		
				23°21	21°65	.92	.06	44	04	60	78	+ 5		
				23°23	21°68	.96	.10	47	01	54	70	− 3		
Mean.	....	....	....	....	....	....	....	....	....	....	73	4	1.01822_6	+ 1
Feb. 18	L4	L3	M3	24°10	23°47	5999.99	5999.83	0.858335	0.858426	91	106	+ 5		
				24°78	24°07	.27	.10	283	372	89	104	+ 3		
Feb. 20				24°69	23°82	.46	.37	302	87	85	99	− 2		
				24°72	23°89	6000.03	.93	42	428	86	100	− 1		
				24°75	23°98	5999.93	.82	34	23	89	104	+ 3		
Feb. 21				24°64	23°85	6000.13	6000.02	49	31	82	96	− 5		
				24°64	23°87	.14	.02	49	27	78	91	−10		
Feb. 25				24°70	23°99	.19	.05	51	38	87	101	0		
				24°71	24°00	5999.92	5999.78	31	20	89	104	+ 3		
				24°72	24°01	.93	.79	32	20	88	103	+ 2		
Feb. 28				24°72	24°06	.90	.75	29	19	90	105	+ 4		
				24°72	24°07	6000.21	6000.07	52	36	84	98	− 3		
Mar. 1				24°70	24°04	.05	5999.90	40	28	88	103	+ 2		
				24°72	24°05	.05	.90	40	22	82	96	− 5		
				24°72	24°06	.03	.88	39	22	83	97	− 4		
Mean.	....	....	....	....	....	....	....	....	....	....	101	4	1.01819_7	−28
Mar. 3	L2	L1	M3	25°17	24°06	5998.92	5998.91	0.858815	0.858871	56	65	− 7		
Mar. 4				25°15	24°10	5999.49	5999.52	58	918	60	70	− 2		
Mar. 6				25°06	23°85	.53	.60	64	29	65	76	+ 4		
				25°08	23°90	.51	.57	62	25	63	73	+ 1		
				25°10	23°95	.57	.62	65	22	57	66	− 6		
Mar. 7				25°08	23°88	.55	.62	65	28	63	73	+ 1		
				25°06	23°91	.53	.58	63	24	61	71	− 1		
				25°06	23°98	.58	.61	65	21	56	65	− 7		
				25°07	23°99	.48	.51	58	20	62	72	+ 0		
Mar. 8				25°02	23°94	.97	6000.00	93	61	68	79	+ 7		
				25°07	24°00	.94	5999.97	90	50	60	70	− 2		
				25°07	24°01	.86	.89	85	50	65	76	+ 4		
				25°08	24°02	.86	.89	85	49	64	75	+ 3		
Mean.	....	....	....	....	....	....	....	....	....	....	72	4	1.01822_7	+ 2

TABLE XVII—Continued

Date	Coils			Temperatures		Double force		Current (absolute amperes)	Current int. amp. (1911)	Int.—abs.	Parts per 10^6	Excess above mean	Emf of cell (semiabsolute volts)	Excess above mean in millionths
	Upper	Lower	Moving	Fixed	Moving	Observed	Corrected							
1911						mg	mg							
Mar. 13	S1	S2	M3	23°.23	24°.05	5999.73	5999.45	0.840232	0.840290	58	69	−3		
				23°.27	24°.08	5998.68	5998.37	149	07	58	69	−3		
Mar. 14				23°.23	24°.07	5999.77	5999.49	228	86	58	69	−3		
				23°.24	24°.08	6000.05	.74	45	304	59	70	−2		
Mar. 15				23°.22	24°.04	.37	6000.06	67	32	65	77	+5		
				23°.24	24°.07	36	.05	67	29	62	74	+2		
				23°.26	24°.10	.31	5999.99	63	24	61	73	+1		
Mar. 16				23°.22	24°.07	.42	6000.11	81	32	51	61	−9		
				23°.23	24°.05	.29	.02	65	28	63	75	+3		
				23°.23	24°.05	.29	.00	63	26	63	75	+3		
Mar. 17				23°.22	24°.07	.35	.04	66	29	63	75	+3		
				23°.24	24°.07	.34	.03	65	26	61	73	+1		
Mean	....	....	....	....	....	....	....	....	....	....	72	3	1.01822$_7$	+2
Mar. 20	L4	L3	M4	23°.96	21°.82	5999.76	6000.08	0.775864	0.775902	38	49	0		
				23°.94	21°.79	.66	5999.98	57	897	40	52	+3		
Mar. 21				23°.97	21°.84	.14	.45	23	61	38	49	0		
Mar. 22				24°.00	21°.85	.05	.38	18	59	41	53	+4		
				24°.01	21°.89	.77	6000.14	68	902	34	44	−5		
				24°.02	21°.95	.79	.13	67	01	34	44	−5		
Mar. 23				23°.98	21°.81	.90	.28	77	18	41	53	+4		
				23°.97	21°.82	6000.01	.39	84	17	33	43	−6		
				23°.97	21°.83	5999.98	.35	81	14	33	43	−6		
				23°.98	21°.84	.92	.29	77	14	37	48	−1		
Mar. 24				23°.96	21°.76	.93	.32	79	20	41	53	+4		
				23°.96	21°.77	.89	.28	77	18	41	53	+4		
				24°.00	21°.80	.91	.30	78	14	36	46	−3		
Mar. 25				23°.98	21°.87	.92	.29	77	18	41	53	+4		
				23°.96	21°.78	.89	.28	77	16	39	50	+1		
				23°.98	21°.82	.85	.23	73	14	41	53	+4		
Mean	....	....	....	....	....	....	....	....	....	....	49	3	[1.01825$_0$]	
Mean, omitting those obtained with M4													1.01822$_5$	10

34. DISCUSSION OF RESULTS

(*a*) *The final mean value.*—In taking the mean we have omitted the values found by the use of moving coil *M*4, owing to the fact, already stated (p. 316), that the effective sectional dimensions of this coil are not accurately known. The mean of the others is 1.018 22_5. The mean of the 1910 observations is 4 in a million higher and that of the observations of 1911 are 6 in a million lower than the grand mean. The mean variation of the groups from the mean of all is 1 part in 100 000. The mean variation of the individual weighings of any group from the mean of that group is but 4 in 1 000 000. The mean of the four groups of observations obtained with moving coil $M4$ is 1.018 25_0, and the mean variation from this mean is but 3 in 1 000 000, although for three of the groups the coils were independently adjusted. Observations with the other moving coils would probably be as good, but for none of the others have so many distinct groups of observations been taken during this final series. The high value obtained by means of $M4$ is due to the fact that the assumed axial breadth of the coil is greater than the true. From the method of winding we know that this is the case, although it was not foreseen when the coil was wound. There is, however, no way to determine how much too great it is. These values must, therefore, be disregarded in taking the mean. We believe that values found at different periods, but with the same combination of coils, will be relatively correct to within about 5 in 1 000 000, provided the radii are redetermined for each period. The radii should, of course, remain constant, but an occasional redetermination would be necessary to make sure that there has been no change.

The final conclusion is that the electromotive force of the Bureau of Standards' concrete realization of the mean Weston normal cell at 20°.00 C, January 1, 1911, was

1.01822 Semiabsolute [23] Volts

The probable error of the mean 1.018 22_5 as given in the table is 3 parts in 1 000 000; and we believe it is a conservative estimate to assign to the value given above a possible uncertainty, due to all causes, of 2 in 100 000, a quantity equal to six times the computed probable error.

This differs from the result obtained in 1907 at the National Physical Laboratory (1.018 18 volts at 20°) by 4 parts in 100 000.

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[23] A "semiabsolute volt" is that potential difference which exists between the terminals of a resistance of one international ohm when the latter carries a current of one absolute ampere.

This difference may, of course, be due to a great many sources combined, but doubtless an appreciable part may be due to a real difference in the cells measured.

(*b*) *Comparison of values obtained with different combinations of coils.*—In Table XVII there are given values obtained with five combinations of coils. Following are the mean values in each of four combinations, omitting the one found with moving coil number 4:

		Millionths
1. M2, L3, L4 gives	$1.018\ 22_8$	$\Delta = 4$
2. M3, L3, L4 gives	$1.018\ 20_4$	20
3. M3, L1, L2 gives	$1.018\ 23_0$	6
4. M3, S1, S2 gives	$1.018\ 23_5$	11
Mean........	$1.018\ 22_4$	10

In these four cases there have been employed three pairs of fixed coils, two large and one small, and two moving coils of different radii. The excellent agreement of the results, in which the separate values differ from the mean by only ten parts in a million, shows that the outstanding errors in the determination of the constants of the coils and in the adjustment of the coils in the balance is very small. The weight to be attached to the final mean is far greater than if the work had been done with a single pair of fixed coils and a single moving coil.

35. ACCELERATION OF GRAVITY

The acceleration of gravity in a basement room of the physical laboratory of this Bureau was determined in terms of its value at the gravity pier at the United States Coast and Geodetic Survey by means of relative pendulum observations taken by Mr. Wm. H. Burger, of the Survey, in August, 1910. The value of the acceleration of gravity at the Survey pier is known in terms of that at Potsdam from the relative pendulum observations made in 1900 by Mr. G. R. Putnam, of the Survey. Hence the value at this Bureau can be referred at once to that at Potsdam. As the result of a long series of determinations, the absolute value of the acceleration of gravity at the latter place is believed to be 981.274 cm sec^{-2} with a mean error of but 3 in a million.[24] On this basis

[24] Jahresbericht Preus. Geod. Inst., 1906–7, p. 6.

the value in the basement of this laboratory is 980.094. The balance is 9 m above the point where these observations were taken, and hence the value at the balance is 980.091 cm per second. This is the value that has been used in the reduction of the results. It is probably correct absolutely to within 5 or 6 in a million. The value of the acceleration of gravity used in the reduction of the results obtained at the National Physical Laboratory, and already mentioned several times, is referred to this same basis.

