THE LUMINOUS EQUIVALENT OF RADIATION.

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I. INTRODUCTION.

1. DISCUSSION OF THE GENERAL PROBLEM.

Light has yet to be accurately defined in precise mathematical terms. The word is applied sometimes to the subjective sensation, sometimes to the objective stimulus causing or capable of causing the sensation. While the precise definition of light, like the defini-

tion of heat or of temperature, is more or less arbitrary, the relation of light to radiation involves primarily the precisely definable concept of the luminous equivalent of radiation. The general problem is to determine how much light and of what quality is represented by a given amount of radiation having a given spectral distribution.

But, while the heat equivalent of mechanical energy is a fixed quantity, the luminous equivalent of radiation varies enormously with the wave length of the radiation concerned and to a slight extent with its intensity as well. The principle of the conservation of energy does not apply to light, either objective or subjective. Light is not a simple multiple of radiation, but a complicated function of its quality, quantity, and duration. This article is devoted to the formulation of these complex relations.

The general problem of the luminous equivalent of radiation has been within the range of investigators for decades if not for centuries, but has been as a whole practically untouched, partly because it does not lend itself to treatment by infinitesimal analysis based on the conservation of energy, but largely no doubt because of its variability and extra-physical nature. Within the last few years many special problems in photometry, pyrometry, and spectroscopy have arisen which insistently demand the establishment of some definite relation between light and radiation.

Data are available of sufficient accuracy and diversity to determine the character of the functions involved and to determine their form to within a slight uncertainty, but not to fix the values of their parameters. This paper deals chiefly with the essential interrelations between functions rather than with their precise determination. If very exact data, constant for different individuals at all times, were available, how might that data be used in determining the luminous equivalent of radiation? That is the problem dealt with in this paper.

Visual sensibility of various kinds is subject to considerable variations with the individual, with fatigue, attention, and expectation, so that special problems will often require its redetermination by the individual at the time he is working. Hence the value of a foreknowledge of short cuts and pitfalls in this field. On the other hand, many general problems in illumination and luminous effi-

ciency require the specification of the mean properties of the average normal human eye. Existing data are sufficient for outlining such specifications.

Radiation, as now known and studied, includes a wide diversity of wave lengths, ranging from kilometers to ten-millionths of a meter. It is interpreted by many different classes of instruments—electric wave resonators, radiometers of various types, and by the eye and the photographic plate; some are absorbers, some resonators, and others act by mechanical pressure. None of these instruments is equally sensitive over the whole range of wave lengths; thermal receivers have the widest range, the eye and electric-wave resonators the narrowest. Some give indications very nearly proportional to the intensity of the radiation, while others depart widely from proportionality. Resonating receivers are very sensitive to differences in wave length, while thermal. mechanical. and chemical receivers are not.

Light is the interpretation given to radiation by the eye, just as the electric current, mechanical pressure, heat, and chemical effects are interpretations by other forms or radiometers. In any case, the interpretation of radiation by any kind of radiometer depends upon the properties of that form of radiometer as a physical instrument. The amount of light in a given quantity of radiation of a given quality is determined by the sensibility of the eye, sensibility to the same amounts of radiation of different wave lengths, and sensibility to radiation of constant wave length but varying intensities.

In other words, in order to express light in terms of mechanical units or radiation in terms of light units, three relations must be known in mathematical form: (1) The spectral distribution of the radiation in question, (2) the sensibility of the eye to radiation in different spectral regions, and (3) the sensibility of the eye to variations in the intensity of the radiation; that is, if $E(\lambda)$ is the radiation E expressed as a function of wave length λ, then $V(\lambda)$ is a function that converts this radiation $E(\lambda)$ into light $L(\lambda)$, or, in other words, that weights the radiation according to its visibility. Then, having obtained $L(\lambda)$, the sensation of visual brightness produced by the radiation E or the light L is some function $B(L)$ of the light L. The chief objective points of this investigation are the determination of V and B and the correlation of E, L, and B.

2. OUTLINE OF TREATMENT.

Given the sensibility of a physical instrument at all parts of its scale or for all values of the exciting stimulus, the scale reading corresponding to each value of the stimulus may be determined. Hence, by the aid of the general theory of sensibility of physical instruments, light may be expressed in terms of radiation when the sensibility of the eye is known.

Another method of attacking the problem is that of synthetic function theory. The sensation of visual brightness B is a function $B(E,\lambda,t)$ of the intensity of radiation $E(\lambda)$, the wave length λ and the duration of the impression t. By varying λ alone or t alone or E alone, three separate functions of these single variables are constructed and then these three functions may be combined to form the complete function.

A third method is that of setting up an arbitrary rational light scale and determining its relation to the radiation scale of watts per unit wave length. It is necessary to identify the zero of each scale on the other, to determine the relative magnitudes of scale divisions and ultimate values approached on each scale. Because the luminous sensation is not proportional to illumination (Purkinje phenomenon), over a considerable range, it is necessary to construct an auxiliary intermediate scale. In all, three scales, each in two dimensions, are involved.

Available data may be thus summarized: $E(\lambda)$ for B constant and $P(L)$ for λ constant. In other words, the relative amounts of radiation of different wave lengths necessary to produce the same sensation of brightness and the least perceptible increment P as a function of intensity at constant wave length. Besides these there are $E_o(\lambda)$, the threshold value or least perceptible intensity measured as radiation, and $L_o(\lambda)$, the same quantity measured as light. To determine the time function there are rough direct determinations of $B(t)$ and more precise measurements of the critical frequency for no flicker, each for various wave lengths. From these sets of data the general function $B(E(\lambda), \lambda, t)$ is to be constructed. Of related interest but of little assistance are some careful measurements of $\delta\lambda(\lambda)$, the least perceptible increment of wave length as a function of wave length.

3. SUMMARY OF THE PROPERTIES OF THE EYE AS A PHYSICAL INSTRUMENT.

The more important characteristics of the visual response to radiation may be thus summarized:

a. **Sensibility to Radiation of Various Wave Lengths.**—(*1*) The eye responds to radiation between ill-defined limits at about 300 and 1000 $\mu\mu$. Its sensibility is highest between 500 and 600 $\mu\mu$. Good seeing requires radiation between 410 and 760 $\mu\mu$. Radiation is easily visible to most eyes out as far as 330 $\mu\mu$ in the violet and 770 $\mu\mu$ in the red.

(*2*) The spectral sensibility curve has a single maximum in the green, slopes off very steeply at first and then very gradually toward the extreme wave lengths.

(*3*) At low intensities the maximum of the curve lies between 500 and 520 $\mu\mu$ for perhaps 90 per cent of all persons. It is approximately symmetrical and nearly or quite independent of color blindness, partial or complete. It is coincident with the reciprocal of the threshold value of the radiation if reduced to the same maximum ordinate.

(*4*) At moderate and high intensities the maximum of the visibility curve broadens and shifts slightly toward the yellow, varying considerably with color blindness in the subject.

b. **Sensibility to Radiation of Varying Intensity.**—Sensibility falls off steadily with increasing intensity. It is approximately inversely proportional to the intensity over a wide range. The ratio of optical intensity to intensity of radiation increases more rapidly for red than for blue and green. (Purkinje phenomenon.)

c. **Sensibility to Small Differences in Intensity.**—The least perceptible increment, measured as a fraction of the whole, is approximately—

(*1*) Independent of intensity (Fechner's Law). It is about 0.016 for moderate and high intensities and greater for very low and extremely high intensities.

(*2*) Independent of wave length (König's Law), at constant luminosity—extremes again excepted.

(*3*) Independent of the individual.

d. **Sensibility to Slight Differences in Wave Length** has two pronounced maxima, one in the yellow and one in the green, and two slight maxima in the extreme blue and red. These maxima vary considerably with the individual and probably also with the intensity of the radiation used.

e. **Visual acuity** or resolving power, so far as studied, appears to follow the same laws as does sensibility to small intensity differences, (3) namely, it is approximately proportional to the luminosity and independent of color and of the individual.

f. The **growth and decay** of the visual responses **with time**, so far as studied, appear to follow the ordinary exponential law. The parameters of the time functions vary with wave length and intensity. A steady impression is the resultant of a pure reception and a fatigue.

4. GENERAL THEORY OF SENSIBILITY OF PHYSICAL INSTRUMENTS.

A physical instrument, responsive to stimulus, indicates in general the quality, intensity, duration, or extension of that stimulus, and in some cases two or more of these properties at once. Simple instruments like chronometers and cathetometers indicate but a single property; most psycho-physical receivers indicate all four simultaneously. An ideal physical instrument is entirely free from subjective variations (pressure, temperature, attention, experience, fatigue, etc.) affecting its constants.

Let the quality, intensity, duration, and extension of any stimulus be measured in the variables q, i, t, v, respectively; let S_q, S_i, S_t, and S_v be the sensibilities of the instrument for these four properties of the stimulus and let R be the response or scale reading indicated by the instrument when subject to stimulus of quality q, intensity i, for a time t or over a space v, as the case may be. Then these nine variables are related by the defining equations:

$$S_q = K_q \left[\frac{R(q, i)}{i(q)}\right]_q^{q+dq} \quad \text{hence } R(q, i) = \frac{S_q}{K_q}\, i(q)$$

$$S_i = K_i \frac{\partial R}{\partial i} \qquad \text{"} \quad R(i) = \frac{1}{K_i}\int S(i)\, di + \text{const.}$$

$$S_t = K_t \frac{\partial R}{\partial t} \qquad \text{"} \quad R(t) = \frac{1}{K_t}\int S(t)\, dt + \text{const.}$$

$$S_v = K_v \frac{\partial R}{\partial v} \qquad \text{hence } R(v) = \frac{1}{K_v}\int S(v)\,dv + \text{const.}$$

K is a constant of proportionality dependent upon the unit of sensibility or scale magnification (or both) chosen. The integration constants depend upon the choice of scale zero.

In the special case of a uniform scale, S is constant and hence R is a linear function of the stimulus. When S is a linear function of R, R is a quadratic function of the stimulus. If a scale has been calibrated and is free from corrections, K is unity, S is unity, and the integration constant zero.

In general i is a function of q, otherwise all these relations would be exceedingly simple. Either S or R is determined experimentally as a function of one variable at a time and the other calculated. Sensibility to quality S_q is not to be confused with sensibility to differences in quality, its derivative. S_i is inversely proportional to the least perceptible difference in intensity. An instrument usually does not begin to respond until the stimulus has attained a finite value (i_o) called the threshold value, which is a function of q. The least perceptible increment is also related to the uncertainty of the readings due to subjective errors, but the experimental determination of S_i from the probability curve of errors is too laborious to be of much service except as a last resort.

Evidently S_q and S_i can be correctly determined only by null methods. In finding S_q, for instance, the scale reading R is held constant, while i and q are varied. If i were held constant and R varied, it would be impossible to distinguish between the two variations $R(q)$ and $R(i)$; much data on visual sensibility is worthless for this very reason.

5. GENERAL METHODS OF SYNTHETIC FUNCTION THEORY.

Instead of treating the eye as a physical instrument to find the relation of light to radiation, the necessary functions may be constructed directly from observed data. The problem is simpler in plan but more difficult to carry out by the methods of synthetic function theory. The magnitude (B) of the visual sensation is a function of the three objective variables, wave length (λ), spectral energy $E(\lambda)$, and time of action (t) of the radiation. We are

required to find $B(\lambda, E(\lambda), t)$ not by approximation to numerical data but from the general properties (as regards maxima, minima, and singular points) of the functions. This function $B(\lambda, E, t)$, like a potential function, is useful not so much for itself as a whole as for its derivatives, invariants, and subsidiary functions.

The general method of constructing functions is as follows: Given that the function required is finite and single valued (as nearly all physical functions are), for all finite positive values of the argument, we set up the most general function possessing these properties. Given further that the function has no finite maxima or a single maximum or a given number of given maxima, the general function is limited in accordance with this condition. Then if the function be known to approach certain limiting values for very high or very low values of the argument, the general function may be still further restricted in form. Finally, if the properties of the required function be sufficiently well known, there remains but to determine its parameters from numerical data in a limited region.

At any stage of construction the simplest possible instead of the most general function may be chosen as a special working basis, much as a particular solution of a differential equation may be chosen as a working solution. But it must be borne in mind that in passing from the general to the special forms we are passing from necessary to sufficient functions.

Such a process finally gives us a function satisfying certain general conditions and fitting numerical data to within experimental uncertainty. Unlike a mere power series fitted to numerical data, such a function may be used with care even outside the range of the experimental data which it fits and may even be integrated and differentiated without introducing any large additional uncertainty.

