

THE SELF-INDUCTANCE OF A COIL OF ANY LENGTH WOUND WITH ANY NUMBER OF LAYERS OF WIRE.

By Edward B. Rosa.

The self-inductance of a coil or short solenoid wound with any number of layers of wire is given by the formula of Weinstein, or Stefan's modification of Weinstein's formula, when one applies the proper corrections for the thickness of the insulation and the shape of the section of the wire. But for a long solenoid this formula is not very accurate. I have elsewhere shown how to obtain the self-inductance of a long solenoid wound with a single layer of round covered wire or with bare wire wound at any given pitch. I propose now to show how one may obtain accurately the self-inductance of a solenoid of any length having a uniform winding of any number of layers; this will include the case of short coils as well as those where the length is too great to be calculated by the formulæ of Weinstein or Stefan.

Mr. Cohen gives elsewhere in this Bulletin an approximate formula for the self-inductance of relatively long coils of more than one layer. His formula is convenient in calculation when the number of layers is not large, and is accurate enough for most practical cases, notwithstanding it assumes the current to be distributed in current sheets, taking no account of the shape of the cross section of the wire or the thickness of the insulation. I shall now show how to calculate accurately the self-inductance of a coil of any length and any number of layers, wound with insulated round wire, taking account of the shape of the section as well as the thickness of the insulation of the wire.

Let Fig. 1 be the section of such a winding of mean radius a , length l , and depth of winding t , and having m layers. If n is the number of turns per centimeter,

$$m = nt$$

$$N_1 = nl = \text{number of turns per layer}$$

$$N = mnl = \text{whole number of turns.}$$

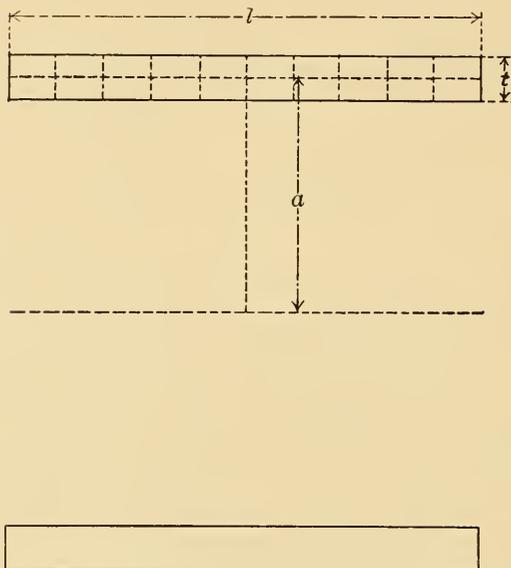


Fig. 1.

Suppose the section is divided into squares, as shown in Fig. 1, of which there will be $l/t = n'$. We shall find first the self-inductance of a single layer winding of n' turns of square wire, which completely fills the section; that is, when the insulation between turns is of infinitesimal thickness. To do this we first find the self-inductance L_s of a thin current sheet of n' turns, radius a and length l , by Lorenz's formula, which is as follows:

$$L_s = \frac{4\pi n'^2}{3l^2} \left\{ d(4a^2 - l^2)E + dl^2F - 8a^3 \right\} \quad (1)$$

where $d = \sqrt{4a^2 + l^2}$, a , l and n' having the meanings explained above, and F and E are the complete elliptic integrals to modulus k ,

$$\text{where } k = \frac{2a}{d} = \frac{2a}{\sqrt{4a^2 + l^2}}$$

The next step is to find the difference between the self-inductance L_s for the thin current sheet and the self-inductance L_t for the thick current sheet, of thickness t and n' turns. Following the method given in a previous paper¹ for a single layer winding of

¹This Bulletin, 2, p. 161, 1906.

round wire, we have to find

$$\Delta L = \Delta L_1 + \Delta M \quad (2)$$

where ΔL_1 is the difference in the self-inductances of the n' turns of square wire making up the thick current sheet and the n' turns of thin tape making up the thin current sheet; and ΔM is the corresponding difference in the mutual inductances. The correction ΔL is to be subtracted from L_s .

CORRECTION FOR SELF-INDUCTANCE.

The self-inductance of one turn of thin strip of width t is given approximately by the expression

$$L'_s = 4\pi a \left\{ \log \frac{8a}{R_s} - 2 \right\}$$

and for one turn of wire of square section

$$L'_t = 4\pi a \left\{ \log \frac{8a}{R_t} - 2 \right\}$$

In the above equations R_s is the geometrical mean distance of a line of length t from itself; R_t is the geometrical mean distance of a square of side t .