36. THE STANDARD CELLS

As stated in a former section, the cell actually used in these measurements was some one of a group of four Weston normal cells, which were kept in a thermostatically controlled oil bath in the same room as the balance. These, in situ, were compared almost daily with a group of seven Weston normal cells main-

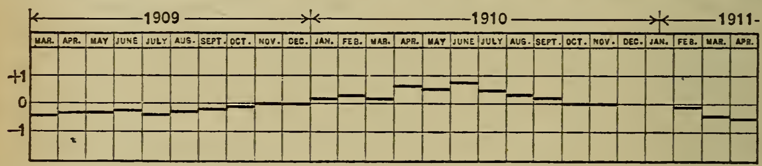

Fig. 19.—*Variation of standard cell W 56 relative to E_7, the reference set of this Bureau*
One division of the ordinates represents ten microvolts

tained continuously at 28°.00 C, and forming the working standard of the Bureau.

These seven cells are known as *W* 7, 10, 13, 20, 21, 23, and 25, and are selected cells. They were chosen in December, 1907, from a large number of cells, and at that time their mean electromotive force was 14 parts in a million lower than the mean of the 12 cells used in the work in this laboratory by F. A. Wolff and C. E. Waters on the Clark and Weston standard cells.[25] The mean of the 12 cells E_{12} has been referred to as the "old basis" of reference.

With reference to the above-named seven cells, those used in the actual measurements have changed very slightly. In Fig. 19 is shown the relative variation during a period of two years of

[25] This Bulletin, 4, pp. 1–81; 1907.

$W56$, the cell which has, except upon rare occasions, been used. It will be seen that throughout this period it has first increased 11 parts in a million and then decreased 12 parts in a million with reference to the seven reference cells.

In the spring of 1910 delegates from the National Physical Laboratory, the Physikalisch-Technische Reichsanstalt, the Laboratoire Central d'Électricité, and of the Bureau of Standards met at this laboratory for cooperative study of the silver voltameter and of the standard cell. Each delegate brought a number of cells from his laboratory, and it was agreed that the mean of the means of each of the four groups of cells (one group from each laboratory) should be regarded as the value of the mean Weston normal cell. The electromotive force of the cell at 20°.00 C thus determined has been defined by the International Committee as 1.0183 international volts.

The difference between this value and the mean electromotive force of the seven cells named above was found to be 40 microvolts, the seven being the lower. Hence, denoting the mean electromotive force of the seven by E_7 and that of the mean Weston normal cell by E_N, we have—

$$E_N = E_7 + 40 \text{ microvolts}$$
$$E_N = E_{12} + 26 \text{ microvolts}$$

on June 1, 1910.

From numerous intercomparisons of various groups of cells at this laboratory and from intercomparisons with other laboratories it appears probable that the mean electromotive force of those cells (E_{36}) which were used in the determination of the mean cell and which have since been kept at this laboratory has decreased by about 12 parts in a million in one year; or, assuming a uniform drift, we may say that they have decreased at the rate of one in a million per month.

The difference between E_{36} and E_7 has been determined from time to time by direct comparison; hence, assuming the uniform drift for E_{36}, we can find the value of E_7 in terms of the mean Weston normal cell (E_N). This is done in Table XVIII.

TABLE XVIII

Values of E_7 in terms of E_N

	Oct., 1910	Feb., 1911
E_7	$=E_{36}-32^m$	$=E_{36}-22^m$
E_{36}	$=E_N-\ 4^m$	$=E_N-\ 8^m$
E_7	$=E_N-36^m$	$=E_N-30^m$

From these observations and inferences we must conclude that E_7 has gone up by 6 in a million from October to February, while the balance observations seemed to indicate that it has gone down by about 10 in a million—a discrepancy of 16 in a million.

This discrepancy is very small and is scarcely more than the limits of accuracy with which it is at present possible to perpetuate a given electromotive force by means of standard cells. For this reason we have assumed that $E_7 = E_N - 36$ microvolts throughout the period covered by this work.

In reducing the values to 20°.00 C. we have used the coefficients adopted by the London Conference, viz:

$$E_t = E_{20} - 0.000\,0406(t-20^\circ) + 0.000\,000\,95(t-20^\circ)^2 + 0.000\,000\,001\,(t-20^\circ)^3$$

37. THE RESISTANCES

The standard of resistance at this Bureau is furnished by a set of sealed wire coils which have been frequently intercompared among themselves, and, by means of traveling standards, compared with the standards of the European laboratories. They have been found to remain very constant. At the above-mentioned meeting in 1910 at Washington a comparison was made of the values of the resistances used at the different national laboratories as representative of the international ohm, and the departures of these various representations from what appeared to be the most probable value of the mean international ohm was determined. The unit as formerly used at this laboratory was found to have a resistance 7 parts in 1 000 000 greater than the adopted value of the mean international ohm. Correction for this has been made in the present work, the results therefore being expressed in terms of the mean international ohm as thus fixed.

38. FINAL RESULT IN TERMS OF THE ELECTROCHEMICAL EQUIVALENT OF SILVER

The above values of the results obtained from the current balance are in terms of the electromotive force of the Weston normal cell and the international ohm. We can also express these results in terms of the electrochemical equivalent of silver, and thereby eliminate the international ohm.

In the course of an extended investigation of the silver voltameter in the laboratories of the Bureau of Standards by Rosa, Vinal, and McDaniel a large number of experiments has been made, comparing different forms of voltameters and different methods of procedure. In three series of deposits during the past 12 months, using very pure electrolyte and following what they consider the best procedure, the following results were obtained for the emf of the Weston normal cell:

		Volts
126 deposits in the porous-pot form of voltameter	mean..	1.01826_2
33 deposits in the nonseptum form of voltameter	mean..	1.01826_5
Mean of all		1.01826_3

This value, of course, is based on the value 1.11800 for the electrochemical equivalent of silver and the international ohm. The above value differs but 4 in 100 000 from the value found from the current balance and international ohm. It follows from the above that the value of the electrochemical equivalent of silver, as derived from our current balance measurements and two types of the silver voltameter as used at the Bureau of Standards, is

$$1.11804 \text{ mg per coulomb}$$

instead of 1.118 00, as adopted by the London Conference. In other words, the international ampere as defined by means of the silver voltameter as used in this Bureau differs from the absolute ampere as realized here by means of our current balance by only 4 parts in 100 000, the international ampere being slightly smaller than the absolute ampere.

APPENDIX

1. EFFECT OF HUMIDITY ON THE RADII OF THE COILS

As stated in the body of this paper, certain abnormalities in the results, which at first seemed inexplicable, were finally found to be due to the moisture absorbed by the thin paper placed between the layers of the coil for the purpose of improving the insulation and of facilitating a uniform winding. Since this effect is quite interesting and of considerable importance, it has seemed well to give the observations with some detail.

In the latter part of April, 1910, using coils $L3$, $L4$, and $M2$, and a current always of the same value, the forces, corrected for coil temperatures, etc., set forth in Table XIX were observed.

TABLE XIX

Corrected Forces, Coils L3, L4, and M2

Date	Force
	mg
April 27, 1910	6000.45
28, 1910	.48
29, 1910	.43
May 9, 1910	.66
10, 1910	.65

Here we see that the force changed by about 0.20 mg between April 29 and May 9, which is 3.3 in 100 000 of the total of 6 g. This is, of course, a very small change, but nevertheless too great to be considered accidental. Previous to April 29 everything had been going nicely and the lower value persisted for several interchanges of the coils; similarly, the higher value of the force persisted after May 10. The insulation was tested (though it was difficult to see how bad insulation of the balance could *increase* the force) and was found to be good, but not quite so good as usual. Then $M2$ was replaced by $M3$, and it also gave a force that was greater than usual. Then the resistance standard was replaced by another, with no change in the result. Other resistance standards were introduced at different points in the circuit so that the current could be measured before going to the balance, between the fixed coils and the moving coil, and after leaving the balance. All three gave the same result. The potentiometer was changed without effect. As usual, the cells were compared daily and were found to be normal. These observations seemed to prove conclusively that the trouble must be in the balance itself.

With the same current the force that had formerly been 5999.77 mg was now found to be 5999.94 mg; it increased as the damp weather continued and then decreased from May 27 to June 1, when the atmosphere became drier. This suggested that the effect was due to changes in humidity.

Later the humidity again became very high and remained so from June 13 to June 24. Dishes of calcium chloride were then placed in the coil case, and the latter was sealed. The force immediately decreased.

TABLE XX

Force for Constant Current, Coils L3, L4, M3 (1910)

Date	Relative humidity	Weight	Force (normal value 5999.77 mg)
	Per cent	mg	mg
May 18	44		5999.94
19	39		6000.03
20	61		6000.24
	Nearly saturated		
26	40		6000.34
27	40		6000.35
31	41		6000.01
June 1	41		5999.98
13	64		6000.53
14	60		6000.60
15	67		6000.58
16	62		6000.72
17	59		6000.66
18	60		6000.68
20	64		6000.75
21	55		6000.78
22	51	299	6000.72
23	49	286	6000.70
24	40	290	6000.78
25	35	119	6000.47
27	42	22	6000.38

These observations are recorded in Table XX. In the third column are also given the fractional weights in the counterpoise for the moving coil. The decrease in these numbers is exactly equal to the decrease in the *weight* of the coil, but does not, of course, affect the measured force.