Consider two functions of x, $A(x)$ and $B(x)$ which differ by no more than a certain small amount within a given region, from $x=x_1$ to $x=x_2$ say, then the definite and indefinite integrals as well as the derivatives of A and B will differ by no more than a small quantity of the same order within that same region. Hence, even when the function constructed is merely an approximation to the actual physical relation sought, it may be used in all ordinary mathematical operations without introducing any large additional uncertainty. The chief liability to serious error lies in the interpretation

of parameters as functions of other arguments and operations on parameters in general.

After separate functions have been constructed for each argument (variable), these may if desired be combined in any manner such that the form of none of the subsidiary functions is affected in form by the combination. Functions of related arguments are usually combined as a function of some combination of these arguments, while functions of independent arguments are multiplied or added together.

II. OBJECTIVE LIGHT.

6. CHROMATIC SENSIBILITY AT LOW INTENSITY.

Chromatic sensibility or visibility is a function of wave length which fixes the relation of radiation to objective light; that is, to light measured by photometric or other zero methods. The spectral light curve is the product (EV) of the spectral energy $E(\lambda)$ and the visibility $V(\lambda)$ at each wave length. Its dimensions are light units per watt, hence its factorial parameter is determinable only by experiment. At low intensities (below 0.1 m-c.) vision appears to be accomplished by the retinal rods alone and to be achromatic, while chromatic vision at higher intensities brings into play the less sensitive but more efficient retinal cones. The two functions overlap widely so that the transition from rod to cone vision is very gradual; it is about half and half at ordinary working intensities, 50 meter-candles.

Visibility is determined by measuring the relative spectral energy necessary to produce a given luminous sensation throughout the visible spectrum. The reciprocal of this energy is then taken as proportional to the relative visibility. The constant luminous sensation used may range from the threshold of vision up.

Sufficient data is available for at least roughing out the visibility curve at low intensity for the individual and for the average eye. The best data is by König,[1] while that by Pflüger[2] and Langley[3] is useful for the determination of constants. Other data is useless; some because the energy of the source used was not recorded, some because of defective methods.

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[1] König and Dieterici, Zs. Psy. Phys. Sinnesorgane, **4**, pp. 241–347; 1893.

[2] A. Pflüger, Ann. Ph. **9**, p. 185; 1902.

[3] S. P. Langley, Am. Jour. Sc., **36**, p. 359; 1888.

König's data on visibility at low intensity is reproduced below. It relates to nine individuals, of whom three were totally color blind (monochromats), two were partially color blind (dichromats), and four of normal (trichromatic) vision. As indicated, some of the curves were obtained by stepping off a luminosity curve at low intensity, while others were obtained as reciprocals of threshold

König's Data on Visibility at Low Intensity.

Subject	Type	Method	λ=430	450	470	490
A. Beyssell	Monoch.	Step	0.015	0.074	0.27	0.57
E. Hering	"	"	0.10	(0.27)	(0.67)	0.90
				0.30	0.64	
F. C. Donders	"	"	0.038	(0.18)	0.31	0.58
				0.14		
R. Ritter	Red Bl.	"	0.062	0.25	0.50	0.74
"	"	Threshold	0.059	0.26	(0.50)	(0.75)
					0.52	0.85
E. Brodhun	Gr. Bl.	Step	0.064	0.21	0.47	0.80
A. König	Normal	"	0.059	0.21	0.49	0.60
"	"	Threshold	0.047	0.23	0.50	0.86
E. Köttgen	"	Step	0.069	0.23	0.49	0.69
F. Hillebrand	"	"	0.31	0.47	0.74	(0.93)
						0.96
Pereles	"	"	0.089	0.31	(0.73)	(0.77)
					0.58	0.84

Subject	505	520	535	555	575	590	605	λ max
A. Beyssell	.....	1.00	0.78	0.45	0.20	0.085	0.035	518
E. Hering	1.00	(0.98)	0.77	0.47	0.20	0.076	.....	507
		0.96						
F. C. Donders	0.85	1.00	0.85	0.49	0.25	0.15	0.09	521
R. Ritter	1.00	0.85	0.64	0.33	0.12	0.051	0.022	
"	1.00	0.83	0.60	0.33	0.12	0.033	0.019	504
E. Brodhun	(1.00)	(0.98)	0.74	0.41	0.17	0.067	0.026	512
	0.98	0.97						
A. König	1.00	0.99	0.77	0.42	0.17	0.061	0.022	
"	1.00	0.98	0.75	0.36	0.12	0.045	0.015	511
E. Köttgen	1.00	0.87	0.66	0.38	0.14	0.071	0.024	506
F. Hillebrand	1.00	0.90	0.71	0.44	0.21	0.10	0.048	502
Pereles	0.99	1.00	0.81	0.47	0.13	.....	.....	514

values. All are reduced to the same maximum ordinate unity. All values are means of several determinations. Values lying off from the smooth mean curve are inclosed in parentheses, these values being largely in the blue and being due apparently to a slight absorption in the eye media and to larger errors of measurement in that region. The values given below the numbers in parentheses as well as values of the maximum were read from carefully plotted curves and are used in subsequent work.

Pflüger measured threshold values directly with a Rubens thermopile. The data given below is from smoothed means of recip rocals of these threshold values. As his investigation was chiefly of variations due to fatigue and ill health and his curves diverge widely from one another, not much weight can be attached to them. Still his values add weight to König's and supply additional data on the position of the maximum.

Pflüger's Data on Visibility.

λ=	420	440	460	480	500	520	540	560	580	λ max
La	0.03	0.13	0.29	0.52	0.79	1.00	0.61	0.19	0.07	517
Ha	0.03	0.12	0.24	0.48	0.81	0.99	0.62	0.26	0.06	516
G	0.03	0.13	0.32	0.63	0.98	0.79	0.34	0.12	0.02	504
Cl	0.04	0.10	0.23	0.51	0.94	0.93	0.58	0.27	0.10	509
M	0.08	0.25	0.50	0.83	0.99	0.80	0.43	0.20	0.07	501
K		0.09	0.21	0.53	0.98	0.80	0.42	0.18		504
A	0.08	0.21	0.45	0.78	1.00	0.68	0.31	0.15	0.06	500
S	0.05	0.17	0.48	0.87	0.99	0.81	0.34	0.12	0.06	499
E	0.04	0.22	0.52	0.80	0.98	0.87	0.54	0.21	0.05	505

Langley's bolometric measurements of the amounts of energy of various wave lengths necessary for making print legible are given below. The values given are means of widely divergent data, and as these are for wave lengths 50 $\mu\mu$ apart they are of no great value in determining the exact form of the visibility curve. Values reduced to unit maximum ordinate are given below Langley's values.

The chief value of Langley's data lies in the fact that it is in absolute measure. The number 5.79 for F. W. V. at 550 represents, he says, the reciprocal of 360×10^6 ergs acting for ½ second. Hence the unit of visibility employed in the above data is roughly

2.4×10^{-10} light units per watt. Langley's own eye is seen to possess but about one-tenth the maximum sensibility of the other eyes investigated (due evidently to a deficiency of retinal rods), so that we are not surprised to find this maximum shifted over to 563 $\mu\mu$ in the region where it ordinarily lies at moderate and high intensities.

Langley's Data on Visibility.

$\lambda =$	400	450	500	550	600	650	700	λ max
S.P.L.		0.042	0.194	0.706	0.475	0.073	0.004	
" max = 1		0.056	0.26	0.94	0.63	0.097	0.008	563
F.W.V.	0.104	1.50	7.90	5.79	0.551	0.036	0.005	
" max = 1	0.011	0.16	0.84	0.62	0.06	0.004	0.0005	516
B.E.L.	0.139	3.75	10.10	6.31	1.17	0.089	0.009	
" max = 1	0.013	0.35	0.96	0.60	0.11	0.008	0.0008	510
D.M.	0.140	2.85	4.73	4.04	1.14	0.084	0.023	
" max = 1	0.03	0.59	0.97	0.83	0.23	0.02	0.005	513

Position of Maximum at Low Intensity.—The mean position of the maximum from the foregoing data is as follows:

Data		Weight
König's data	510.8	5
Pflüger's data	506.1	2
Langley's ex (S. P. L.)	513.0	1
Weighted mean of all data	509.7	

Pflüger's data has been assigned less weight than König's on account of its wide diversity, Langley's on account of its representing fewer subjects and on account of the uncertainty in the position of the maximum due to the few points of the curve given. Langley's own value (563 $\mu\mu$) has been excluded for reasons stated in the preceding paragraph.

We may accept, then, for the present 510 $\mu\mu$ as the position of the maximum of sensibility of the average normal human eye at low intensities approaching the threshold value, with a probable error of not more than 1 or 2 $\mu\mu$. Maxima for different individuals lie

usually within 10 $\mu\mu$ of this mean. At higher intensities, with the establishment of cone vision, the maximum shifts toward the yellow at least as far as 560 $\mu\mu$.

Form of the Visibility Curve.—To determine roughly the form of the visibility curve at low intensities, König's data has been carefully plotted and then replotted to a common maximum ordinate. Of the two sets of data given for Ritter and König, those obtained by the threshold value method have been chosen as not being subject to correction for the spectral intensity of the source.

Reduced Data on Visibility at Low Intensity (König).

$\lambda-\lambda_m=$	−100	−80	−60	−40	−20	0.0	+20	+40	+60	+80	+100
Beyssell	0.005	0.030	0.14	0.38	0.72	1.00	0.72	0.39	0.17	0.045	0.008
Hering		0.09	0.28	0.61	0.89	1.00	0.88	0.57	0.29	0.09	
Donders	0.015	0.09	0.22	0.44	0.77	1.00	0.74	0.41	0.20	0.10	0.055
Ritter		0.025	0.18	0.44	0.76	1.00	0.77	0.48	0.22	0.055	0.015
Brodhun		0.075	0.22	0.49	0.83	1.00	0.81	0.47	0.19	0.055	0.02
König		0.055	0.23	0.52	0.85	1.00	0.82	0.44	0.15	0.035	0.01
Köttgen		0.04	0.19	0.45	0.76	1.00	0.80	0.51	0.23	0.08	0.025
Hillebrand			0.38	0.59	0.88	1.00	0.89	0.64	0.36	0.16	0.05
Pereles		0.07	0.24	0.50	0.87	1.00	0.85	0.48	0.16		
Means	0.02	0.06	0.23	0.49	0.81	1.00	0.81	0.49	0.22	0.077	0.026

7. THE VISIBILITY FUNCTION. LUMINOSITY OF RADIATION.

This data is graphically reproduced in the accompanying figure. Calculated curves of the form

$$V = V_0 e^{-\kappa(\lambda-\lambda_m)^2}$$

with $\kappa=4$ and $\kappa=5$ have been plotted in the figure in dotted lines.

Numerical values are given below.

$\lambda-\lambda_m=$	−100	−80	−60	−40	−20	max	+20	+40	+60	+80	−100
Observed data (mean)	0.03	0.06	0.23	0.49	0.81	1.00	0.81	0.49	0.22	0.08	0.03
Probability, $\kappa=4.0$	0.02	0.08	0.24	0.53	0.85	1.00	0.85	0.53	0.24	0.08	0.02
" $\kappa=4.5$	0.01	0.06	0.20	0.49	0.83	1.00	0.83	0.49	0.20	0.06	0.01
" $\kappa=5.0$	0.01	0.04	0.16	0.45	0.82	1.00	0.82	0.45	0.16	0.04	0.01
Goldhammer, $n=235$	0.01	0.03	0.16	0.46	0.83	1.00	0.86	0.54	0.27	0.11	0.03

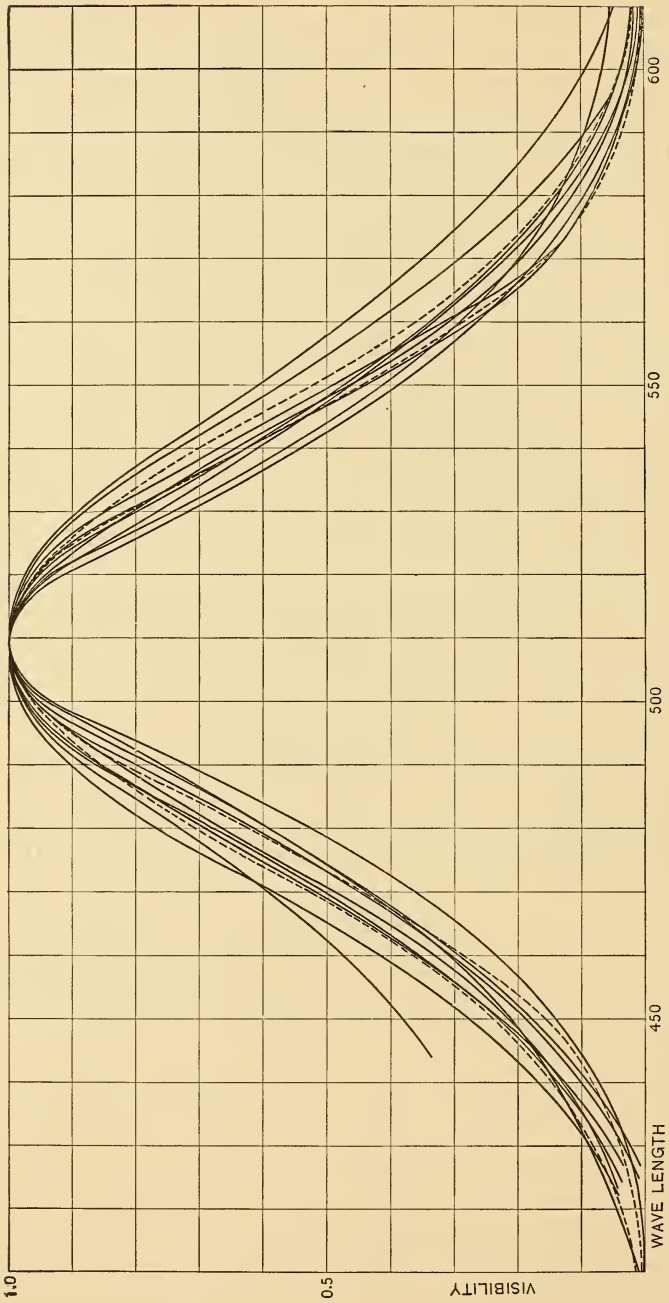

Fig. 1.