The difference between these two expressions is

$$L'_s - L'_t = 4\pi a \left(\log \frac{R_t}{R_s} \right) \quad (3)$$

$$R_t = .44705 \quad R_s = .22313$$

$$\therefore \frac{R_t}{R_s} = 2.0035 \text{ and } \log \frac{R_t}{R_s} = 0.6949$$

Hence the difference in the self-inductances of the n turns of the thick and the thin current sheet is

$$\Delta L_1 = 4\pi a n [0.6949] = 4\pi a n A \quad (4)$$

We may derive the value of the constant A somewhat more accurately otherwise as follows:

The self-inductance of one turn of thin tape of width b by Rayleigh's formula is

$$L'_s = 4\pi a \left\{ \log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2} \left(\log \frac{8a}{b} + \frac{1}{4} \right) \right\} \quad (5)$$

The self-inductance of one turn of square wire (of side b) by Weinstein's formula is

$$L'_t = 4\pi a \left\{ \log \frac{8a}{b} - 1.1949 + \frac{b^2}{24a^2} \left(\log \frac{8a}{b} + 0.8777 \right) \right\} \quad (6)$$

The difference between these two is

$$L'_s - L'_t = 4\pi a \left\{ 0.6949 - \frac{b^2}{96a^2} \left(\log \frac{8a}{b} + 2.76 \right) \right\}, \text{ nearly.} \quad (7)$$

The first term of this expression is the same as found above (4); the second term, depending on the dimensions of the coil, is small and may often be neglected.

If $a=5$, and $b=1$, this is

$$\begin{aligned} L'_s - L'_t &= 4\pi a (0.6949 - .0027) \\ &= 4\pi a (0.6922) \end{aligned}$$

$$\therefore \Delta L_1 = 4\pi a n A, \text{ where } A = 0.6922$$

Thus the value of the constant A is 0.6949 when b/a is very small, and decreases slightly (to 0.6922 in this particular case) as b/a increases. Equation (7) may be used to find its value accurately for any given case.

TABLE I.

Value of the Constant A as a Function of t/a , t being the depth of the Winding and a the Mean Radius.

t/a	A
0.	0.6949
0.10	0.6942
0.15	0.6933
0.20	0.6922
0.25	0.6909

CORRECTION FOR MUTUAL INDUCTANCE.

The correction for mutual inductance is nearly the same as for a single layer winding of round wire. It would be exactly the same if the geometrical mean distance of square areas from each other were the same as for circular areas. This is true almost exactly except for very near areas, as the adjacent turns. Table VIII of my previous paper² gives the values of the constant B for round wires, and these values may be used in this case without serious error. The correction depending on mutual inductances is

$$\Delta M = 4\pi anB \quad (8)$$

To obtain somewhat more accurate values of B we should use the values of the geometrical mean distances of square areas, as shown in column 2, Table II. These are sensibly the same as for round areas beyond a distance equal to five times the diameter of the squares. Hence only the first five values of δ in Table I differ from those of Table III² of the previous paper, referred to above.

TABLE II.

Table of Geometric Mean Distances and Corrections Depending upon Them.

G. M. D. R for Tapes	G. M. D. R' for Squares	Ratio, $\frac{I}{k} = \frac{R'}{R}$	$\delta = \log \frac{I}{k}$
$R_1 = 0.89252$	1.00655	1.12776	0.12025
$R_2 = 1.95653$	2.00102	1.02274	.02249
$R_3 = 2.97171$	3.00030	1.00962	.00958
$R_4 = 3.97890$	4.00013	1.00531	.00530
$R_5 = 4.98323$	5.00007	1.00337	.00337
$R_6 = 5.98610$	6.00003	1.00233	.00233
$R_7 = 6.98806$	7.0000	1.00171	.00171
$R_8 = 7.98957$	8.0000	1.00131	.00131
$R_9 = 8.99076$	9.0000	1.00103	.00103

From these values of δ and the succeeding values as given in Table III² we can calculate the table of values of B , corresponding

²This Bulletin, 2, p. 161.

to Table VIII for round wires. The constant B is given by

$$B = \frac{2}{n} \sum_{m=n-1}^{m=1} \left(m \log \frac{1}{k} \right) \quad (9)$$

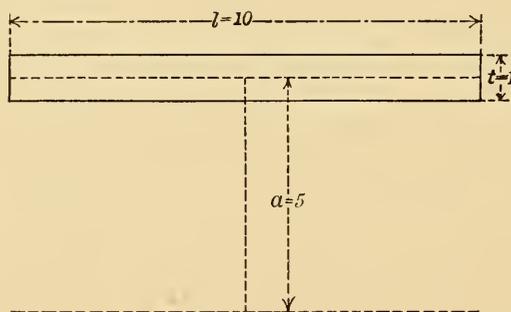
TABLE III.