It will be noticed that while on May 18th the force was 0.17 mg greater than normal, it increased in one week by 0.4 mg; then it went back, but later increased again until on June 24th it was 1.01 mg greater than normal; that is, it was *17 parts in 100 000* greater than earlier in the season when the atmosphere was dry. Drying the case decreased it 0.4 mg in two days.

An increase of 17 in 100 000 in the force corresponds in this case to an increase in the radius of the moving coil of 7 in 100 000, assuming that the fixed coils remained unchanged; that is, to an increase of 0.007 mm in the radius of the moving coil. A layer of water of this thickness spread over the bottom of the wire channel would have a volume of about 44 cubic millimeters. As seen from the few observations on the changing weight of the coil, this is much smaller than the change in the weight of the coil, but, of course, the moisture absorbed would be chiefly in the outer layers, and hence would produce far less effect than if it all penetrated to the bottom of the coil. From these observations we concluded that the observed changes in the force were due to a very slight swelling of the thin paper between the layers of the moving coil when more moisture was absorbed, and a shrinkage in the volume of the paper as moisture was given off, the latter occurring when the weather became drier. This conclusion was confirmed by the fact that after the coils were sealed air-tight the phenomenon disappeared, no such changes in the force being observed as the humidity of the atmosphere varied.

A similar change in the fixed coils would, of course, act in the opposite direction. Hence, the actual change in the moving coils must have been greater than the above figures indicate. The fixed coils had about three times as many layers of wire as the moving coil and hence would be less affected.

2. THE SECTIONAL DIMENSIONS OF A COIL

The value of the galvanometer constant of a coil of axial breadth 2α, of radial depth 2ρ, of mean radius A, and of n turns is usually deduced from the expression [26]

$$G=\frac{2\pi n}{A}\left\{1-\frac{1}{2}\left(\frac{\alpha}{A}\right)^2+\frac{1}{3}\left(\frac{\rho}{A}\right)^2+\frac{3}{8}\left(\frac{\alpha}{A}\right)^4+\frac{1}{5}\left(\frac{\rho}{A}\right)^4-\frac{\alpha^2\rho^2}{A^4}+\ldots\right\}$$

For the mutual inductance of two such coils Maxwell has given the expression [27]

$$M=G_1g_1P_1(\theta)+G_2g_2P_2(\theta)+\ldots\ldots$$

where G_n g_n are given functions of the sectional dimensions of the coils and of their distances from the origin and $P_n(\theta)$ is the zonal harmonic of the angle θ between the axes of the coils.

Lord Rayleigh and Mrs. Sidgwick in the computation of the constant of the current balance used by them made use of the approximation method (due to Purkiss) expressed by the equality [28]

$$\begin{aligned}6F=&f(A+\rho_1,\ a,\ B)+f(A-\rho_1,\ a,\ B)+f(A,\ a+\rho_2,\ B)\\&+f(A,\ a-\rho_2,\ B)+f(A,\ a,\ B+\alpha_1)+f(A,\ a,\ B-\alpha_1)\\&+f(A,\ a,\ B+\alpha_2)+f(A,\ a,\ B-\alpha_2)-2f(A,\ a,\ B)\end{aligned}$$

Lyle's method of approximation by means of the use of equivalent radii has already been mentioned in Section IV.

These and other methods of approximation are based upon an integration over the section of the coil and thus assume that to the order of approximation desired the actual coil is so constructed that the distribution of current may be regarded as uniform over the entire section covered by the integration.

Hence in the application of these formulae of approximation two questions of prime importance arise, viz: (1) Is the construction of the coil such that the effect of the actual distribution of the current is the same as that of a strictly uniform distribution of the same number of ampere turns? (2) What are the boundaries of the area over which the equivalent uniform distribution is spread?

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[26] Rayleigh: Scientific Papers, vol. II, p. 291.

[27] Maxwell: Electricity and Magnetism, § 700.

[28] Maxwell: Electricity and Magnetism, Vol. II, App. II, Ch. XIV, also this Bulletin, 2, p. 370; 1906.

If the wires are uniformly spaced and the distance between adjacent wires is small as compared with the degree of nonhomogeneity (in the neighborhood of the wires) of that magnetic field with which we are concerned, then the answer to the first question is *yes*, otherwise *no*. In the case of the galvanometer constant and of the force between two coils not very close together (that is, in the cases with which we are at present concerned), it is evident that the condition specified above is amply fulfilled by coils wound as closely as those used in this work. On the other hand, in the case of the self-inductance of a coil or of the mutual inductance of coils that are very close together, this condition is not fulfilled and the question of the corrections that are thus necessitated has been considered by several writers from Maxwell down to the present time.[29]

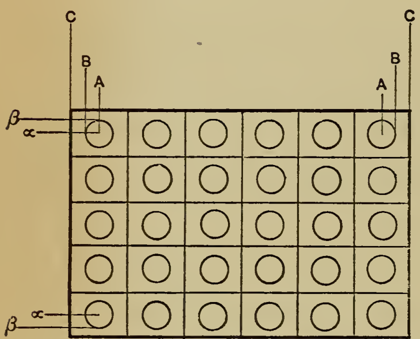

Fig. 20.—*A section of a winding of round wires; the circles represent sections of the conductor and the spaces between are insulation*

When we come to the second question we find considerable difference of opinion among those who have had occasion to use the various approximation formulas. This question has already been considered by one of us,[30] but it appears that another discussion of the subject will not be superfluous.

Limiting ourselves for the present to the case of a uniformly wound coil with the wires arranged in a rectangular array, the distance between the wires in adjacent layers being in general different from that between adjacent wires in the same layer, we may represent a section of the coil as in Fig. 20. The actual boundaries of the channel holding the wire is not shown, as it is immaterial.

Some authors have considered AA and $\alpha\alpha$, the distance measured to the axes of the limiting wires, as the true dimensions of the ideal coil; some have used distances slightly greater than these, but still

[29] Maxwell: Electricity and Magnetism, §§ 691, 692, 693. This Bulletin, **2**, p. 161, 1906; **3**, p. 1, 1907; **4**, pp. 149, 369, 1907–8.

[30] This Bulletin, **2**, pp. 77, 413, 1906; **3**, p. 235, 1907.

measured to points inside the conductor of the bounding wires; others have considered that the correct dimensions are BB and $\beta\beta$ measured to the outer boundaries of the conducting portions of the terminal wires; some have measured to the outside of the insulation of the terminal wires; in at least one case one dimension was measured from axis to axis and the other from outside of insulation to outside of insulation of the bounding wires. Though apparently a compromise, this in reality introduces a greater error than any of the other methods mentioned above. The correct dimensions, namely, those of the heavy rectangular boundary of the figure, have seldom been used.

That this is the proper boundary for the ideal coil assumed in the integration is readily seen from a consideration of the fact that the only uniform distribution that can possibly be equivalent to the wires in the interior of the section is that having such a density that the amount of current which flows through each of the small rectangles is equal to the current carried by the wire. But in order that such a uniform distribution shall carry the same total ampere turns as the actual coil, it must be extended over the area bounded by the heavy rectangle (Fig. 20). Hence the latter is the correct boundary of the equivalent uniform distribution.

In order to exhibit the nature of the errors introduced by an erroneous choice of section, we shall proceed to a calculation of a few cases by means of the method given by Lyle.[31] He has shown that the magnetic field on the axis of an ideal circular coil of mean radius A and of square section of side b, and over which the current is uniformly distributed, is the same as that due to a linear circular circuit of radius equal to $A\left(1+\frac{b^2}{24A^2}\right)$ to within quantities of the magnitude of $\left(\frac{b}{A}\right)^4$. Hence for $A=10$ and $b=0.3$ the accuracy of replacement is of the order of 1 in a million. This radius of the equivalent linear circular circuit he calls the "equivalent radius" of the coil, and we shall denote it by A_e.

If $b=0.1$ and $A=10$, $A_e=10\ (1+4.16\times10^{-6})$, as b becomes smaller A_e approaches A. But it is evident that the effect of a

[31] Philosophical Mag., 3, p. 310; 1902.

circular wire must lie between those of the square wires having sections of such size that they may be circumscribed about or inscribed in the section of the circular wire. Hence, at least to an accuracy of 4.16 in a million in the radius, a circular filament of 10-cm radius, or any round wire bent so that its axis forms a circle of 10-cm radius and having a sectional radius not exceeding a half a millimeter, may be replaced by a square conductor of 1 mm on the side and bent so that its axis forms a circle of 10-cm radius. Hence for simplicity we shall consider the case of square wires.

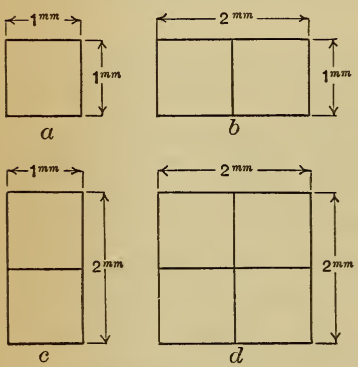

Fig. 21

We shall limit our discussion to a consideration of the galvanometer constant; it can readily be extended to other cases. In Fig. 21 we show sections of but one side of the coils, and shall always assume that the mean radius is 10 cm and that the axes of the coils are horizontal.

(*a*)—For the case represented by (*a*), Fig. 21, we have seen $A_e = 10 \times (1 + 4.16 \times 10^{-6})$.