The probability function with $\kappa = 4.0$ or $\kappa = 4.5$ represents the data quite satisfactorily considering the large differences in the individual curves. The value $\kappa = \pi$ would be advantageous mathematically since it gives the curve unit area for unit maximum ordinate, but while it might be adopted arbitrarily, it is more nearly an envelope than a mean value of the experimental data.

Goldhammer has proposed the formula

$$\left(\frac{\lambda_m}{\lambda} e^{1-\frac{\lambda_m}{\lambda}}\right)^n$$

to represent the variation of sensibility with wave length. The wave length for which the sensibility is a maximum is indicated by λ_m while n is a number varying from 150 to 300. This is identical with the well-known Wien emission function for "black" radiation, which is

$$\frac{E}{E_m} = \left(\frac{\lambda_m}{\lambda} e^{1-\frac{\lambda_m}{\lambda}}\right)^n$$

when written in terms of its maxima E_m and λ_m in order to be independent of units of energy and wave length.

Goldhammer's function has the same number of parameters (n and λ_m) as the probability function, so that it is no more flexible. It is always steeper on the short wave length side, while the experimental curves are unsymmetrical on either side in about equal numbers. The discrepancy is not great, however, as may be seen from the data appended to the above table.

Hertzsprung,[4] in discussing the relation between the luminosity and temperature of "black bodies," adopts the following numerical factors to convert radiation into light:

$\lambda =$	400	450	500	550	600	650	700
Factor	0.044	0.083	0.340	1.000	0.631	0.107	0.0065

These values he takes as rough means of observed data at all intensities. He does not attempt to formulate any relation between them.

[4] E. Hertzsprung, Zs. Wiss. Phot., **IV**, pp. 43–54; 1906.

8. OTHER POSSIBLE FORMS OF VISIBILITY FUNCTION.

The probability function and the formula for the radiation of black bodies, with altered parameter, were both found to be fairly satisfactory visibility functions, but perhaps rather too simple in form; that is, too inflexible to be made to represent some of the wide individual variations observed. Let us consider more general forms, which include both of these and other possible forms. The ultimate choice of a function is to be determined not so much by its simplicity, flexibility, or precise agreement with data, as by the form of variation of its parameters with intensity.

The more important characteristics of the visibility function are that it has a single maximum, it approaches zero for very long and very short waves, and that it is everywhere smooth, continuous, and single valued. The same may be said of its first derivative except that it has the single root $\lambda=\lambda_m$, and of the second derivative except that it has two roots which are in general not symmetrical with respect to λ_m.

Our data relates not to luminosity at constant spectral energy but to the inverse function, spectral energy at constant luminosity $E(\lambda)$, L constant. This function has a minimum instead of a maximum and is infinite where the visibility function is zero, otherwise the description of the preceding paragraph applies. Since the first derivative has a single root $\lambda=\lambda_m$ and is infinite at $\lambda=0$ and $\lambda=\infty$,

$$\frac{dE}{d\lambda}=a(E)(\lambda-\lambda_m)\left(\lambda,\frac{1}{\lambda}\right)$$

where (E) and $\left(\lambda,\frac{1}{\lambda}\right)$ denote functions of these quantities having no finite roots.

The function of E and indicated by (E) must be exceedingly simple, else $E(\lambda)$ would be so complex as to be quite useless in practice. The form $(E)=E$ is the simplest possible and yet sufficiently general to make $E(\lambda)$ include the probability and black body functions as well as a wide variety of others. The factor $\lambda-\lambda_m$ might of course be replaced by $f(\lambda)-f(\lambda_m)$ where f denotes any function (including $E(\lambda)$, but we wish to consider only the sim-

pler and more specific of the possible forms of $E(\lambda)$. As to the function $\left(\lambda, \frac{1}{\lambda}\right)$ we shall consider in turn various simple possible forms.

$$\left(\lambda, \frac{1}{\lambda}\right) = \text{const. gives } E = \alpha e^{\beta(\lambda-\lambda_m)^2}$$

$$= E_m e^{\kappa\left(\frac{\lambda}{\lambda_m}-1\right)^2}$$

$$\left(\lambda, \frac{1}{\lambda}\right) = \frac{1}{\lambda} \quad \text{"} \quad E = E_m\left(\frac{\lambda_m}{\lambda} e^{\left(\frac{\lambda_m}{\lambda}-1\right)}\right)^n$$

$$\left(\lambda, \frac{1}{\lambda}\right) = \frac{1}{\lambda^2} \quad \text{"} \quad E = E_m\left(\frac{\lambda}{\lambda_m} e^{\left(\frac{\lambda_m}{\lambda}-1\right)}\right)^n$$

The forms in positive powers of λ are inadmissible.

The first of these three functions is the reciprocal of the probability function proposed by the author. The third is the Wien-Goldhammer function above described. The second is a new one which may be used instead of the third to represent unsymmetrical sensibility curves in which the dissymmetry is reversed.

It is to be noted that writing the function $E(\lambda)$, (or any other physical function), in terms of its maxima, E_m and λ_m, gives a function $\frac{E}{E_m}\left(\frac{\lambda}{\lambda_m}\right)$ which is of zero dimensions throughout; i. e., independent of the units employed in the measurement of the quantities involved. Such a relation between pure numbers is often the simplest and always the safest to deal with.

9. CHROMATIC SENSIBILITY AT HIGHER INTENSITIES.

The sensibility of the eye to the same amount of radiation of various wave lengths varies considerably with variations in the intensity of the radiation. The maximum shifts from bluish green over to greenish yellow, the visibility curve at the same time becoming much broader near the maximum. The shift of the maximum toward longer wave lengths with increasing intensity may be easily observed with a small, low-power grating spectroscope and sunlight. With the slit all but closed the brightest part of the spectrum is in the pure green, but on opening the slit wider and wider, the brightest portion moves over nearly to the *D* lines.

This variation of the visibility of radiation with intensity is of the utmost importance. It shows that the assumption of luminosity proportional to intensity can be true only under special conditions and within certain limitations. In the relation $L = VE$, while L, V, and E are all functions of wave length λ, V is further a function of E. And the greatest variation occurs within the range of illumination in most common use, 1 to 100 meter-candles.

But scant data exists in this field. A. König appears to have been the only one to take up this problem, and his data[5] is complete only for his own eye. His long experience in work on visual sensibility and the fact that his eye appears to have been perfectly normal with sensibilities very nearly those of the average eye make his data of considerable value. König chose a standard of (visual) intensity in the green at 535 $\mu\mu$ and with that compared 13 other

König's Data on Visibility—Intensity

Intensity Steps	T	A	B	C	D	E	F	G	H
Int. (meter-candles)	.00024	.00225	.0360	.575	2.30	9.22	36.9	147.6	590.4
Ratio to Preceding Step		9.38	16	16	4	4	4	4	4
Wave Length	Equivalent Slit Widths								
670	296.1	189.0	29.08	14.53	4.56	1.955	1.403	1.23	1.17
650	178.3	87.6	13.9	5.98	1.95	0.991	0.667	0.547	0.420
625	34.8	20.55	6.20	2.93	1.004	0.497	0.392	0.307	0.289
605	12.1	8.60	4.29	2.073	0.869	0.451	0.321	0.289	0.274
590	5.65	4.29	2.75	1.86	0.787	0.523	0.376	0.346	0.330
575	2.76	2.00	1.68	1.42	0.868	0.568	0.449	0.459	0.424
655	1.40	1.232	1.10	1.037	0.876	0.702	0.621	0.603	0.590
535	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
520	1.07	1.09	1.12	1.16	1.40	1.71	1.76	1.77	1.80
505	1.54	1.58	1.59	1.81	2.30	2.78	3.75	4.06	4.46
490	2.34	3.00	3.06	3.33	4.93	7.97	8.91	9.53	10.06
470	5.92	6.49	7.22	7.75	11.0	20.7	22.9	26.6	.....
450	19.21	21.5	20.95	24.2	40.7	57.3	62.9	.....	.....
430	131.6	115.4	85.9	119.2	138.	151.	.....	.....	.....

parts of the spectrum by widening an auxiliary slit until luminosities matched. These slit widths, corrected for variable prismatic dispersion, gave an inverse visibility curve for the source (a triplex gas flame) used. Then choosing a new standard of intensity, 4 or

[5] A. König, Helmholz Festgrüss, Hamburg and Leipzig, pp. 309–388; 1891.

16 times as great, the process was repeated until he had spectral curves at nine different intensities ranging from the threshold of vision up to about 600 meter-candles. König's data are given below. The different steps of intensity are indicated by $T, A, B—H$; T being the threshold value. The actual intensity (approximately in meter candles) and ratio to preceding step are also given.

These data refer to the gas flame spectrum while the visibility curves proper refer to a spectrum of constant (energy) intensity throughout. Fortunately the spectral energy curve of such a gas flame was determined bolometrically by Langley and counter-checked photometrically by König by means of the solar spectrum. His relative values for the energy are

λ =	670	650	625	605	590	575	555	535	520	505	490	470	450	430
E =	13.0	8.88	5.38	3.99	2.97	2.27	1.48	1.00	0.720	0.488	0.370	0.251	0.169	0.114

To obtain the data of the following table each slit width of the preceding table was first multiplied by the spectral energy of the source for that wave length. The reciprocals for these corrected slit widths were then plotted as functions of wave length. These curves all passed through the value unity at 525 $\mu\mu$, but had maxima of different heights. Finally all curves were reduced to the same maximum ordinate unity and the data read from these curves.

Visibility-Intensity, Reduced Data

	T	A	B	C	D	E	F	G	H	C:T	P:T	F:T	H:T
430	.047	.057	.076	.060	.063	0.57	.....	.....	.....	1.28	1.34		
450	.23	.21	.21	.20	.14	.10	.085	.....	.....	0.92	0.61	0.37	
470	.50	.46	.42	.42	.36	.19	.16	.13	.....	0.84	0.72	0.32	0.26
490	.85	(.78)	(.75)	(.70)	(.63)	(.40)	.28	.25	.24	0.82	0.74	0.30	0.34
505	.99	.99	.98	.94	.88	(.63)	.50	.45	.40	0.95	0.89	0.50	0.45
520	.97	.96	.94	.99	.97	(.85)	.72	.69	.67	1.02	1.00	0 74	0.71
535	.75	.76	.76	.83	.98	.98	.91	.88	.87	1.11	1.31	1.21	1.17
555	.36	.42	.47	.54	(.83)	(.97)	.99	.99	.99	1.50	2.30	2.75	2.75
575	.12	.17	.20	.26	(.58)	(.85)	(.97)	(.98)	(.98)	2.17	4.83	8.08	8.17
590	.045	.059	.095	.15	.42	(.71)	(.88)	(.92)	(.92)	3.00	9.40	19.6	20.4
605	.015	.022	.044	.10	.28	.54	.71	.79	.79	6.6	18.6	47.5	52.6
625	.....	.....	.022	.050	.17	.35	.42	.52	.54				
650	.....	.....	.....	.....	.057	.11	.15	.18	.23				
670	.....	.....	.....	.....	.....	.038	.050	.055	.058				
λ max	511	511	511	515	529	545	560	565	565				

In plotting these curves it is noticeable that those values which lie off from the smooth curve of the probability type (inclosed in parentheses in the table) nearly all lie near the two wave lengths at which the three fundamental sensibility curves of the Young-Helmholtz theory cross. The plot of the wave length of the maximum against the logarithm of the intensity is a remarkable curve.