Values of Correction Term B, Depending on the Number of Turns of Square Conductor on Single Layer Coil.

No. of Turns	B	No. of Turns	B
1	0.0000	16	0.3017
2	.1202	17	.3041
3	.1753	18	.3062
4	.2076	19	.3082
5	.2292	20	.3099
6	.2446	21	.3116
7	.2563	22	.3131
8	.2656	23	.3145
9	.2730	24	.3157
10	.2792	25	.3169
11	.2844	26	.3180
12	.2888	27	.3190
13	.2927	28	.3200
14	.2961	29	.3209
15	.2991	30	.3218

EXAMPLE 1.

As a first example let it be required to find the self-inductance of a coil of mean radius $a = 5$ cm, length $l = 10$ cm, depth of winding $t = 1$ cm, and uniformly wound with 10 turns of wire per cm, having



therefore 10 layers of 100 turns per layer, or a total of 1,000 turns. We first find the self-inductance L_s of a current sheet of 10 turns, radius $a = 5$, $l = 10$.

Using formula (1), we have the following values:

$$\begin{aligned} d &= \sqrt{4a^2 + l^2} = \sqrt{200} & k &= \frac{2a}{d} = \frac{1}{2}\sqrt{2} = \sin \gamma \\ 4a^2 - l^2 &= 0 & & \therefore \gamma = 45^\circ \\ n' &= 10 \\ n'/l &= 1 \end{aligned}$$

Hence

$$L_s = \frac{4\pi}{3} \left\{ 1000\sqrt{2} F - 8 \times 125 \right\} = \frac{4000}{3} \pi \left\{ \sqrt{2} F - 1 \right\} \quad (10)$$

The value of F for $\gamma = 45^\circ$ is 1.85407. Substituting in

$$L_s = \frac{4000}{3} \pi \left\{ 1.62205 \right\} = 2162.73\pi \text{ cm}$$

To find the correction ΔL we must have the two constants A and B . By equation (7), or Table I, $A = .6922$. By Table II, $B = .2792$. Hence

$$\begin{aligned} \Delta_1 L &= 4\pi a n (A + B) \\ &= 200\pi (0.6922 + .2792) \\ &= 194.28\pi \text{ cm} \\ L_u &= L_s - \Delta_1 L = 1968.45\pi \text{ cm.} \end{aligned} \quad (11)$$

Thus the thick current sheet (10 cm x 1 cm) of 10 turns has a self-inductance L_u about 9 per cent less than that of a thin current sheet L_s of the same length and radius equal to the mean radius of the thick sheet.

If the coil were wound with 1,000 turns of square wire (the insulation being of infinitesimal thickness), then the current would be uniformly distributed over the entire section of the winding, and the self-induction would be $\overline{100}^2$ times as great as for 10 turns, being proportional to the square of the number of turns. In general, we get the self-inductance for m layers by multiplying the value for one layer by m^4 . This, of course, supposes a uniform winding such that m layers occupy as much space vertically as m turns horizontally on

the coil. In this case the self-inductance will be

$$\begin{aligned} L_u &= 1968.45\pi \times 10^4 \text{ cm} \\ &= 19.6845\pi \text{ millihenrys} \\ &= 61.841 \text{ millihenrys} \end{aligned} \quad (12)$$

The subscript u signifies uniform distribution of current over the entire cross section of the coil. We must now find the self-inductance L for this coil when wound with 1,000 turns of *round insulated* wire. Suppose the bare wire to be 0.8 mm in diameter, the covering being 0.1 mm thick; there will be three correction³ terms to apply, C , F , and E , each of which gives an increase in L . The first expresses the difference in L , due to the wire being round instead of square, as previously assumed; the second expresses the difference in L , due to the wire being of smaller diameter than if there was no appreciable thickness of insulation; the third expresses the corresponding difference in the mutual inductances.

The sum $\Delta_2 L$ is then as follows:

$$\begin{aligned} \Delta_2 L &= 4\pi na(C + F + E) \\ C &= 0.1381 \text{ for the case of round wire.} \\ F &= 0.2231 = \log_e \frac{D}{d} = \log_e 1.25 \text{ in this case.} \\ E &= 0.0172 \text{ for this size and shape of section.}^3 \\ C + F + E &= 0.3784 \\ \therefore \Delta_2 L &= 4\pi \times 5000 [0.3784] \\ &= 7568\pi \text{ cm} = 23776 \text{ cm} \\ &= 0.0238 \text{ millihenrys} \end{aligned}$$

This correction $\Delta_2 L$ amounts in this particular case to about 4 parts in 10,000 of L . We have now finally

$$L = L_u + \Delta_2 L = 61.865 \text{ millihenrys} \quad (13)$$

The same method may be used for a coil of any length and any number of layers.