(*b*)—For each square of (*b*), $A'_e = 10 \times (1 + 4.16 \times 10^{-6})$, but each is displaced 0.05 cm from the mean plane. Hence, in this case, the galvanometer constant for each square is

$$G' = \frac{2\pi}{A'_e}\left\{1 - \frac{3}{2}\left(\frac{0.05}{10}\right)^2 + \dots\right\} = \frac{2\pi}{A'_e}\left\{1 - 37.5 \times 10^{-6} + \dots\right\}$$

Hence, for the complete coil of two turns the equivalent radius is $A_e = 10\ (1 + 41.66 \times 10^{-6})$.

(*c*)—In the case represented by (*c*), the two equivalent radii are $A_e' = (10 + 0.05)(1 + 4.16 \times 10^{-6})$ and $A_e'' = (10 - 0.05) \times (1 + 4.16 \times 10^{-6})$, and the galvanometer constant is

$$G = \frac{2\pi}{2}\left\{\frac{1}{A'_e} + \frac{1}{A_e''}\right\} = 2\pi \frac{10(1 + 4.16 \times 10^{-6})}{(100 - 0.0025)(1 + 4.16 \times 10^{-6})^2}$$

$$= \frac{2\pi}{10\ (1 - 25 \times 10^{-6})(1 + 4.16 \times 10^{-6})}$$

Hence, for this coil of two turns the equivalent radius is

$$A_e = 10(1 - 20.84 \times 10^{-6})$$

(*d*)—The square coil shown is equivalent to two coils of the kind just considered, each displaced a distance 0.05 from the mean plane. Hence, for the galvanometer constant, the equivalent radius of this coil is $A_e = 10\ (1 - 20.84 \times 10^{-6})(1 + 37.5 \times 10^{-6}) = 10\ (1 + 16.66 \times 10^{-6})$.

In this case there is no doubt that the sectional dimensions of the coil are the dimensions of the large bounding square, and applying Lyle's method to this large square we find $A_e = 10\ (1 + 16.64 \times 10^{-6})$, sensibly the same as we found by considering the separate squares.

But we have seen that the effect of any square differs by not over about 4 in a million from the effect of a small round wire at its axis. Hence, this large square is what is to be understood when we speak of the section of a coil composed of 4 small wires, each coinciding with the center of one of these squares. This is exactly the conclusion at which we arrived in the early portion of this discussion.

Were we to take the section as determined by the axes of the bounding wires, this last case (*d*) would be equivalent to the first (*a*), and the equivalent radius found would be *too small* by 12.5 in a million.

Were we to regard the axial breadth of the coil as given by the breadth of the large square, and its depth as determined by the axes of the wires forming the upper and the lower layer, it would be equivalent to the second case (*b*) considered, and the equivalent radius found would be *too great* by 25 in a million.

Were we to take the radial depth as determined by the large square and the axial breadth as measured to the axes of the bounding wires, it would be equivalent to the third case (*c*), and the equivalent radius found would be *too small* by 37.5 in a million.

Hence, it is most important to choose the correct dimensions. They are given by the very simple expression $b = ns$, where b is the length of the side desired, n is the number of wires along this side, and s is the distance along this side between the axes of adjacent wires.

3. THE EFFECT OF ERRORS IN THE SECTIONAL DIMENSIONS OF THE COILS

In addition to introducing an error in the radius of a coil as deduced from its galvanometer constant, which effect has been fully considered on page 314, an error in the sectional dimensions of a coil will also introduce an error in the computed value of the force which this coil will exert upon a second coil, though the correct radii are used in the computation. It is this second correction which we desire to consider in this place.

Taking the series given by Maxwell[32] for the coefficient of mutual induction of two coaxial circles, and differentiating with respect to b, we find for the force between two coaxial circular currents the following expression, which was given by Lord Rayleigh.[33]

$$F=\frac{\pi^2A^2a^2}{C^4}\left\{1.2.3\frac{B}{C}+2.3.4\frac{\left(B^2-\frac{1}{4}A^2\right)b}{C^3}+3.4.5\frac{B\left(B^2-\frac{3}{4}A^2\right)\left(b^2-\frac{1}{4}a^2\right)}{C^5}\right.$$
$$+4.5.6\frac{\left(B^4-\frac{3}{2}B^2A^2+\frac{1}{8}A^4\right)\left(b^2-\frac{3}{4}a^2\right)b}{C^7}$$
$$\left.+5.6.7\frac{B\left(B^4-\frac{5}{2}B^2A^2+\frac{5}{8}A^4\right)\left(b^4-\frac{3}{2}b^2a^2+\frac{1}{8}a^4\right)}{C^9}+\ldots\ldots\right\}$$

where A and a are the radii of the two circles, B and b are the distances of their planes from some fixed point upon their common axis, and $C^2=A^2+B^2$.

If instead of linear circular circuits we are concerned with circular coils of axial breadths 2α and $2\alpha'$ and of radial depth 2ρ and $2\rho'$ (the accented letters referring to the smaller coil, assumed to be the one of radius a, and to which b refers), then the total force between the coils per unit current turn in each is given by the expression

$$F_t=\frac{1}{16\alpha\alpha'\rho\rho'}\int_{A-\rho}^{A+\rho}dA\int_{a-\rho'}^{a+\rho'}da\int_{B-\alpha}^{B+\alpha}dB\int_{b-\alpha'}^{b+\alpha'}F\,db$$

[32] Electricity and Magnetism, § 699.

[33] Phil. Trans., 175, pp. 411–460; 1884. Rayleigh, Scientific Paper, II, p. 282.

Performing this integration we shall obtain for the force an expression of the form

$$F_t = F + \lambda_1'\left(\frac{\rho}{A}\right)^2 + \lambda_2'\left(\frac{\alpha}{a}\right)^2 + \lambda_3'\left(\frac{\rho'}{a}\right)^2 + \lambda_4'\left(\frac{\alpha'}{a}\right)^2 + \ldots\ldots$$

whence by Maclaurin's theorem we find

$$\frac{\Delta F}{F} = \frac{F_t - F_{to}}{F_{to}} = \frac{1}{F_{to}}\left\{2\lambda_1'\left(\frac{\rho_o}{A}\right)^2\frac{\delta\rho}{\rho_o} + 2\lambda_2'\left(\frac{\alpha_o}{a}\right)^2\frac{\delta\alpha}{\alpha_o} + 2\lambda_3'\left(\frac{\rho_o'}{a}\right)^2\frac{\delta\rho'}{\rho_o'} \right.$$
$$\left. + 2\lambda_4'\left(\frac{\alpha_o'}{a}\right)^2\frac{\delta\alpha'}{\alpha_o'} + \ldots\ldots\right\}$$
$$= \lambda_1\left(\frac{\rho_o}{A}\right)^2\frac{\delta\rho}{\rho_o} + \lambda_2\left(\frac{\alpha_o}{a}\right)^2\frac{\delta\alpha}{\alpha_o} + \lambda_3\left(\frac{\rho_o'}{a}\right)^2\frac{\delta\rho'}{\rho_o'} + \lambda_4\left(\frac{\alpha_o'}{a}\right)^2\frac{\delta\alpha'}{\alpha_o} + \ldots\ldots\ldots$$

where F_{to} is the true value of F_t; ρ_o, α_o, ρ_o', α_o', are the true values of the half sectional dimensions, and F_t is the value of the force computed from the assumed dimensions $\rho = \rho_o + \delta\rho$, etc.

Though it is impracticable to calculate with sufficient accuracy F from the series given, or to calculate the coefficients λ_1', etc., so as to derive F_t from F, yet for small values of $\frac{\delta\rho}{\rho}$, etc., we can without undue labor determine λ_1, etc., with sufficient accuracy to determine the effects of slight errors in the dimensions and to calculate the amount by which the force exerted by a coil of rectangular and nearly square section differs from that exerted by one of strictly square section.

This procedure is perfectly straightforward and is capable of application, but the determination of the coefficients λ_1, etc., necessitates somewhat tedious integration, so that it is expedient to adopt the method used in Maxwell §700, in which the coefficients are determined by differentiations. There it is shown that the mutual induction between two circular coils of dimensions $2\rho_o$, $2\alpha_o$, $2\rho_o'$, $2\alpha_o'$ is given by the expression

$$M = G_1 g_1 P_1(\theta) + G_2 g_2 P_2(\theta) + \ldots\ldots$$

where

$$G_n = G_{no} + \frac{1}{6}\left(\rho_o^2\frac{\partial^2 G_{no}}{\partial A^2} + \alpha_o^2\frac{\partial^2 G_{no}}{\partial B^2}\right) + \frac{1}{120}\left(\rho_o^4\frac{\partial^4 G_{no}}{\partial A^4} + \alpha_o^4\frac{\partial^4 G_{no}}{\partial B^4}\right) + \ldots\ldots$$

and an exactly similar equation connects g_n with g_{no}. The functions G_{no} and g_{no} are the coefficients of the solid zonal harmonic $r^n P_n(\theta)$ or $\frac{P_n(\theta)}{r^{n+1}}$ in the expansions of the solid angles subtended by the mean turns of the two coils at the point from which B and b are measured; θ is the angle between the axes of the coils. Expanding these functions in terms of the radii and the distances, and introducing the conditions that $\theta = 0$, and that the origin is to be taken at the center of the moving coil, we find as before

$$F_{to} = \frac{\partial M}{\partial b} = F + \lambda_1'\left(\frac{\rho_o}{A}\right)^2 + \lambda_2'\left(\frac{\alpha_o}{a}\right)^2 + \lambda_3'\left(\frac{\rho_o'}{a}\right)^2 + \lambda_4'\left(\frac{\alpha_o'}{a}\right)^2$$

where the λ's are functions of the radii of the coils and of their distance apart.