Some of these curves are reproduced in the accompanying figures. In Fig. 2 are shown three curves of visibility as a function of wave length at constant intensities for the steps *T* (about 0.0002 m-c.), *D* (2.3 m-c.), and *H* (600 m-c.). The curves for steps *B* (0.002 m-c.) and C (0.036 m-c.) are nearly coincident with the curve for step *T*, the threshold value. These curves show the shift of the maximum and the broadening of the visibility curve with increasing intensity.

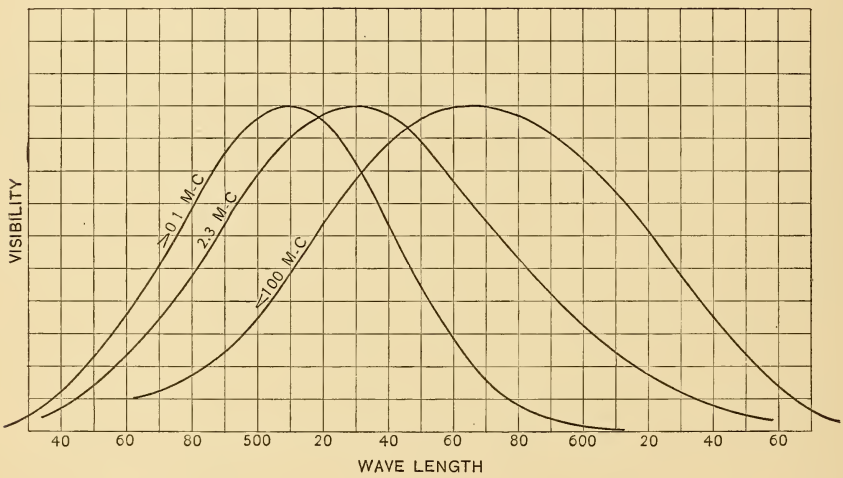

Fig. 2.

In Fig. 3 are shown seven curves of visibility as functions of intensity at constant wave length, the reproductions being for the wave lengths 430, 470, 505, 535, 590, 605, and 670 $\mu\mu$. These monochromatic curves represent the ratio of light to radiation in its relation to intensity under the assumption of maximum ratio equal to unity. The curves for yellow and orange ascend sharply with increasing intensity, while those for blue descend, and those for extreme violet and extreme red as well as for green do not vary

much. These results show that the factor (visibility, V) converting radiation into light must vary with both color and intensity, although in the integrated effect for white light that factor may approach a constant value.

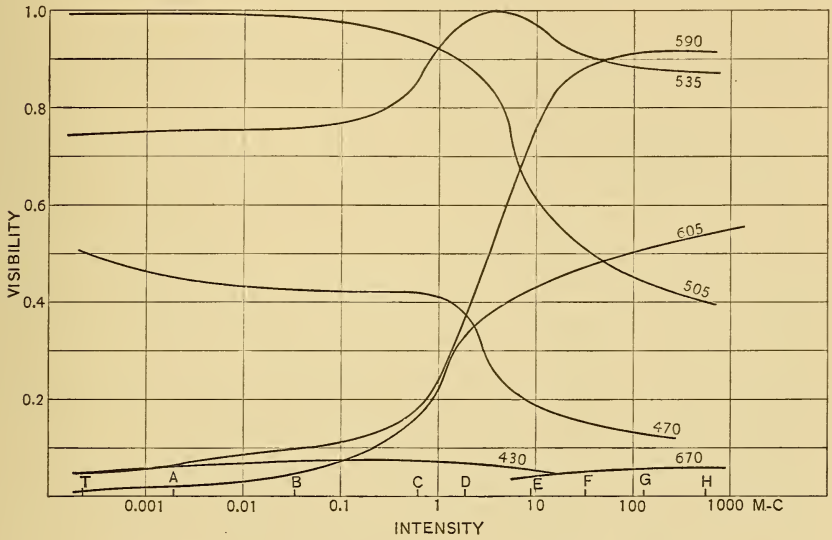

Fig. 3.

10. VARIATION OF THE PARAMETERS OF THE VISIBILITY FUNCTION.

Taking König's data as reduced in the last table as a working basis, let us see how the wave length of maximum sensibility and the parameters giving the visibility curve breadth and height vary with the intensity of the radiation. The wave lengths of the maxima are given in the table as read from the original reduced curves. They are plotted as a function of logarithmic intensity in Fig. 4.

In the plot the uncertainty in each value is indicated by the diameter of the circle inclosing each point. The shift in the position of the maximum of sensibility is seen to occur between intensities of about 0.5 and 50 meter-candles, just the range within which the eye is most commonly used. At very high and very low intensities there appears to be little or no shift. It is very remarkable how closely the shift that does occur is proportional to the logarithmic intensity. It is to be noted that this relation rests directly on experimental data without any assumption as to the form of any subsidiary function.

In Fig. 4 is also a plot of the variation of the parameter κ in

$$V = V_o\, e^{-\kappa(\lambda-\lambda_m)^2}$$

with logarithmic intensity. These values were obtained by calculation from the values of λ and λ_m for the various relative intensities.

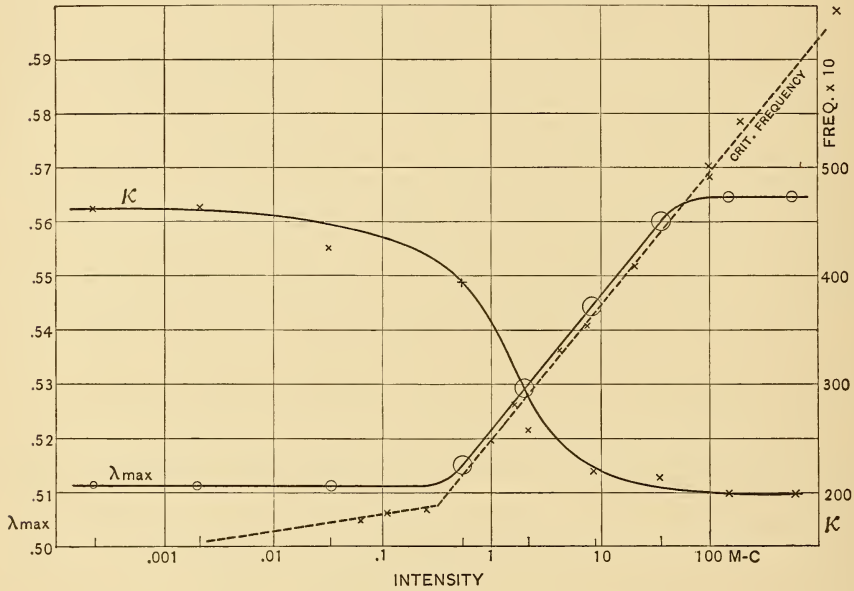

Fig. 4.

Each is the mean of the values at the different wave lengths for each intensity, the two middle values nearest the maximum being

	Luminous Intensity.	Wave Length Maximum Sensibility.	κ Calculated	
			$\text{Log}\,\frac{L}{EV_o}$	$\kappa\,(\lambda-\lambda_m)^2=\text{const.}$
T	0.00024	5.11	4.61	4.6
A	0.00225	5.11	4.66	4.6
B	0.0360	5.11	4.45	4.6
C	0.575	5.15	3.95	4.4
D	2.30	5.29	2.54	3.4
E	9.22	5.45	2.21	2.6
F	36.9	5.60	2.14	2.1
G	147.6	5.65	1.98	2.0
H	590.4	5.65	1.97	2.0

omitted in each case on account of their great uncertainty; they involve logarithms of numbers very nearly equal to unity.

Appended for subsequent discussion are values of κ calculated under the assumption $\kappa(\lambda-\lambda_m)^2=\text{constant}$; that is, illumination proportional to radiation.

There remains the parameter V_0 to be determined. This is the value of V for $\lambda=\lambda_m$, or in other words the maximum ratio of light to radiation at a given luminosity. To determine V_0 it is only necessary to measure the same monochromatic radiation as light and radiation, photometrically and radiometrically in meter candles and in meter watts.

The author[6] used a Rubens thermopile which was arranged to be interchangeable with the white screen of a Lummer-Brodhun photometer in determining the spectral intensity of a Nernst lamp. The thermopile was calibrated in watts by direct exposure to a Nernst lamp filament whose potential drop and current were then measured. A correction was then applied for the equatorial distribution of radiation from a linear source. Losses by convection were found to be less than half a per cent at most.

The results obtained are given in the following table. The accuracy is not high, the aim was rather to establish the order of magnitude of V_0 beyond any question.

Wave length	$\frac{\text{Watt}}{\text{cm}^2}$	Meter-candles	$\frac{\text{Meter-candles}}{\text{watt/cm}^2}$	$\frac{\text{Candles}}{\text{watt}}$
620	.000162	64	390000	3.1
566	.000042	64	1520000	12.1
566	.000036	61	1700000	13.5
595	.000123	182	1480000	11.8

These amounts of energy are low to measure radiometrically. The value of V_0 for pure rod vision appears to be almost hopeless of direct determination.

Drysdale[7] determined the ratio of light to radiation units by a somewhat similar method. Using a bolometer and glow lamps, he obtained the value 16.7 cand./watt for "yellow-green" light.

[6] Electrical World, June 26, 1908.

[7] C. V. Drysdale, P. R. S., **80**, pp. 19–25; 1907.

11. SENSIBILITY TO DIFFERENCES IN WAVE LENGTH.

Of related interest in the theory of visual sensibility and in measurements of color are the minimum perceptible wave length or color differences in different regions of the spectrum and for different intensities. The best recent data is that of Steindler[8] on twelve subjects. The curves differ considerably in detail with the individual, but all are of the same general type. That for Steindler's own eye is reproduced in Fig. 5, the reciprocal of the least perceptible increment being plotted as ordinate to represent sensibility.

Fig. 5.

The two large central maxima may easily be located with a small spectroscope and daylight. They are supposed to correspond with the intersections of the three fundamental (red, green, and blue) sensibility curves. It would be important in visual theory to know whether these maxima shifted with varying intensity, but no one appears to have studied that phase of the problem.

[8] O. Steindler, Wien Sitz., IIa, **115**, pp. 1–24; 1906.

III. SUBJECTIVE LIGHT.

12. PHOTOMETRIC SENSIBILITY.

Visual sensibility to variations in intensity decreases continually with increasing intensity. Compared with other physical instruments the eye has a most extraordinary range. It may be used with ease at intensities a million times the minimum perceptible. Its great power of accommodation is due chiefly to the decrease of sensibility with intensity. It is like an ammeter or galvanometer with an automatic shunt continuously variable.

As a measure of relative sensibility the best data theoretically would be the mean errors in a long series of determinations of given fixed intensities in various parts of the spectrum. This has never been done and would be prohibitively tedious. Next best is probably the determinations of the least perceptible increment or "difference limen" by König. These include six different wave lengths and about twenty intensities of each. The least perceptible increment is greater than the mean error of observations of equality but proportional to it, although the exact relation between them is still undetermined.

König referred all his intensities to a single large unit (about 600 meter candles) for all six wave lengths, and then at each wave length stepped down to intensities of 1/2, 1/5, 1/10, etc., of the large standard value. We shall refer to his intensities by the letter I, the threshold value being I_0 measured as a fraction of that high value. But although, on this basis, an illumination of 600 meter-candles in the red is of the same apparent brightness as 600 m-c. in the blue, 1 m-c. in the red is by no means of the same apparent brightness as 1 m-c. in the blue. At lower intensities the divergence is still greater. König's scale of intensities is, however, the only rational one available for experimental work, and from it must be deduced a tempered scale free from such anomalies.

König and Brodhun[9] determined the least perceptible increment for light ranging in intensity from just above the threshold value to about 100,000 meter-candles and in color from violet to deep red. The data relating to König's own eye—a normal trichromat—are reproduced in the accompanying table and figure.

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[9] König and Brodhun, Sitz. Ak. Berlin, July 26, 1888.

Photometric Sensibility.