EXAMPLE 2.

As a second illustration of this method of calculating the self-inductance of a coil of any number of layers let us take a very short

³This Bulletin, 3, pp. 34 and 37, 1907.

coil and check the result by the formula of Stefan. Let the mean radius be 10 cm and the cross section of the winding 1 x 1 cm. For the self-inductance L_s of the current sheet of radius a and length b we will this time use Rayleigh's formula, which is very accurate where b/a is as small as 1/10. Rayleigh's formula is

$$L_s = 4\pi a N^2 \left\{ \log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2} \left(\log \frac{8a}{b} + \frac{1}{4} \right) \right\}$$

Here $\frac{8a}{b} = 80$ $\log_e 80 = 4.382027$

$$\frac{b^2}{32a^2} = \frac{1}{3200} \therefore \frac{b^2}{32a^2} \left(\log_e 80 + \frac{1}{4} \right) = \frac{0.001448}{4.383475} - \frac{.50}{3.883475}$$

$$\therefore L_s = 4\pi a N^2 \times 3.883475$$

This is the self-inductance if the current is condensed into a ring of zero thickness; that is, $a=10$, $b=1$, $c=0$. By equation (7) the correction term $A=0.69415$ (see also Table I) and $B=0$, since $n'=1$ (Table III).

$$\begin{aligned} \therefore \Delta_1 L &= 4\pi a N^2 (.69415) \\ \therefore L_u &= L_s - \Delta_1 L = 4\pi a N^2 \times 3.18932 \text{ cm.} \end{aligned} \quad (14)$$

If $N=400=20$ turns per cm and 20 layers, we have for L_u , a being 10 cm,

$$\begin{aligned} L_u &= 6400000 \times 3.18932 \text{ cm} \\ &= 64.125 \text{ millihenrys.} \end{aligned} \quad (15)$$

This is the self-inductance if the wire is square and fills the entire section. The corrections C , F , and E must be applied as in example 1 to reduce to the actual case of a winding of round, insulated wire. In this case, supposing the bare wire is 0.3 mm in diameter, and hence $D/d=5/3$.

$$\begin{aligned} C &= 0.1381 \\ F &= 0.5108 \\ E &= 0.0176 \\ \hline C+F+E &= 0.6665 \end{aligned}$$

$$\begin{aligned}
 \therefore \Delta_2 L &= 4\pi n a (C + F + E) \\
 &= 16000\pi \times 0.6665 \\
 &= 33,502 \text{ cm.} \\
 &= .0335 \text{ millihenrys} \\
 \therefore L &= L_u + \Delta_2 L = 64.1585 \text{ millihenrys.} \quad (16)
 \end{aligned}$$

We can check the value of L_u by the formula of Weinstein or that of Stefan. These formulae assume the current to be uniformly distributed over the section of the coil; that is, that the wire is square and fills the entire section.

Stefan's formula is as follows:

$$L_u = 4\pi a n^2 \left\{ \left(1 + \frac{3b^2 + c^2}{96a^2} \right) \log_e \frac{8a}{\sqrt{b^2 + c^2}} - y_1 + \frac{b^2}{16a^2} y_2 \right\} \quad (17)$$

In this case $a = 10$, $b = c = 1$ $y_1 = 0.84834$ $y_2 = 0.8162$

$$\log_e \frac{8a}{\sqrt{b^2 + c^2}} = \log_e \frac{80}{\sqrt{2}} = 4.03545$$

$$\left(1 + \frac{3b^2 + c^2}{96a^2} \right) \log_e \frac{80}{\sqrt{2}} = 4.03713$$

$$\frac{b^2}{16a^2} y_2 = \frac{0.00051}{4.03764}$$

$$y_1 = \frac{0.84834}{3.18930}$$

$$3.18930$$

$$\therefore L_u = 4\pi a N^2 \times 3.18930 \quad (18)$$

This differs from the value found above (14) for L_u by less than 1 part in 100,000.

It is of course to be expected that these two methods of obtaining L_u would agree, from the method of obtaining the correction term A . But an actual numerical test is nevertheless not superfluous.

When the coil is long, so that Stefan's (or Weinstein's) formula does not apply, the method given above for obtaining L_u from L_s is the same and just as accurate provided the correction term B is accurately known. That term here differs very slightly from its value for single layer coils where it has been already verified, so we

may be sure that L_u and so also L for any winding may be obtained with great precision by the above method.