Writing

$$f_o = 2 \cdot 3$$

$$f_1 = \frac{3 \cdot 5}{4} \cdot \frac{a^2(4B^2 - 3A^2)}{C^4}$$

$$f_2 = \frac{3 \cdot 5 \cdot 7}{4 \cdot 8} \cdot \frac{a^4(8B^4 - 20B^2A^2 + 5A^4}{C^8}$$

$$f_3 = \frac{5 \cdot 7 \cdot 9}{4 \cdot 8 \cdot 16} \cdot \frac{a^6(64B^6 - 336B^4A^2 + 280B^2A^4 - 35A^6)}{C^{12}}$$

the expressions for the various terms in F_{to} may be put in the form

$$F = \frac{\pi^2 a^2 A^2 B}{C^5}\left\{f_o - f_1 + f_2 - f_3 \ldots\ldots\right\}$$

$$\lambda_1' = \frac{1}{6} \cdot \frac{\pi^2 a^2 A^2 B}{C^5}\left\{\frac{2 \cdot 3(2B^4 - 21B^2A^2 + 12A^4)}{C^4} - \frac{3 \cdot 5a^2(8B^6 - 200B^4A^2 + 395B^2A^4 - 90A^6)}{4C^8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8} \cdot \frac{a^4(16B^8 - 728B^6A^2 + 3066B^4A^4 - 2345B^2A^6 + 280A^8)}{C^{12}} + \ldots\right\}$$

$$\lambda_2' = \frac{1}{6} \cdot \frac{\pi^2 a^2 A^2 B}{C^5}\left\{2 \cdot 4 f_1 - 4 \cdot 6 f_2 + 6 \cdot 8 f_3 \ldots\right\}$$

$$\lambda_3' = \frac{1}{6} \cdot \frac{\pi^2 a^2 A^2 B}{C^5}\left\{1 \cdot 2 f_1 - 3 \cdot 4 f_2 + 5 \cdot 6 f_3 \ldots\right\}$$

$$\lambda_4' = \frac{1}{6} \cdot \frac{\pi^2 a^2 A^2 B}{C^5}\left\{2 \cdot 4 f_1 - 4 \cdot 6 f_2 + 6 \cdot 8 f_3 \ldots\right\}$$

As required by symmetry, we find that $\lambda_2' = \lambda_4'$.

These coefficients being of zero dimensions are determined solely by the ratio of the radii, provided that the distance B is always so chosen as to make the force a maximum.

Multiplying these coefficients by $\frac{2}{F_{to}}$ we obtain the coefficients in the variation formula. The values of these coefficients are given in Table XXI.

TABLE XXI

Coefficients in the Expression

$$\frac{\Delta F}{F}=\lambda_1\left(\frac{\rho_1}{A}\right)^2\frac{\delta\rho_1}{\rho_1}+\lambda_2\left(\frac{\alpha_1}{a}\right)^2\frac{\delta\alpha_1}{\alpha_1}+\lambda_3\left(\frac{\rho_2}{a}\right)^2\frac{\delta\rho_2}{\rho_2}+\lambda_4\left(\frac{\alpha_2}{a}\right)^2\frac{\delta\alpha_2}{\alpha_2}$$

$\frac{A}{a}$	λ_1	$\lambda_2=\lambda_4$	λ_3
2	+2.523	−0.8478	+1.704
2.5	+1.954	−0.4361	+1.201

TABLE XXII

Reduction of the Computed Forces to the Forces for Fixed Coils of 25 cm Radius and 2 cm Square Section, or of 20 cm Radius and 1.6 cm Square Section

Coils	Dimensions of fixed coil			Computed force	Reduced force
	Radius	2ρ	2α		
M3 L1	25.03121	2.035	2.027	3.138051	3.147097
M3 L2	25.03056	2.054	2.027	3.138331	96
M3 L3	24.99767	1.943	1.969	3.147744	98
M3 L4	25.00247	1.925	1.965	3.146282	98
M4 L3	Same as above			5.317906	5.316741
M4 L4				5.315175	42
M2 L3				5.355558	5.354384
M2 L4				5.352799	75
M3 S1	19.97510	1.528	1.580	5.417100	5.400429
M3 S2	19.96611	1.522	1.579	5.423282	06

By means of these coefficients and those in Table X we can reduce the calculated forces (Table XI) to what they would be were the fixed coils, that are nominally equivalent, of exactly the same mean radius and of the same square section, thus obtaining a check upon the calculations. These reduced forces are given in Table XXII.

In the first four combinations, $M3\ L1$, $M3\ L2$, $M3\ L3$, and $M3\ L4$, the fixed coils are reduced to the same radius and cross section, hence the reduced forces should be the same. The variations are less than 1 in a million. The same reductions have been made for the other moving coils excepting that only two combinations can be compared. The maximum difference occurs in the comparison of the combinations $M3\ S1$, and $M3\ S2$, and amounts to 4 in a million in the force or 2 in a million in the resulting value obtained for the current. This may be due to a slight error of calculation or to the considerable range of the reduction.

4. RELATION BETWEEN THE OBSERVED AND THE TRUE MAXIMUM FORCE

In the body of this paper, p. 348, we have shown from physical considerations that the slope of the line representing the relation between the position of the moving coil and the difference in the forces exerted by the two fixed coils is a measure of the departure of the distance between the fixed coils from that which corresponds to the maximum force; that the observed difference in the forces for that position of the moving coil which corresponds to the axis of the parabola, which represents the relation between the sum of the forces and the position of the moving coil, is the value of the difference in the two true maximum forces; that the distance from the axis of this parabola to that point at which the tangent to the curve has the same slope as the line representing the difference in the forces is equal to one-half the amount by which the spacing of the fixed coils is in error; and that the value of the true maximum of the sum of the forces is as much above the vertex of the observed curve as the latter is above the point just mentioned. All of this is strictly true if the distances are so small that the variation in the force of either fixed coil alone can be regarded as strictly parabolic. It now remains to show mathematically that these statements are true and to see what modifications must be made if the distances are somewhat greater, so that the cube of the distance from the maximum must be considered.

Suppose that the fixed coils are very nearly identical, are coaxial with one another and with the moving coil, and are separated from one another by a greater distance than that corresponding to the maximum force for a given current. Let u and l denote quantities belonging to the upper and to the lower coil, respectively, let the dotted horizontal lines (Fig. 22) denote the

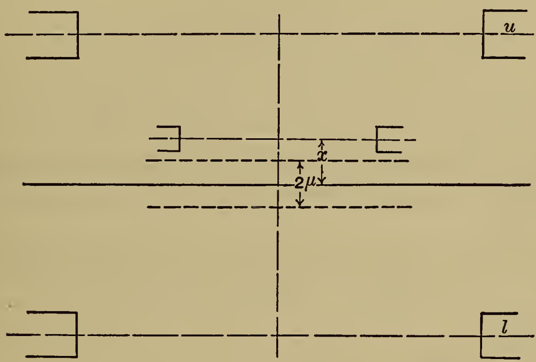

Fig. 22

positions of the moving coil for which the force exerted upon it by the respective fixed coils taken singly is a maximum. Denote the distance between these lines by 2μ. Let x denote the distance of the moving coil above the plane midway between these lines. Then, for any position of the moving coil the electromagnetic forces acting upon the moving coil will be

$$F_u = F_{uo}\{1 - \gamma(x-\mu)^2 - \delta(x-\mu)^3 \ldots\}$$

$$F_l = F_{lo}\{1 - \gamma(x+\mu)^2 + \delta(x+\mu)^3 \ldots\}$$

By hypothesis $F_{lo} - F_{uo}$ is a small quantity as compared with either force, and γ and δ will be essentially the same for both coils. Hence, the sum and the difference of the forces acting on the moving coil will be very approximately given by the expressions

$$\Sigma = F_u + F_l = \Sigma_o\{1 - \gamma(x^2+\mu^2) + \delta(3x^2\mu + \mu^3)\} + \varDelta_o\{2\gamma\mu x - \delta(x^3 + 3x\mu^2)\}$$

$$\varDelta = F_u - F_l = \varDelta_o\{1 - \gamma(x^2+\mu^2) + \delta(3x^2\mu + \mu^3)\} + \Sigma_o\{2\gamma\mu x - \delta(x^3 + 3x\mu^2)\}$$

where $\Sigma_o = F_{uo} + F_{lo}$ is the true maximum force, and $\varDelta_o$ is the difference between the two maxima.

Now γ and δ are small quantities (Table X, p. 330), μ can be made small, and x in this work never exceeded 2 or 3 mm. Hence, in practice, $\varDelta_o\{2\gamma\mu x - \delta(x^3 + 3x\mu^2)\}$ will be very small as compared with Σ_o; likewise $-\gamma(x^2+\mu^2)+\delta(3x^2\mu+\mu^3)$ is sufficiently small as compared with unity to be neglected where $\varDelta$ is concerned. Consequently, we have to a very high order of accuracy

$$\Sigma = \Sigma_o\{1 - \gamma(x^2+\mu^2) + \delta\mu(3x^2+\mu^2)\}$$

$$\varDelta = \varDelta_o + x\Sigma_o\{2\gamma\mu - \delta(x^2+3\mu^2)\}$$

Hence, the vertex of the (Σ, x) curve lies at $x = o$, and at this point $\varDelta = \varDelta_o$. That is, the value $\varDelta_o$ is the value of $\varDelta$ observed for that value of x which corresponds to the vertex of the observational (Σ, x) curve.

Also, if $\mu = o$ the $(\varDelta, x)$ curve will be a very flat cubic, the curvature being given by the very small coefficient δ.