λ=	.670	.605	.575	.505	.470	.430	White
I_0=	0.060	0.0056	0.0029	0.00017	0.00012	0.00012	0.00072
I	δI:I						
1,000,000							0.0358
500,000							0.0273
200,000		0.0425					0.0267
100,000		0.0241	0.0325				0.0195
50,000	0.0210	0.0255	0.0260				0.0173
20,000	0.0160	0.0183	0.0205	0.0195			0.0175
10,000	0.0156	0.0163	0.0179	0.0181			0.0176
5,000	0.0176	0.0158	0.0166	0.0160			0.0179
2,000	0.0165	0.0180	0.0180	0.0175	0.0180		0.0181
1,000	0.0169	0.0198	0.0185	0.0184	0.0167	0.0178	0.0178
500	0.0202	0.0214	0.0180	0.0194	0.0184	0.0214	0.0192
200	0.0220	0.0225	0.0225	0.0220	0.0215	0.0245	0.0222
100	0.0292	0.0278	0.0269	0.0244	0.0225	0.0246	0.0298
50	0.0376	0.0378	0.0320	0.0252	0.0250	0.0272	0.0324
20	0.0445	0.0460	0.0385	0.0295	0.0320	0.0345	0.0396
10	0.0655	0.0610	0.0582	0.0362	0.0372	0.0396	0.0477
5	0.0918	0.103	0.0888	0.0488	0.0464	0.0494	0.0593
2	0.1710	0.167	0.136	0.0655	0.0715	0.0600	0.0939
1	0.258	0.212	0.170	0.0804	0.0881	0.0740	0.123
0.5	0.376	0.276	0.208	0.0910	0.096	0.0966	0.188
0.2		0.332	0.268	0.110	0.127	0.116	0.283
0.10			0.396	0.133	0.138	0.137	0.377
0.05				0.183	0.185	0.154	0.484
0.02				0.251	0.209	0.223	0.695
0.01				0.271	0.289	0.249	
0.005				0.325	0.300	0.312	
0.002						0.369	

König and Brodhun did not include the increment (δI) in the total light (I) in calculating the values of $\delta I:I$. Logically the increment should be included, otherwise the ratio $\delta I:I$ becomes meaningless as the threshold value is reached. The values quoted have been recalculated with increment included. In the figure, $\delta I:I$ is plotted as a function of the natural logarithm of I, the curves being extended from the data of König and Brodhun to the point $\delta I:I=1$ for $I=I_0$, the threshold value, shown as a dotted ordinate in the figure.

We have now to determine the mathematical form of these curves by the methods of synthetic function theory. For brevity, call

$\delta I : I = P\ (I)$ or simply P, the photometric function. Denote the abscissa log I by x and the minimum value of P (about 0.016) by P_m.

All the curves have evidently the threshold value—the abscissa of the dotted ordinate—as natural origin of coordinates; that is, they would be strictly comparable if plotted as functions of $x - x_0$ or log $(I : I_0)$ instead of x or log I. Further, they are all of the general class of functions included in

$$P = a + be^{-z}$$

when a and b are constants (independent of x) and z is a function of x. As a special case we may consider the particular function $z\ (x) = n\ (x - x_0)$ just as we should choose a particular integral of a differential equation as the practical working solution.

Taking, then, the particular function

$$P = a + be^{-n\ (x - x_0)}$$

is a working basis of probably the simplest possible form consistent with observed data; the parameter a is identified with P_m (=about 0.016, a pure number) the minimum value toward which $P\ (I)$ approaches for I very large. The constant b is $1 - P_m$ or about 0.984, since at $x = x_0$, $P = a + b = 1$. Both a and b are of course independent of x and hence of I, and the data shows them to be very nearly, if not quite, independent of wave length as well. And being pure numbers they as well as P are of course independent of the units employed. The parameter n is a fraction varying between ¼ and ½ with different wave lengths.

Since $x - x_0$ is $log\ I - log\ I_0$ or $log\ (I : I_0)$, P as a function of L is

$$P = P_m + (1 - P_m)\ (I_0 : I)^n$$

to determine n it is convenient to reduce this equation to the form linear in n

$$log\ \frac{1 - P_m}{P - P_m} = n\ log \frac{I}{I_0}$$

Below is given for each wave length the value of the threshold value I_0 (approximately in meter candles) as determined by König

together with the calculated mean values of n and their computed probable errors:

	I_0	n	$P_m^{1/n}$
670	0.060	0.584±0.005	0.000832
605	0.0056	0.388±0.020	0.0000234
575	0.0029	0.364±0.020	0.0000116
505	0.00017	0.335±0.005	0.00000434
470	0.00012	0.323±0.008	0.00000276
430	0.00012	0.323±0.003	0.00000276

In the next table are given for each wave length and intensity:

1. The value of P observed by König.
2. The value of n computed from P with $P_m=0.016$.
3. The value of P computed from $P_m=0.016$ and the mean value of n, in the curves reproduced in the accompanying figure discrepancies between observed and calculated values do not show as well as in the table.

Wave Lengths.

	670	605	575	505	470	430
I	P, n calculated and P calculated.					
1000	0.017	0.020	0.019	0.018	0.017	0.018
	.....	0.456	.....	.....	.....	.....
	0.019	0.025	0.025	0.021	0.022	0.022
500	0.020	0.024	0.018	0.019	0.018	0.021
	.....	0.464	.....	.....	.....	0.333
	0.021	0.028	0.028	0.023	0.023	0.023
200	0.022	0.023	0.023	0.022	0.022	0.025
	0.591	0.464	0.445	0.348	0.282	0.333
	0.024	0.033	0.033	0.025	0.026	0.026
100	0.029	0.028	0.027	0.024	0.023	0.025
	0.580	0.450	0.430	0.364	0.362	0.344
	0.029	0.038	0.038	0.028	0.028	0.028
50	0.038	0.038	0.032	0.025	0.025	0.027
	0.565	0.418	0.423	0.368	0.363	0.346
	0.038	0.045	0.045	0.031	0.031	0.031
20	0.045	0.046	0.039	0.030	0.032	0.035
	0.602	0.418	0.431	0.372	0.350	0.327
	0.049	0.057	0.056	0.036	0.037	0.037

Wave Lengths—Continued.

	670	605	575	505	470	430
I	P, n calculated and P calculated.					
10	0.066	0.061	0.058	0.036	0.037	0.040
	0.586	0.413	0.388	0.354	0.339	0.327
	0.065	0.070	0.068	0.041	0.042	0.042
5	0.092	0.103	0.089	0.049	0.046	0.049
	0.583	0.357	0.344	0.328	0.327	0.320
	0.090	0.086	0.082	0.048	0.048	0.048
2	0.171	0.167	0.136	0.065	0.071	0.060
	.....	0.307	0.323	0.317	0.294	0.320
	0.142	0.117	0.108	0.057	0.058	0.058
1	0.258	0.212	0.170	0.080	0.088	0.074
	.....	0.313	0.317	0.315	0.288	0.314
	0.204	0.147	0.134	0.069	0.069	0.069
0.50	0.376	0.276	0.208	0.091	0.096	0.097
	.....	0.296	0.314	0.323	0.301	0.301
	0.300	0.187	0.166	0.084	0.073	0.073
0.20		0.332	0.268	0.110	0.127	0.116
		0.299	0.323	0.332	0.295	0.309
		0.261	0.237	0.108	0.105	0.105
0.10			0.396	0.133	0.138	0.137
			0.270	0.334	0.311	0.312
			0.286	0.131	0.128	0.128
0.05				0.183	0.185	0.154
				0.314	0.292	0.325
				0.161	0.155	0.155
0.02				0.251	0.209	0.223
				0.297	0.320	0.305
				0.215	0.204	0.204
0.01				0.271	0.189	0.249
				0.322	.....	0.326
				0.265	0.253	0.253
0.005				0.325	0.300	0.312
				0.343	.....	0.322
				0.331	0.291	0.291

The agreement between the function chosen and the data is quite satisfactory in the blue and violet, fair in the red and green, and poor in the yellow. One would expect the poorest agreement at 490 and 575 $\mu\mu$ where the primary sensibility curves intersect. König states that he chose 575 as one wave length purposely to test the color vision theories. Another cause of poor agreement lies in

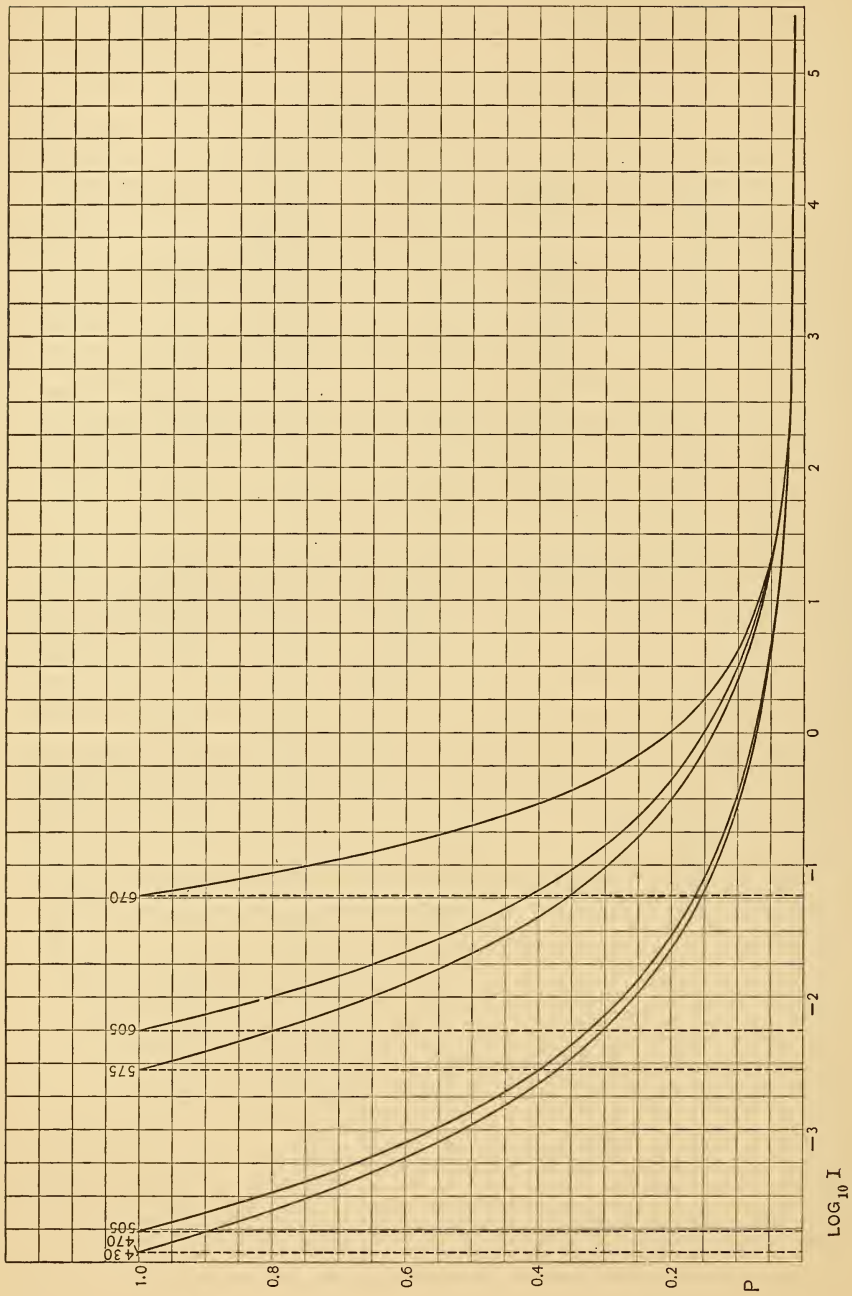

Fig. 6.

the uncertainty in the roughly determined values of the threshold of visibility, I_o which enters as a vital part of the function. However, the function may be considered as a serviceable first approximation to the intensity sensibility of the average human eye over a wide range of color and intensity.

13. THE LUMINOUS SENSATION, FECHNER'S LAW.

Accepting, then, for the photometric function or least perceptible increment the form

$$P = P_m + (1 - P_m)(I_o/I)^n$$

as a tentative working basis, we may apply the theory of sensibility to find the scale reading—which is luminous sensation in this case.

P is expressed as a fraction ($P = \delta I/I$) in the above function, so that sensibility, being inversely proportional to δI, is inversely proportional to IP. Calling sensibility S_I and the constant of proportionality K_I, then

$$S_I = \frac{K_I}{\delta I} = \frac{K_I}{IP} = \frac{K_I}{I(P_m + (1 - P_m)(I_o/I)^n)}$$

But from the theory of sensibility the sensation of visual brightness B is the general integral of the sensibility with respect to the stimulus, hence

$$B = \int S dI = \int \frac{K_I dI}{I(P_m + (1 - P_m)(I_o/I)^n)}$$

$$= \frac{K_I}{P_m} \log(1 + P_m(I^n I_o^{-n} - 1))^{1/n}$$

plus an integration constant which is zero, since the sensation B approaches zero as the stimulus I approaches the threshold value I_o.