Furthermore,

$$\frac{d\Sigma}{dx} = -\Sigma_o x(2\gamma - 6\delta\mu)$$

$$\frac{d\varDelta}{dx} = +\Sigma_o\mu\left(2\gamma - 3\delta\frac{\mu^2+x^2}{\mu}\right)$$

Hence, for $x = -\mu$, $\dfrac{d\Sigma}{dx} = \dfrac{d\varDelta}{dx}$.

That is, even when it is necessary to consider the cubic term, the conclusions stated at the beginning of this section are correct except for the fact that the $(\varDelta, x)$ curve is not a straight line, but a very flat cubic. The latter fact involves the necessity of determining experimentally the slope of the $(\varDelta, x)$ curve at that particular value of x for which this slope is the same as that for the (Σ, x) curve, if we desire to determine with extreme accuracy the error introduced by the error in the spacing of the coils.

This practically involves an experimental determination of the exact shape of the $(\varDelta, x)$ curve through the region under consideration. This is impracticable, but another very approximate method can be developed from the recognition of the fact that the values of Δ at the points $x = -x_1$, o, $+x_1$ all lie upon a straight line of which the slope is

$$S = \Sigma_o\{2\gamma\mu - \delta(x_1^2 + 3\mu^2)\}$$

But δ is always small as compared with γ, and x_1 can be made large as compared with μ, hence very approximately

$$\mu = \frac{S}{2\gamma\Sigma_o} + \frac{\delta x_1^2}{2\gamma}$$

All the quantities on the right being known, we can calculate μ and apply the proper correction to Σ. Or, by equating S to $\frac{d\Sigma}{dx}$ we find that the point at which the tangent to the (Σ, x) curve has the slope S is given by

$$x = -\mu + \frac{1}{2}\frac{\delta}{\gamma}\cdot\frac{x_1^2 - 3\mu^2}{1 - 3\frac{\delta}{\gamma}\mu}$$

$$= -\mu + \frac{1}{2}\frac{\delta}{\gamma}x_1^2 \qquad \text{approximately.}$$

This gives the best ocular estimate of the error in spacing.

Since in no case with which we are concerned does $\frac{\delta}{\gamma}$ exceed 0.1, and x_1 has never exceeded 0.3 cm, the absolute value of the second term on the right has never exceeded 0.045 mm. Hence, if μ is small, the method of procedure outlined for the simpler case will yield a sufficiently approximate value for the correction for the error in spacing. In the present work μ has always been very small, so that this approximate method has been of ample accuracy.

Since the force satisfies Laplace's equations, it will be a minimum with respect to horizontal displacements of the moving coil when the coils are properly adjusted. Hence, the above discussion applies in full to errors due to a noncoaxiality of the fixed coils, the planes of the latter being assumed horizontal. The only change being the multiplication of γ by the factor -0.5, and of δ by 0.0 (the values, for $\theta = 90°$, of the second and of the third zonal harmonics).

5. DIRECT MEASUREMENT OF THE COILS

As the coils were wound the mean diameter of each layer (of each third layer in the case of *L*3 and *L*4) was determined from the measurements of a number of diameters chosen so that their extremities were uniformly distributed around the circumference of the coil. These mean diameters, corrected to a common temperature, are given in Table XXIII and XXIV. The measurement of the individual diameters of the bottom (*B*) of the wire channel and of the top (*T*) of the outer layer are given in Table XXV. These numbers give an insight into the circularity of the forms and into the variations in the depth of the windings.

By referring to Table VIII, page 322, and comparing the direct and the electrical measurements of the radii, it will be seen that, though the direct measurements were made with great care, the accuracy attained is not sufficient to give an accuracy of even 1 in 10 000 in the current.

The only coils upon which such measurements were made as are capable of yielding information in regard to the compression of the forms by the wire wound upon them are the large coils *L*3 and *L*4 and the moving coil *M*4. Unfortunately, the finish upon the latter was accidentally abraded during the winding, so that the value found for its compression is too great by an unknown amount. On the other hand, forms *L*3 and *L*4 have each been wound twice, the first winding having been damaged by a slight leak from the water channel in the forms; so that we have two measurements for the compression of them. These measurements are given in Table XXVI and show that the forms of the large fixed coils may be so compressed as to shorten their mean diameters by about 0.07 mm, but it is probable that the actual compression will not exceed a half of this. As now wound, *L*3 and *L*4 are each compressed by the same amount—0.027 mm. It is believed that no coil has been wound under a greater tension than *L*3 when first wound, for then the wire fitted so tightly that it was difficult to draw into place the last turn of each layer. Also, *M*4 was wound under a greater tension than any of the other moving coils, and even then its compression is less than that corresponding to 0.022 mm in the diameter.

TABLE XXIII

Diameters of Fixed Coils (Direct Measurement, mm)

Layer N	L1(t=20°0) Micrometer setting S	L1 Δ	L2(t=25°0) Micrometer setting S	L2 Δ	L3(t=20°0) Micrometer setting S	L3 Δ	L4(t=20°0) Micrometer setting S	L4 Δ	S1(t=25°0) Micrometer setting S	S1 Δ	S2(t=25°0) Micrometer setting S	S2 Δ
0	−19.879	1.075	−20.091	1.113	−19.275	1.021	−19.221	1.073	−15.897	1.107	−15.904	1.093
1	−18.804	1.056	−18.978	1.152	−18.254	......	−18.150	......	−14.790	1.073	−14.811	1.083
2	−17.748	1.184	−17.826	1.176	....	1.047	....	1.073	−13.717	1.093	−13.728	1.071
3	−16.564	1.229	−16.650	1.127	....	......	....	......	−12.624	1.093	−12.657	1.064
4	−15.335	1.131	−15.523	1.140	−15.112	......	−14.932	1.053	−11.531	1.073	−11.593	1.042
5	−14.204	1.047	−14.383	1.122	....	1.056	....	......	−10.458	1.084	−10.551	1.075
6	−13.157	1.090	−13.261	1.148	....	......	....	......	− 9.374	1.091	− 9.476	1.050
7	−12.067	1.122	−12.113	1.149	−11.943	......	−11.774	......	− 8.283	1.100	− 8.426	1.112
8	−10.945	1.219	−10.964	1.165	....	1.083	....	1.051	− 7.183	1.111	− 7.314	1.088
9	− 9.726	1.113	− 9.799	1.125	....	......	....	......	− 6.072	1.104	− 6.226	1.162
10	− 8.613	1.179	− 8.674	1.142	− 8.693	......	− 8.622	......	− 4.968	1.098	− 5.064	1.086
11	− 7.434	1.042	− 7.532	1.306	....	1.060	....	1.080	− 3.870	1.098	− 3.978	1.107
12	− 6.392	1.171	− 6.226	1.169	....	......	....	......	− 2.772	1.136	− 2.871	1.160
13	− 5.221	1.057	− 5.057	1.155	− 5.513	......	− 5.381	......	− 1.636	1.212	− 1.711	1.067
14	− 4.164	1.325	− 3.902	1.144	....	1.073	....	1.064	− 0.424	1.089	− 0.644	1.068
15	− 2.839	0.984	− 2.758	1.160	....	......	....	......	+ 0.665	1.211	+ 0.424	1.124
16	− 1.855	1.197	− 1.598	1.118	− 2.293	......	− 2.189	1.068	+ 1.876	1.054	+ 1.548	1.050
17	− 0.658	1.122	− 0.480	1.111	....	1.057	....	......	+ 2.930	1.097	+ 2.598	1.079
18	+ 0.464	1.101	+ 0.631	1.166	....	......	....	......	+ 4.027	1.074	+ 3.677	1.080
19	+ 1.565	1.252	+ 1.797	1.095	....	......	+ 1.019	......	+ 5.101	1.049	+ 4.757	1.098
20	+ 2.817	1.012	+ 2.892	1.143	+ 1.936	1.063	....	1.075	+ 6.150	1.082	+ 5.855	1.064
21	+ 3.829	1.146	+ 4.035	1.155	....	......	....	......	+ 7.232	1.078	+ 6.919	1.107
22	+ 4.975	1.059	+ 5.190	1.140	+ 4.062	......	+ 4.243	......	+ 8.310	1.067	+ 8.026	1.109
23	+ 6.034	1.086	+ 6.330	1.125	....	1.129	....	1.077	+ 9.377	1.056	+ 9.135	1.021
24	+ 7.120	1.104	+ 7.455	1.093	....	......	....	......	+10.433	1.067	+10.156	1.096
25	+ 8.224	1.239	+ 8.548	1.120	+ 7.449	......	+ 7.473	......	+11.500	1.044	+11.252	1.072
26	+ 9.463	1.134	+ 9.668	1.083	....	1.159	....	1.087	+12.544	1.082	+12.324	1.053
27	+10.597	1.130	+10.751	1.126	....	......	....	......	+13.626	1.046	+13.377	1.155
28	+11.727	1.108	+11.877	1.126	+10.925	......	+10.733	......	+14.672	......	+14.532	......
29	+12.835	1.251	+13.003	1.118	....	1.054	....	1.089	....	......	....	......
30	+14.086	0.967	+14.121	1.119	....	......	....	......	....	......	....	......
31	+15.053	1.096	+15.240	1.190	+14.087	......	+13.999	......	....	......	....	......
32	+16.149	1.054	+16.430	1.139	....	1.073	....	1.050	....	......	....	......
33	+17.203	1.104	+17.569	1.112	....	......	....	......	....	......	....	......
34	+18.307	1.127	+18.681	1.148	+17.305	1.093	+17.149	1.050	....	......	....	......
35	+19.434	1.337	+19.829	1.136	....	......	....	......	....	......	....	......
36	+20.771		+20.965		+19.492		+19.248		....		....	
Mean	$+0.973_0$		$+1.091_3$		$+1.034_3$		$+0.985_8$		$+0.026_5$		-0.159_6	

TABLE XXIII—Continued

	L1(t=20°0)	L2(t=25°0)	L3(t=20°0)	L4(t=20°0)	S1(t=25°0)	S2(t=25°0)
Mean, 1 to n......	$+0.973_0$	$+1.091_3$	$+1.034_3$	$+0.985_8$	$+0.026_5$	-0.159_6
—wire..............	-0.587_5	-0.556_5	−1.021	-0.990_5	-0.546_5	-0.545_5
End standard......	500.028	500.056	500.028	500.028	400.042	400.042
Cor to 22°00........	+0.018	−0.027	+0.019	+0.019	-0.020_4	-0.021_1
$D_{22°00}$............	500.432	500.564	500.060	500.042	399.502	399.316
$A_{22°00}$............	25.0216 cm	25.0282 cm	25.0030 cm	25.0021 cm	19.9751 cm	19.9658_0 cm

N=The number of the layer on top of which the diameter is measured.