About eighty years ago E. H. Weber observed that the least perceptible increment to a stimulus affecting several of the sense organs, under fixed subjective conditions of attention, expectation, and fatigue, bore a definite relation to the amount of that stimulus.

G. T. Fechner, thirty years later, extended Weber's observations and formulated his results in mathematical terms. Calling the

stimulus I and the least perceptible increment δI, then Fechner's statement of what he termed Weber's Law is that $\delta I:I=$constant. This law appears to hold over a wide range of ordinary working intensities, breaking down only for very low and for excessively intense stimulation.

Fechner proceeded further by assuming that the above constant was proportional to the corresponding increment δB to the sensation. Hence $\delta I/I=c\delta B$ and by integration

$$B=c\ (\log I-\log I_o).$$

In this form or in similar forms differing only in the choice of integration constant, Fechner's law has been accepted by phychologists for half a century.

There are two very serious if not fatal defects in this deduction. In the first place, the increments δI and δB are finite quantities and by no means infinitesimal increments approaching zero as a limit, such as would be required for such an integration. The least perceptible increment to the stimulus (δI) is determined by the sensibility of the sensory organ concerned. At the threshold value it is as large as I itself, while at moderate intensities it bears a fixed ratio to I. The value of δB is entirely arbitrary, dependent upon the unit chosen in which to measure it. It may be greater than unity in special cases. In the second place, c is not a constant but a function of I. At low intensities approaching the threshold value it varies rapidly with I. The method used in this paper is free from these defects. Weber's Law $\delta I/I=$constant is extended to apply to high and low intensities, in the form

$$\delta I:I=P_m+(1-P_m)\ (I_o/I)^n$$

or some other form of function not differing greatly from this. Similarly Fechner's Law $B=c\ (log\ I-log\ I_o)$ is more widely applicable in the more general form derived in this paper.

The argument

$$L=(1+P_m(I^nI_o^{-n}-1))^{1/n}$$

of the visual brightness function is of such form that, however the radiation is varied in either wave length or intensity, so long as L

is constant the sensation is constant. This, then, is the proper measure of light measured as light in fractions of some high intensity. Luminosity is light in meter-candles tempered to a rational self consistent scale. But for the Purkinje effect, L would be identical with I and all parameters of both functions would be constant. $Ld\lambda$ is integrable to give total light while I is not.

14. THE GROWTH OF VISUAL IMPRESSION WITH TIME.

The visual sensation as a function of time may be constructed directly from the data of Broca and Sulzer without recourse to sensibility as a function of time. This function has, of course, the value zero at zero and approaches a constant value after a sufficient lapse of time. Broca found that with blue light there was a very pronounced maximum at 0.07 sec., with red a slight maximum farther

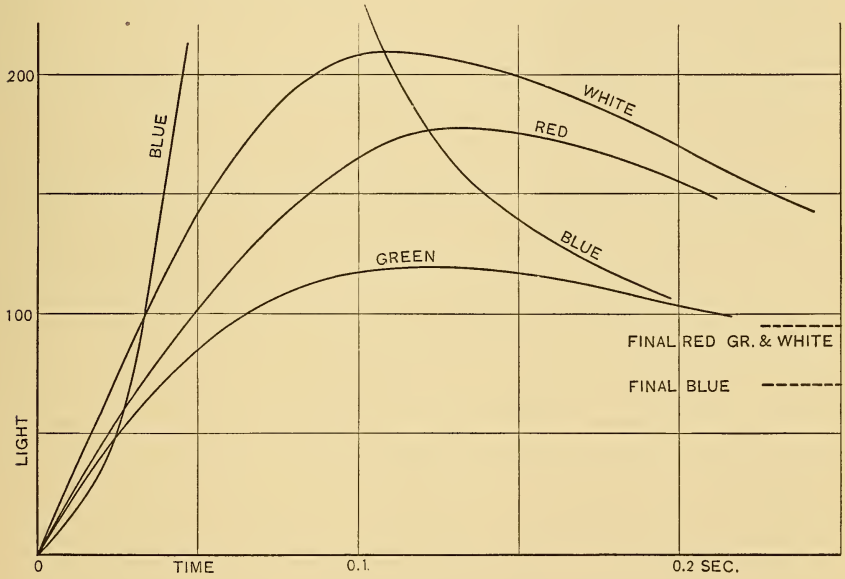

Fig. 7.

out (at 0.12 sec.), while with green the curve rose steadily to its final value. These values are for intense illuminations; for lower intensities the maxima are lower and occur after a much greater lapse of time. These results may be represented by very simple functions. Consider the actual sensation as the resultant of a pure positive sensation (no fatigue) represented by the ordinary law of

growth $(1-e^{-at})$ together with a decrement $1-e^{-\beta(t-t_0)}$ due to fatigue such that the fatigue is greater the greater the intensity of the light to which the eye is exposed, but affected with a lag t_0.

Hence,

$$l=L\,(1-e^{-at})\,(1-e^{-\beta(t-t_0)})$$

will serve as a tentative working basis. This satisfies the conditions $l=o$ for $t=o$ and $l=L$ at $t=\infty$ The derivative dl/dt has the value

$$La\,(1-e^{-\beta t_0})$$

for $t=o$ and has roots at $t=\infty$ and one finite point given by the value of t in

$$\frac{a}{e^{at}-1}+\frac{\beta}{e^{\beta(t-t_0)}-1}=o$$

Hence, for fatigue coefficient β, very small, the maximum will occur only after a lapse of considerable time.

Accordingly Broca's results would indicate very slight fatigue in the green, more in the red, and most of all in the blue and violet.

Work by E. S. Ferry[10] and by T. C. Porter[11] on the critical frequency at which flicker is just imperceptible is of great interest in this connection. Radiation of known intensity and quality is sent through a rotating sectored desk. The frequency at which the flicker disappears appears to be a function of optical intensity alone.

Flicker will evidently vanish when the difference between successive maxima and minima of visual impression becomes less than the least perceptible difference or photometric constant P. Hence, the observation that critical frequency is independent of color is confirmatory to König's Law that the photometric constant is independent of wave length.

Porter found further that the critical frequency is a linear function of the logarithm of the intensity except for a sharp variation at about 0.2 m-c. For intensities above and below this intensity these relations are

$$N_1=12.4\log L+24.9$$
$$N_2=1.56\log L+19.6$$

in meter-candles and common logarithms.

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[10] E. S. Ferry, Am. J. Sci., **44**, p. 192–207; 1892.

[11] T. C. Porter, P. R. S., **70**, p. 313–329; 1902.

These relations are shown graphically by the dotted line in Fig. 4. The elbow corresponds closely with the sharp turn in the curve of wave length of maximum sensibility, the lower branch evidently representing rod vision.

Critical frequency varies from 36 to 50 per second within the ordinary working range of intensities 10 to 100 m-c. Critical periods range from about 0.02 to 0.03 second for these intensities. These short periods of time lie back on the first slope of Broca and Sulzer's curves for $l\ (t)$, the parts that are sensibly straight lines. The corresponding range of intensity is roughly 0.016×50 or 0.8 m-c., while from the initial part of any of these curves it is over 40 m-c. Hence we must conclude that at vanishing flicker the eye is operating up near the maximum of a lower curve. It must then be a crude energy integrator but one with very small lag. Much remains to be done in this interesting field, particularly in relation to Talbot's Law. The measurement of intensity in terms of time alone is well worth further study, particularly as to its relation to the Purkinje effect and variation due to different observers.

The relation between critical frequency and luminosity is not at first apparent but may easily be deduced. Let $t_2 - t_1$, be the short interval of time during which the eye is exposed and $t_3 - t_2$ be the period of recovery. Then,

$$\frac{1}{t_3 - t_1} = n = f(L).$$

But,

$$\frac{dB}{dt} = \frac{dB}{dL} \quad \frac{dL}{dt}$$

and

$$\frac{dB}{dL} = \frac{K}{LP} \text{ and } \frac{dL}{dt} = LPns,$$

where s is the fractional sector opening.

Hence,

$$\frac{dB}{dt} = Ksn = Ks\,f(L),$$

where $f(L)$ is some function of L experimentally determined. For instance, that found by Porter to be $a \log L + b$.

15. OTHER FORMS OF SENSATION FUNCTION.

Since no sensation can be directly measured nor more than roughly estimated, the sensation function must always be derived from some subsidiary related function. We have here derived it by integrating photometric sensibility, its derivative, so that it is subject to the same order of uncertainties as the latter. In the photometric function small divergences were smoothed over to obtain a general function of simple form for general use. It may be that this should be broken up into two separate functions, one for cone vision and the other for rod vision, with a different threshold value for each.

There are several forms of photometric function $P(I)$ that would fit the data nearly or quite as well as the one here proposed, but imposing the further conditions of integrability in simple form and that Fechner's logarithmic law must be approached at moderate intensities excludes all forms that the writer has yet been able to construct.

Hertzsprung,[12] in a paper but recently come to my notice, has constructed a radically different function to represent König's data on white light. He chooses as dependent variable not the least perceptible increment but the ratio of the just noticeably different intensities, and as independent variable the geometric mean intensity. Plotting $\log(\log I_2 - \log I_1)$ against $\frac{1}{2}(\log I_2 + \log I_1)$ he notes that the observed points lie near a parabola whose equation is

$$-\log\log\frac{I_1}{I_2} = 1.3138 + 0.43595\frac{\log I_1 I_2}{2} - 0.05796\left(\frac{\log I_1 I_2}{2}\right)^2$$

The agreement is excellent for white light except for low intensities. Unfortunately this function is not integrable nor sufficiently simple to prove very useful. It is approximately equivalent to

$$P = a\,e - (1 + bI - cI^2)$$

[12] E. Hertzsprung, Zs. Wiss. Phot., 3, p. 468; 1906.

16. THE PURKINJE EFFECT.

If monochromatic radiation, from a slit, be thrown on a screen, it is clear that both the radiation and the illumination produced by it must be proportional to the width of the slit, and hence proportional to each other. The constant of proportionality will, however, in general, vary with the wave length. This is one way of stating the Purkinje effect. If $I_\lambda = V_\lambda E_\lambda$, then, this phenomenon is involved in the fact that the constant of proportionality varies with the wave length.

At illuminations of over 200 m-c. and below 0.2 m-c. down to the threshold of vision, visibility appears from experimental data to vary little, if any, with change of intensity. (See Fig. 4.) In the blue and violet it is nearly or quite independent of wave length; in the green it is slightly greater, while in the yellow and red it increases more and more rapidly. Starting, then, with the constant values at low intensities, an expression for the total integral Purkinje effect may be easily obtained.

At low intensities, calling the maximum unity,

$$V_1 = e^{-4.6(\lambda-5.11)^2}$$

At high intensities where V is again constant

$$V_2 = e^{-2(\lambda-5.65)^2}$$

Hence, V_2/V_1 is approximately

$$\frac{V_2}{V_1} = e^{2.6(\lambda-4.70)^2}$$

These relations are shown graphically in the accompanying figure.

If at low intensity vision is by the retinal rods alone and at high intensity by both rods and cones, then the variation in V is due to the varying proportion of cones to rods in action and the above ratio represents the relative sensibility of rods and cones together to the sensibility of rods alone.

Relative sensibility for slight differences of intensity may be obtained by reading the corresponding values of κ and λ_m from tables or plotted curves such as these in Fig. 4. The assumption of light proportional to radiation for all intensities leads to the expression $\kappa(\lambda-\lambda_m)^2=\text{constant}$, and hence to κ as calculated in the last column of the table on page 282. The discrepancy is considerable in the range between ½ and 20 meter-candles, but high and low values agree well.

The relation $L=(1+P_m\,(I^nI_o^{-n}-1)^{1/n}$ between luminosity and illumination gives two more quantitative expressions for the Purkinje effect. This expression is based on I measured downward from a high intensity at which radiation in different spectral regions were brought to the same luminosity. Hence the measured value of the threshold limen I_o as a function of wave length is an expression for the total Purkinje effect. Again, if luminosity is proportional to illumination at high intensities (over 200 m-c.), then $P_m^{1/n}$ is another expression for the total Purkinje effect. Appended to the table on page 288 are computed values of $0.016^{1/n}$.

In the table below are given series of values for the total Purkinje effect relative to λ 4.70 s-m together with their common logarithms. They are respectively the exponential of $2.6\ (\lambda-4.7)^2$, $P_m^{1/n}$, the observed values of I_o measured down and H/T from the table on page 279.