N=0 is the diameter of the bottom of the wire channel.

S=Diameter—End Standard, in mm and at the temperature t.

L1, L2, L3, L4, S1, S2, M1, M2, M3, M4, are the coils.

Δ=Difference between consecutive diameters.

Mean 1 to n=Mean of all measurements except the first.

Wire=Amount by which "Mean 1 to n" exceeds the mean diameter if uniformly wound.

=Diameter of wire if all diameters are measured.

=2 diameter of wire in case of L3.

$=\frac{23}{14}$ diameter of wire in case of L4.

A22°00=Mean radius at 22°00.

TABLE XXIV

Diameters of Moving Coils (Direct Measurement, mm)

Layer N	M1 (t=20°0) Micrometer setting S	M1 Δ	M2 (t=25°00) Micrometer setting S	M2 Δ	M3 (t=23°50) Micrometer setting S	M3 Δ	M4 (t=25°05) Micrometer setting S	M4 Δ
0	−9.808	1.560	−9.539	1.526	− 9.592	1.394	−11.351	1.820
1	−8.248	1.498	−8.013	1.565	− 8.198	1.425	− 9.531	1.770
2	−6.750	1.511	−6.448	1.636	− 6.773	1.457	− 7.761	1.900
3	−5.239	1.706	−4.812	1.585	− 5.316	1.470	− 5.861	1.710
4	−3.533	1.506	−3.227	1.598	− 3.846	1.446	− 4.151	1.770
5	−2.027	1.494	−1.629	1.594	− 2.400	1.433	− 2.381	1.820
6	−0.533	1.455	−0.035	1.606	− 0.967	1.445	− 0.561	1.765
7	+0.922	1.384	+1.571	1.568	+ 0.478	1.461	+ 1.204	1.685
8	+2.306	1.590	+3.139	1.593	+ 1.939	1.510	+ 2.889	1.970
9	+3.896	1.494	+4.732	1.574	+ 3.449	1.543	+ 4.859	1.755
10	+5.390	1.514	+6.306	1.605	+ 4.992	1.567	+ 6.614	1.655
11	+6.904	1.465	+7.911	1.591	+ 6.559	1.450	+ 8.269	1.778
12	+8.369	....	+9.502	....	+ 8.009	1.483	+10.062	....
13	....	....	....	....	+ 9.492	1.435	....	....
14	....		....		+10.927		....	
Mean 1 to n	$+0.121_4$		$+0.749_8$		$+1.310_4$		$+0.304_2$	
—Wire	-0.765_0		-0.768_5		-0.701_0		-0.875_0	
End standard	200.004		250.011		200.012		250.011	
Cor. to 22°00	$+0.007_6$		-0.015_2		-0.005_9		-0.014_5	
$D_{22°00}$	199.368		249.977_1		200.616		249.426	
$A_{22°00}$	9.9684 cm		12.4988_6 cm		10.0308 cm		12.4713 cm	

TABLE XXV

Circularity of the Coils and Variations in the Depth of the Windings (Diameter Minus End Standard; for Bottom of Wire Channel and for Top of Outer Layer of Wire)

MOVING COILS

Coil M1			Coil M2			Coil M3			Coil M4		
Bottom	Top	T−B	Bottom	Top	T−B	Bottom	Top	T−B	Bottom	Top	T−B
mm	mm	mm	mm	mm	mm	mm	mm	mm	mm	mm	mm
−9.790	+8.976	18.766	−9.559	+9.566	19.125	−9.591	+10.944	20.535	−11.35	+10.09	21.44
			−9.527	+9.456	18.983	−9.607	+10.947	20.554	−11.39	+10.04	21.43
−9.845	+8.070	17.915	−9.524	+9.492	19.016	−9.586	+10.893	20.479	−11.38	+10.05	21.43
			−9.562	+9.555	19.117	−9.581	+10.924	20.505	−11.33	+10.03	21.36
−9.788	+8.062	17.850	−9.528	+9.472	19.000	−9.599	+10.938	20.537	−11.32	+10.03	21.35
			−9.535	+9.467	19.002	−9.588	+10.917	20.505	−11.33	+10.14	21.47
Mean	....	18.177	....	....	19.040	....	....	20.519	....	....	21.41_3

FIXED COILS

Coil S1			Coil S2			Coil L1			Coil L2		
Bottom	Top	T−B	Bottom	Top	T−B	Bottom	Top	T−B	Bottom	Top	T−B
mm	mm	mm	mm	mm	mm	mm	mm	mm	mm	mm	mm
−15.905	+14.680	30.585	−15.937	+14.525	30.462	−19.877	+20.715	40.592	−20.102	+20.945	41.047
−15.899	+14.651	30.550	−15.907	+14.550	30.457	−19.887	+20.867	40.754	−20.083	+20.911	40.994
−15.886	+14.679	30.565	−15.860	+14.556	30.416	−19.879	+20.882	40.761	−20.076	+20.951	41.027
−15.898	+14.678	30.576	−15.913	+14.498	30.411	−19.873	+20.698	40.571	−20.095	+21.059	41.154
						−19.877	+20.768	40.645	−20.098	+20.958	41.056
Mean	....	30.569	....	....	30.436	....	....	40.665[34]	....	....	41.056

[34] This value differs by 15μ from the value given in Table XXII owing to the latter value being based on a greater number of top measurements than is given here.

TABLE XXV—Continued

Coil L3			Coil L4		
Bottom	**Top**	**T−B**	**Bottom**	**Top**	**T−B**
mm	mm	mm	mm	mm	mm
−19.327	+19.577	38.904	−19.150	+19.398	38.548
−19.310	+19.458	38.768	−19.169	+19.268	38.437
−19.261	+19.560	38.821	−19.080	+19.518	38.598
−19.368	+19.697	39.065	−19.290	+19.366	38.656
−19.327	+19.435	38.762	−19.292	+19.366	38.658
−19.180	+19.535	38.715	−19.261	+19.098	38.359
−19.140	+19.595	38.735	−19.248	+19.160	38.408
−19.213	+19.393	38.606	−19.186	+19.162	38.348
−19.262	+19.621	38.883	−19.204	+19.248	38.452
−19.306	+19.631	38.937	−19.266	+19.101	38.367
−19.286	+19.494	38.780	−19.204	+19.094	38.298
−19.238	+19.481	38.719	−19.302	+19.040	38.342
−19.158	+19.374	38.532	−19.165	+19.260	38.425
−19.300	+19.313	38.613	−19.199	+19.402	38.601
−19.343	+19.193	38.536	−19.294	+19.239	38.533
−19.327	+19.278	38.605	−19.232	+19.386	38.618
−19.279	+19.730	39.009	−19.260	+19.106	38.366
−19.355	+19.485	38.840	−19.210	+19.220	38.430
Mean........	....	38.768	....	....	38.469

Since the forms for $L3$ and $L4$ are built up of rolled brass, it was feared that the pressure of the wire might cause the sides of the channel to spread. Consequently, the mean thicknesses of the forms were measured both before and after winding. The observations were so combined as to give the mean thicknesses on the three circles α, β, γ, Fig. 23. The results in millimeters are given in Table XXVII, and show that there is no appreciable spreading except in the case of the first winding of $L3$, which, as stated, was very tight. The other fixed coils have forms of solid cast brass of about the same section, and consequently the spreading for them, though not measured, must be less than for $L3$ and $L4$.

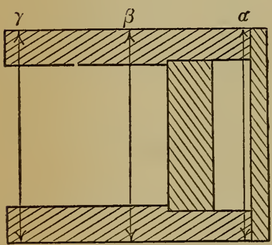

Fig. 23

Owing to the tapering of the sides, no accurate measurement of the spreading of the sides of the channels of the moving coils has been obtained. It is, however, believed to be negligible, as there are relatively few layers on them, and the forms have been designed to secure stiffness.

TABLE XXVI

Compression of Forms (Diameter of Faces of the Forms on which the Wire was Wound)

	Coil L3		Coil L4		Coil M4
	Top	Bottom	Top	Bottom	Top
1st winding	cm	cm	cm	cm	cm
Start	52.4264	52.4238	52.4226	52.4254	26.2140
Finish	52.4201	52.4172	52.4201	52.4220	26.2118
Compression	0.0063	0.0066	0.0025	0.0034	0.0022
2d winding					
Start	52.4256	52.4230	52.4230	52.4256	....
Finish	52.4229	52.4203	52.4203	52.4228	....
Compression	0.0027	0.0027	0.0027	0.0028	....