λ	$V_2:V_1$	$P_m^{1/n}$	I_o	H:T
4.30	1.52 0.183	1.00 0.00	1.00 0.00	
4.70	1.00 0.00	1.00 0.00	1.00 0.00	1.00 0.00
5.05	1.37 0.137	1.57 0.196	1.42 0.152	1.73 0.238
5.75	17.4 1.265	4.20 0.623	24.2 1.382	31.4 1.497
6.05	144. 2.16	8.46 0.927	46.7 1.67	203. 2.307
6.75	302. 2.48	500. 2.70		

These results show roughly the course of the Purkinje effect The numerical quantities being calculated by such different methods from such widely different and uncertain data, no very close agreement could be expected, particularly since exponentials and ratios.

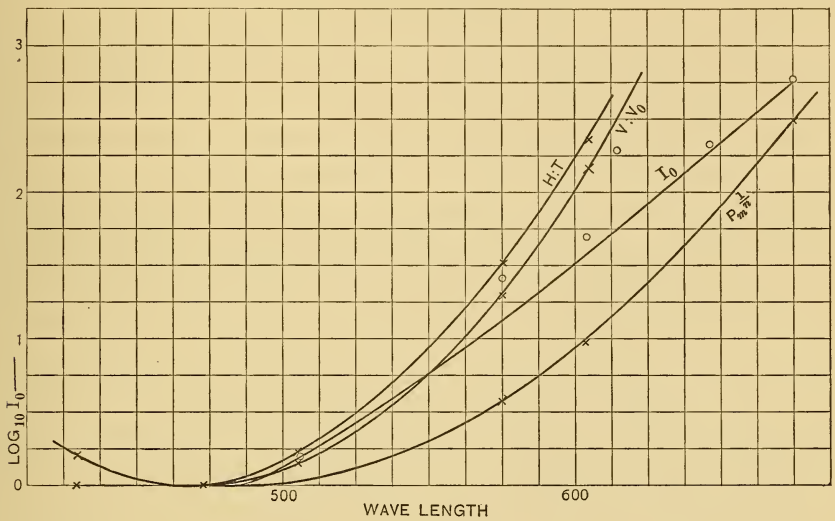

Fig. 8.

of small differences are involved. The Purkinje effect is a direct consequence of the varying of the ratio of light to radiation; no attempt at a quantitative statement of it has heretofore been made.

The ratio of light to radiation as a function of intensity is shown graphically in Fig. 3, page 281, under the assumption of maximum ordinate equal to unity and intensity step measured at λ 535 $\mu\mu$. It is hardly worth while to plot L as a function of E here. These curves would evidently consist of two straight extremities joined by a short curve for each wave length.

IV. CONCLUSION.

17. COMPLETE LIGHT SCALES.

The logical conclusion of such a paper as this would be tables of numerical values of the luminous equivalent of radiation at each wave length and for all intensities together with the corresponding algebraic expressions for the functions involved. To do this would

require a statistical treatment of data relative to the properties of a great number of human eyes. However, fairly complete data on a few eyes have been available, the essential functions have been tentatively constructed, the general problem outlined and the way cleared for incisive experimental work. Eventually it should be possible, as it is highly desirable, to define for general adoption the properties of the average human eye and so completely establish a light scale in terms of radiation.

The functions involved in the luminous equivalent of radiation and its determination are tabulated below:

1. $E(\lambda)$ *Radiation*, spectral energy, in watts per unit wave length.
2. $V(\lambda)$ *Visibility* of radiation, chromatic visual sensibility, a spectral factor converting radiation into (objective) light.
3. $I(\lambda, E)$ *Illumination*, luminous intensity, objective light, proportional to radiation in intensity. $I = VE$.
4. $P(I)$ *Photometric function*, difference limen, least perceptible increment, inverse relative sensibility.
5. $L(I)$ *Luminosity*, subjective corresponding to illumination objective, the argument of the function $B(I)$.
6. $B(I)$ *Visual brightness*, the sensation produced by the illumination I, logarithmic function of L.
7. $l(t)$ *Time function*, lag of sensation behind stimulus.

Of these, E is supposed given; V, P, and $L(t)$ rest entirely upon experimental data, while I, $L(I)$, and B are functions derived from E, V, and P. B is a sensation, hence incapable of precise measurement. Luminosity $L(I)$ is an auxiliary function introduced and defined because of its fundamental importance in practical work as well as in the theory of light.

The assumed and derived forms of these functions are as follows:

$E(\lambda)$, spectral energy curve of source, supposed given.

$V(\lambda) = E/I$ determined experimentally by varying E while holding I (or strictly, L) constant

$$V = V_0 e^{-\kappa(\lambda-\lambda_m)^2}$$

approximately, where V_0, κ and λ_m are functions of E.

$$I = VE$$

for any source once V is known.

$$P = P_m + (1 - P_m)\, I_o^n\, I^{-n}$$

where I is measured downward from a high intensity. I_o (so measured) and n are functions of wave length while Fechner's constant P_m is not.

$$L = (1 + P_m (I^n I_o^{-n} - 1))^{1/n},$$

the argument of $B(I)$. Subject to the same conditions and uncertainties as $P(I)$.

$$B = B_o \log (1 + P_m (I^n I_o^{-n} - 1))^{1/n} = B_o \log L$$

obtained by integrating $1/IP$ under the assumption that IP is inversely proportional to visual sensibility to intensity differences. Subject to the same conditions and uncertainties as $P(I)$.

$$l = L\,(1 - e^{-\alpha t})\,(1 - e^{-\beta(t - t_o)})$$

The form of V at low intensities (rod vision) is fairly well established, but the value of the constant V_o in light units per watt is not known to within a factor of ten at best. If Langley's eye was lacking in retinal rods, as the form of his visibility curve would lead us to suspect, then V_o for rod vision is about twenty times that for cone vision, or roughly, 250 candles per watt. No other data of any kind on this constant is known to the writer. The values of κ and λ_m should rest upon data from hundreds of individuals of diverse nationality instead of upon barely twenty and those chiefly Teutonic.

The form of P and its parameters should be determined for a number of individuals using König's method and the best modern light sources instead of gas flames. Critical frequency and visual fatigue deserve much further study by the most precise methods.

18. THE DEFINITION OF WHITE LIGHT.

Experiment shows that light is sensibly "white" when either (*a*) E = constant throughout the visible spectrum or (*b*) light of three or more colors is combined in certain proportions. According to generally accepted theories of color vision, the visibility curve $V(\lambda)$ is a composite of three elementary visibility curves such that

$$V = V_1 + V_2 + V_3.$$

These elementary curves appear to be indeterminate at present chiefly because there are more variables to be determined than equa-

tions to determine them. By assuming them to be of equal area (König and Dieterici) or equal maximum ordinate or of exactly the same form, they may be determined roughly by mixing known amounts of arbitrary colors to produce white light. Assuming them to be known and taking $I_1 = EV_1$, etc., evidently the criterion for white light is that

$$\int I_1 d\lambda : \int I_2 d\lambda : \int I_3 d\lambda = \int V_1 d\lambda : \int V_2 d\lambda : \int V_3 d\lambda$$

whatever the form of the spectral energy curve E (λ).

One of the best recent determinations of these curves[13] is reproduced in the accompanying figure (Fig. 9), where equality of area has been assumed. All these curves overlap, to some extent, the red and green very widely. From ordinary experience one would say that the blue curve is somewhat too narrow and much too high and too steep at the foot of the short wave length side. From these curves it would appear that the most nearly monochromatic source to give a sensation of white light would be a heavy line located between $\lambda 560$ and $\lambda 580$ balanced up with a little greenish blue of wave length 480 $\mu\mu$.

The question of the best source of white light often arises. No economical source of pure white (E = const. or as defined above) light is known or perhaps possible. Sunlight and the light from carbon at a temperature of 4000 degrees are approximately white, since their spectral energy curves are nearly constant throughout the visible spectrum, but contain a large proportion of radiation of low visibility. The whitest discontinuous sources are the incandescent vapors of sulphur and uranium and helium gas, but these emit a very large proportion of waste energy.

An emission curve, the shape of the visibility curve (compare Fig. 2), would not correspond with white but with decidedly greenish white light.

The best source would probably be not pure white but slightly greenish or yellowish consisting of radiation nearly uniform in intensity between 530 and 610 $\mu\mu$ or else of a spectrum line at or near 570 balanced by a fainter one at 480 $\mu\mu$. It is remarkable to what extent light may differ from white and yet with steady exclu-

[13] F. Exner, Wien. Sitz., **111** (IIa), p. 837; 1902.

sive use appear but slightly tinted. In other words, the visual color accommodation is large.

Highly colored illumination, like that from the mercury glow lamp, although not injurious to the eyes, appears to be of low visual economy, that is a higher luminosity is required to produce the

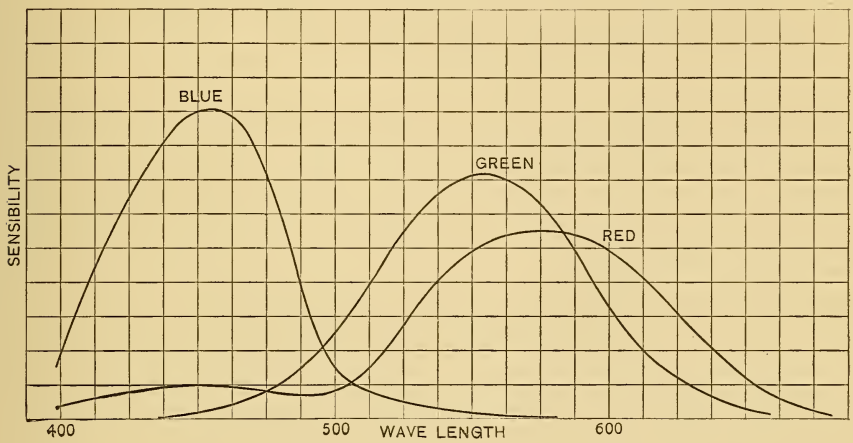

Fig. 9.

same comfort and acuity of vision. There is a real gain in focal achromatism for which the eye is, however, already well provided. Fig. 9 shows that at least two of the three sets of cones are in use with monochromatic illumination in every part of the spectrum except the violet and extreme red.

19. UNITS OF ILLUMINATION.

Neither subjective nor objective light is expressible in purely mechanical units since the magnitude of each is determined by a pure sensation. The units involved being entirely arbitrary, may be chosen according to convenience. Four units of illumination commend themselves:

1. The threshold value is from a mathematical standpoint by far the most suitable, since measuring in terms of this would greatly simplify the mathematical expressions involved. It is much too small, however, for use as a practical unit; it is difficult or impossible of precise determination and is separated from the region of common use by the whole Purkinje effect.

2. The meter candle, lux, and lumen now in ordinary use have the advantage of familiarity, but are small and within a region of intensity heavily burdened by the Purkinje effect.

3. A much higher intensity of, say, 1000 meter candles has much in its favor. This is about half that on a horizontal plane in the open on a cloudy day. It is a good working intensity and safely above the Purkinje effect. It is objectionable chiefly in that ordinary artificial illuminations expressed in terms of it would be fractions.

4. Another unit of similar magnitude, but different character from (3), would be one-thousandth of a watt per square cm of radiation lying in the spectral region between 560 and 580 $\mu\mu$. The last two units would be large enough for accurate measurement as radiant energy as well as light.

20. STANDARDS OF LUMINOSITY.

Evidently the logical primary standard of illumination is one in which the measured quantity is radiation measured as radiation by some form of radiometer and not a rate of combustion, a linear dimension, or an electric current. Such a primary standard was recently proposed by Steinmetz,[14] who advocated the use of the three mercury lines, blue 435, green 546, and red 691 $\mu\mu$. Referring to Fig. 2, the red and blue are seen to be rather too far out, while Fig. 9 indicates that far from equal energy proportions would be required to produce white. The principal objection to Steinmetz's proposition is the difficulty in getting sufficient intensity for radiometry, particularly in the red.

The least intensity of radiation that can be measured with the necessary certainty is about 0.0001 watt/cm^2. This is approximately 200 meter-candles at 570 (green-yellow), 70 m-c. at 640 in the red just beyond the orange, and an equal amount in the blue green at 500 $\mu\mu$. This means quite intense sources, but sources safely above the Purkinje effect. A glow lamp burning in air like the Nernst would appear to be the most practicable. Two or three segments of selected wave lengths from the continuous spectra of such a source could easily be obtained of sufficient intensity and purity.

[14] C. P. Steinmetz, Proc. A. I. E. E., March, 1908.