TABLE XXVII

Spreading of the Sides of the Channel (Thickness of Brass Form in mm at the points α, β, γ, Fig. 23)

	First winding			Second winding		
	α	β	γ	α	β	γ
Coil L3						
Before	30.037	30.052	30.063	30.038	30.052	30.065
After	30.040	30.057	30.080	30.037	30.049	30.062
Spreading	0.003	0.005	0.017	−0.001	−0.003	−0.003
Coil L4						
Before	30.059	30.073	30.081	30.058	30.072	30.080
After	30.057	30.072	30.087	30.059	30.072	30.086
Spreading	−0.002	−0.001	+0.006	+0.001	0.000	+0.006

The effective radial depth of the windings is shown (p. 375) to be ns where n is the number of layers and s is the mean distance between adjacent layers as measured to the axes of the wires.

If during the winding the diameter of the form is decreased by an amount c, and if the diameter of the wire is d, then $s = \frac{(T-B)+c-2d}{2(n-1)}$ where $(T-B)$ is the difference between the diameter measured to the outside of the outer layer, and the diameter of the bottom of the coil channel. The values of $(T-B)$ can be obtained from the tables just given. The computation of the radial depths (2ρ) is given in Table XXVIII.

TABLE XXVIII

Effective Radial Depths of the Windings

	M1	M2	M3	M4	S1
n	12	12	14	12	28
T−B in mm	18.177	19.041	20.519	21.413	30.569
c in mm	....	....	....	0.022	....
2d in mm	1.530	1.537	1.402	1.750	1.093
2 (n−1) s in mm	16.647	17.504	19.117	19.685	29.476
2ρ in cm	0.9080	0.9548	1.0294	1.0737	1.5284

	S2	L1	L2	L3	L4
n	28	36	36	36	36
T−B in mm	30.436	40.650	41.056	38.767	38.469
c in mm	....	....	....	0.027	0.027
2d in mm	1.091	1.075	1.113	1.021	1.073
2 (n−1) s in mm	29.345	39.575	39.943	37.773	37.423
2ρ in cm	1.5216	2.0353	2.0542	1.9426	1.9246

The effective axial breadth of the windings is shown (p. 375) to be ns where n is the number of spires in one layer and s is the axial distance of adjacent spires; this will be the breadth of the channel if the wires are uniformly spaced and the axes of the terminal spires are as far from the sides of the channel as half the distance between the axes of adjacent wires. If the wires are equally spaced, but the terminal spires touch the sides of the channel, then the effective breadth ns becomes $\frac{n}{n-1}(a-d)$ where a is the breadth of the channel and d is the diameter of the wire. These values are given in Table XXIX. It will be seen that the difference between these values is very small, only a few hundredths of a millimeter. For reasons stated in the body of the paper, we believe the conditions are such that *a* is the correct value, except for *M*4, and this is the value we have used in the computations

TABLE XXIX

Effective Axial Breadth of the Windings, in Centimeters

Coils	M1	M2	M3	M4	S1
n	12	12	14	12	28
a	0.9850	0.9564	0.9967	1.0877	1.5803
d	0.0765	0.0768	0.0701	0.0875	0.0546
$\frac{n}{n-1}(a-d)$	0.991_1	0.959_6	0.997_9	1.091_1	1.582_2
$\frac{n}{n-1}(a-d)-a$	0.006_1	0.003_2	0.001_2	0.003_4	0.001_9

Coils	S2	L1	L2	L3	L4
n	28	36	36	35.97	35.97
a	1.5787	2.0267	2.027	1.969	1.965
d	0.0546	0.0538	0.0556	0.051	0.0536
$\frac{n}{n-1}(a-d)$	1.580_5	2.029_3	2.027_7	1.972_8	1.966_2
$\frac{n}{n-1}(a-d)-a$	0.001_8	0.002_6	0.000_7	0.003_8	0.001_2

6. TABLE OF LOG SIN $\gamma\{2F_\gamma-(1+SEC^2\gamma)E_\gamma\}$

TABLE XXX

Table of the logarithms to the base ten of sin $\gamma\{2F_\gamma-(1+\sec^2\gamma)E_\gamma\}$ from $\gamma = 55°$ to $\gamma = 70°$. Compiled by means of the Tables of Legendre and of Vega

γ	Log	Δ₁	Δ₂	γ	Log	Δ₁	Δ₂
°				°			
55.0	$\bar{1}$.91989049	517780	−117	59.0	0.12673397	518270	157
.1	.92506829	517663	111	.1	.13191667	518427	164
.2	.93024492	517552	106	.2	.13710094	518591	173
.3	.93542044	517446	97	.3	.14228685	518764	180
.4	.94059490	517349	93	.4	.14747449	518944	187
55.5	.94576839	517256	85	59.5	.15266393	519131	196
.6	.95094095	517171	80	.6	.15785524	519327	203
.7	.95611266	517091	72	.7	.16304851	519530	211
.8	.96128357	517019	66	.8	.16824381	519741	219
.9	.96645376	516953	60	.9	.17344122	519960	226
56.0	.97162329	516893	53	60.0	.17864082	520186	236
.1	.97679222	516840	47	.1	.18384268	520422	243
.2	.98196062	516793	40	.2	.18904690	520665	252
.3	.98712855	516753	33	.3	.19425355	520917	258
.4	.99229608	516720	27	.4	.19946272	521175	268
56.5	.99746328	516693	20	60.5	.20467447	521443	276
.6	0.00263021	516673	14	.6	.20988890	521719	284
.7	.00779694	516659	6	.7	.21510609	522003	293
.8	.01296353	516653	± 0	.8	.22032612	522296	301
.9	.01813006	516653	+ 6	.9	.22554908	522597	309
57.0	.02329659	516659	14	61.0	.23077505	522906	319
.1	.02846318	516673	21	.1	.23600411	523225	327
.2	.03362991	516694	27	.2	.24123636	523552	335
.3	.03879685	516721	35	.3	.24647188	523887	345
.4	.04396406	516756	40	.4	.25171075	524232	352
57.5	.04913162	516796	49	61.5	.25695307	524584	363
.6	.05429958	516845	55	.6	.26219891	524947	371
.7	.05946803	516900	62	.7	.26744838	525318	381
.8	.06463703	516962	69	.8	.27270156	525699	388
.9	.06980665	517031	76	.9	.27795855	526087	399
58.0	.07497696	517107	84	62.0	.28321942	526486	408
.1	.08014803	517191	91	.1	.28848428	526894	417
.2	.08531994	517282	98	.2	.29375322	527311	426
.3	.09049276	517380	104	.3	.29902633	527737	436
.4	.09566656	517484	113	.4	.30430370	528173	446
58.5	.10084140	517597	120	62.5	.30958543	528619	455
.6	.10601737	517717	127	.6	.31487162	529074	465
.7	.11119454	517844	135	.7	.32016236	529539	475
.8	.11637298	517979	141	.8	.32545775	530014	484
.9	.12155277	518120	150	.9	.33075789	530498	495

TABLE XXX—Continued

γ	Log	Δ_1	Δ_2	γ	Log	Δ_1	Δ_2
°				°			
63.0	0.33606287	530993	505	66.5	0.52563052	555378	924
.1	.34137280	531498	515	.6	.53118430	556302	940
.2	.34668778	532013	525	.7	.53674732	557242	955
.3	.35200791	532538	535	.8	.54231974	558197	968
.4	.35733329	533073	546	.9	.54790171	559165	984
63.5	.36266402	533619	556	67.0	.55349336	560149	999
.6	.36800021	534175	568	.1	.55909485	561148	1015
.7	.37334196	534743	577	.2	.56470633	562163	1029
.8	.37868939	535320	589	.3	.57032796	563192	1046
.9	.38404259	535909	599	.4	.57595988	564238	1061
64.0	.38940168	536508	610	67.5	.58160226	565299	1077
.1	.39476676	537118	623	.6	.58725525	566376	1094
.2	.40013794	537741	632	.7	.59291901	567470	1110
.3	.40551535	538373	644	.8	.59859371	568580	1126
.4	.41089908	539017	656	.9	.60427951	569706	1144
64.5	.41628925	539673	667	68.0	.60997657	570850	1160
.6	.42168598	540340	679	.1	.61568507	572010	1178
.7	.42708938	541019	691	.2	.62140517	573188	1195
.8	.43249957	541710	702	.3	.62713705	574383	1213
.9	.43791667	542412	715	.4	.63288088	575596	1231
65.0	.44334079	543127	726	68.5	.63863684	576827	1249
.1	.44877206	543853	740	.6	.64440511	578076	1268
.2	.45421059	544593	751	.7	.65018587	579344	1285
.3	.45965652	545344	764	.8	.65597931	580629	1305
.4	.46510996	546108	776	.9	.66178560	581934	1325
65.5	.47057104	546884	790	69.0	.66760494	583259	1343
.6	.47603988	547674	803	.1	.67343753	584602	1362
.7	.48151662	548477	814	.2	.67928355	585964	1384
.8	.48700139	549291	829	.3	.68514319	587348	1404
.9	.49249430	550120	842	.4	.69101667	588752	1423
66.0	.49799550	550962	856	69.5	.69690419	590175	1444
.1	.50350512	551818	869	.6	.70280594	591619	1466
.2	.50902330	552687	882	.7	.70872213	593085	1486
.3	.51455017	553569	897	.8	.71465298	594571	1509
.4	.52008586	554466	912	.9	.72059869	596080	1531
				70.0	.72655949	597611	1552

WASHINGTON, September 9, 1911.