21. LUMINOSITY AND TEMPERATURE.

For perfect radiators, the so-called black bodies, $E(\lambda)$ is known as a function of temperature, hence with $V(\lambda)$ known and constant (at high intensities), the light emitted by such a body may be expressed as a function of temperature. Being concerned with the visible spectrum we may safely use the simple Wien-Paschen formula

$$E(\lambda) = C_1 \lambda^{-n} e^{-C_2/\lambda T} \tag{1}$$

This multiplied by visibility gives the optical intensity of the radiation. Using the probability curve visibility function

$$V = V_0 e^{-\kappa(\lambda-\lambda_m)^2} \tag{2}$$

$$I = EV = C_1 V_0 \lambda^{-n} e^{-C_2/\lambda T - \kappa(\lambda-\lambda_m)^2} \tag{3}$$

Hence the isochromatic relation is of the form

$$\log I = A - B/T \tag{4}$$

where

$$A = \log (C_1 V_0 \lambda^{-n}) - \kappa(\lambda - \lambda_m)^2$$

and

$$B = C_2/\lambda$$

The value of the constant B is seen to be independent of the form of visibility function assumed while the value of A is not. Further, the form of function (4), namely $I(T)$, is independent of the form of the visibility curve being determined solely by the form of radiation function $E(\lambda)$ chosen.

Equation (4) assumes a single, constant wave length, hence it can apply to but a very limited spectral region. It further assumes that the form of the visibility function does not change with intensity, so that if used at moderate and low intensities account must be taken of variations in κ and λ_m. It might be found experimentally to hold approximately for the total light emission because about 90 per cent of the light is confined to a region only 0.*1* μ broad, but it could never be considered of general validity for such a case.

The relation between total light emission and temperature is obtained by integrating $EVd\lambda$ from zero to infinity. Function (3)

above is not integrable, but using the alternative form (see page 277) of visibility function

$$V = V_o\left(\frac{\lambda_m}{\lambda}\right)^{\nu} e^{\nu\left(1-\frac{\lambda m}{\lambda}\right)} \tag{5}$$

the product EV is readily integrable in the form

$$I = \int_o^{\infty} EVd\lambda = C_1 V_o \lambda^{\nu}{}_m e^{\nu}\Gamma(n+\nu-1)\left(\frac{C_2}{T}+\nu\lambda_m\right)^{1-n-\nu} \tag{6}$$

which is in the functional three constant form

$$I = A\left(\frac{B}{T}+1\right)^{-s} \tag{7}$$

when

$$S = n+\nu-1$$
$$B = C_2/\nu\lambda_m$$
$$A = C_1 V_o \lambda^{\nu}{}_m e^{\nu}\Gamma(n+\nu-1)(\nu\lambda_m)^{-s}$$

function (7) is subject to the same high intensity limitations as (4); it can be safely used only at intensities of over about 100 meter candles unless variations in the parameters are taken into account.

22. PRACTICAL APPLICATIONS.

To obtain the maximum sensibility in many measurements involving visual observation, some of the above results may be applied to advantage. A few of these applications are briefly outlined below. If necessary, they may be considered more in detail in a later paper.

Photometry, as a determination of luminous intensity, may be accomplished either by bringing known and unknown to equality by a zero method (ordinary photometry) or by the direct measurement of intensity by an empirical method. Only when the spectral distributions in the two sources are identical is the ordinary photometric method strictly a zero method. Readings are then independent of the observer, be he color-blind or of normal vision, and of the absolute intensity at which comparisons are made. Uncertainties in readings vary with the individual, and are least at intensities of illumination of from 500 to 50,000 m-c., but at intensities as low as 5 m-c. are only three times the minimum. When the sources

compared are not spectrally identical, the ordinary photometric method is not strictly applicable, for zero methods can not be used. With corrections for the observer's eye according to a fixed average luminous equivalent, the method could still be used or observations could be made with rod vision and similar corrections applied.

The direct measurement of intensity by observing critical frequency or visual acuity is less accurate than the zero methods but has the advantage of simplicity and freedom from color effects.

In pyrometry or wherever color screens are used, the shift with intensity of the center of luminosity of the transmitted band must be taken account of. Here the transmitted light is a resultant of three curves; the spectral energy of the source, visibility, and the transmission curve for the screen used.

In chemical work, the so-called colorimeters used in comparing the depths of the colors of two solutions are ordinarily used at too low illuminations of field to attain maximum sensibility. An illumination equivalent to a thousand meter candles in the resultant field is still too low for the highest sensibility.

In astronomy the measurement of star magnitude depends upon the color of the star and the visual sensibility of the observer to the extent of a large fraction of a magnitude. Corrections should be applied for color and observer according to visibility and visual properties.

In polarimetry the light intensities used are necessarily low. With a simple analyzing nicol it is, of course, merely a question of the use of the brightest possible source, for the maximum sensibility is the square root of the ratio of the intensity of the source to the threshold value. For a half shade instrument it has been shown[15] that for maximum sensibility the analyzing angle ($=2\delta$) is given by

$$\frac{P}{I}+\frac{dP}{dI}=0 \text{ where } I=E^2\delta$$

I being the light intensity in the visual field, E that at the source, and P being the photometric function as used in this paper. Substituting the value of P, I is given by

$$I^n P_m=(n-1)(1-P_m)I_0^n$$

[15] This Bulletin, 2, p. 258; 1906.

from which the analyzing (or polarizing) angle may be calculated when the intensity E of the source used is known.

In spectroscopy it is customary to record impressions of the relative intensities (visual or photographic) of spectrum lines, some bright line being 10 or 100 and some very faint line unity. These are of course but rough estimates of purely subjective sensations. It was in a search for a rational and practicable intensity scale that this work was undertaken. We have here to deal, not with fixed quantities like stellar magnitudes, but with intensities varying with every variation in either source or set up as well. For the more precise work the best procedure appears to be a radiometric measurement through photometric comparison with an intermediate source of known spectral energy distribution. In less accurate work the simpler method of critical frequency for no flicker might be employed in preference.

It has been the aim of the writer in this paper to establish a scientifically precise yet practically manageable relation between light and its sole measurable constituent, radiation. It is hoped that this preliminary paper may pave the way toward more and better data and to a more complete solution of the general problem in all its details. The next step is the general adoption of a suitable nomenclature. Ultimately we may perhaps be able to define and adopt the visual properties of the average normal human eye. When that has been established, light may be precisely defined in terms of radiation and all values of the luminous equivalent of radiation, as a function of wave length and intensity, completely established.

WASHINGTON, September 3, 1908.

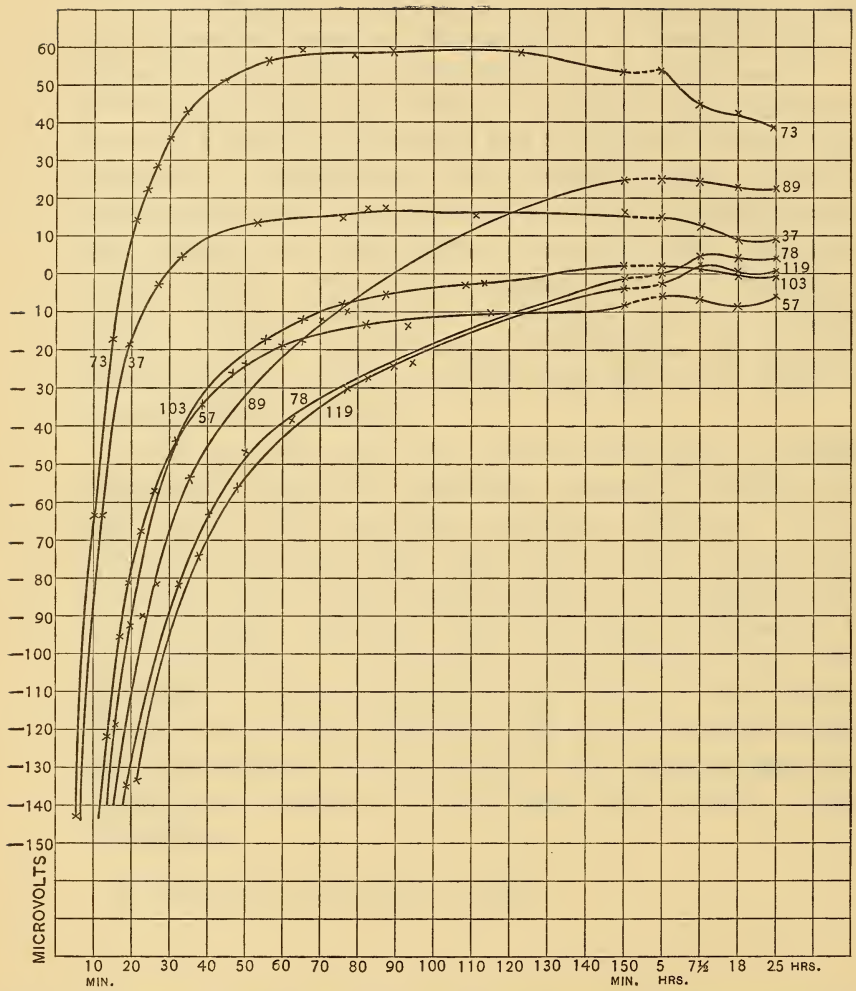

Fig. 1.—*Differences of individual cells from normal values after an abrupt change of temperature from 25° to 4°.8.*

52839—08. (To follow page 308.)

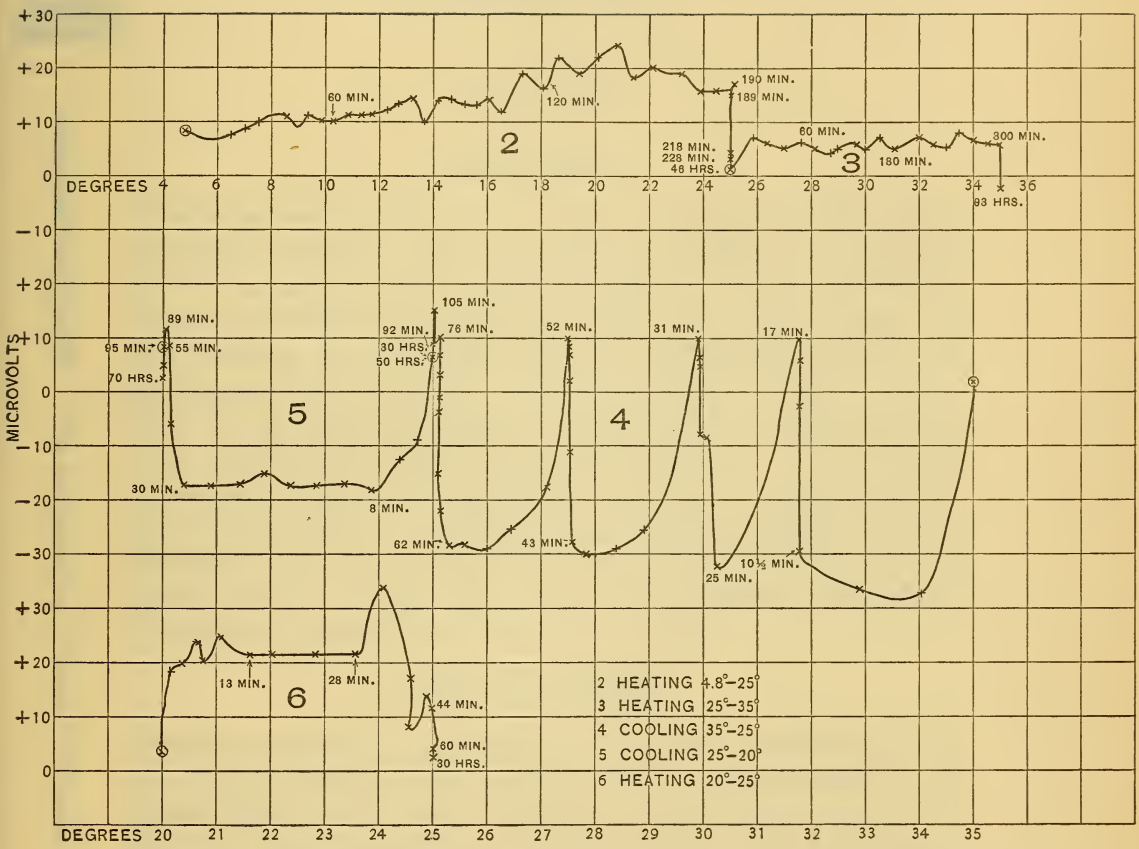

Fig. 2.—(*Curves 2–6.*) *Average differences of seven cells from normal values during temperature changes.*