Let us now use this method to determine the magnitude of the error in results obtained by Stefan's formula when applied to coils longer than those for which this formula is accurate.

EXAMPLE 3.

$$a = 10, b = 10, c = 1$$

By Coffin's formula (extension of Rayleigh's) we have

$$\begin{aligned} L_s &= 4\pi a N^2 (1.57944 + .07280 - .00138 + .00009) \\ &= 4\pi a N^2 \times 1.65095 \end{aligned} \tag{19}$$

$$A = .6942 \quad \text{as before}$$

$$B = .2792 \quad \text{Table II}$$

$$A + B = .9734$$

$$\Delta_1 L = 4\pi a n' \times (.9734)$$

Here $n' = b/c = 10$, and hence $A + B$ must be divided by 10 to put the correction in the same form as L_s . That is, if $n = 10$,

$$\Delta_1 L = 4\pi a n'^2 [.09734]$$

Therefore, if $n' = N$ we have

$$L_u = L_s - \Delta_1 L = 4\pi a N^2 [1.5536] \tag{20}$$

So far we have assumed the section 10×1 cm to be divided into 10 squares. But N may now be any number, and the above expression L_u gives the value of the self-inductance, assuming the current is uniformly distributed over the section.

Another way of putting it is to put $n = m^2 n'$, where m is the number of layers and $m n'$ is the number of turns per layer. In the above example $n = 10$. It is always b/c . The correction $\Delta_1 L$ is

$$\Delta_1 L = 4\pi a n' (A + B) \text{ for } n' \text{ turns}$$

$$\therefore \Delta_1 L = 4\pi a n'^2 \left(\frac{A + B}{n'} \right) \text{ for } n' \text{ turns}$$

But if the winding instead of being one layer of n' turns is m layers of $m^2 n'$ turns, the self-inductance and also the correction $\Delta_1 L$ will be m^4 times as great. Therefore

$$\begin{aligned}\Delta_1 L &= 4\pi a m^4 n'^2 \left[\frac{A+B}{n'} \right] \\ &= 4\pi a N^2 \left[\frac{A+B}{n'} \right] \text{ since } N = m^2 n' \quad (21)\end{aligned}$$

Hence $(A+B)$ is always divided by n' . In example 2, $n' = 1$.

By Stefan's formula we have, since $y_1 = 0.59243$, $y_2 = 0.1325$

$$\begin{aligned}L_u &= 4\pi a N^2 \left\{ \left(1 + \frac{301}{9600} \right) 2.074467 - 0.59243 + \frac{0.1325}{16} \right\} \\ &= 4\pi a N^2 \times 1.55536 \quad (22)\end{aligned}$$

This is 1 part in 900 more than the value obtained above (20) by the more accurate method, showing that Stefan's formula is not seriously in error for a coil whose length is equal to its radius, but it is much less accurate than for short coils, as its method of derivation requires.

For a coil only half as long ($b=5$, $c=1$) the difference between the two methods is only one part in 4,000, while for a coil twice as long ($b=20$, $c=1$) the difference is about 1 per cent. That is, Stefan's very convenient formula is correct to within 1 per cent for a coil whose length is as great as its diameter. Beyond that the error increases rapidly.

SUMMARY.

To recapitulate the method set forth in the preceding pages for obtaining the self-inductance of a coil wound in the ordinary manner with insulated round wire, we have the different self-inductances to consider, L_s , L_u and L , which are related as follows:

$$\begin{aligned}L_s - \Delta_1 L &= L_u \\ L_u + \Delta_2 L &= L \\ \therefore L &= L_s - \Delta_1 L + \Delta_2 L\end{aligned}$$

L_s is the first approximation to the self-inductance required, and is obtained by the current sheet formula of Rayleigh, Coffin or Lorenz.⁴ It is the self-inductance of a coil of length l , radius a ,

⁴This Bulletin, 2, p. 186.

having N turns, but the *depth of the winding* t is supposed reduced to zero.

$\Delta_1 L$ is the correction to apply to L_s to obtain L_u , which is the self-inductance of the coil of same dimensions and number of turns, but with *actual depth* t of winding; the current is, however, now supposed to be uniformly distributed over the cross section of the winding, as though the wire were of square section and had only an infinitesimal covering. $\Delta_1 L$ depends on two terms, A and B . A is found from equation (7) and B from Table II.

The actual self-inductance of the coil L differs from L_u by the correction $\Delta_2 L$, which depends on three correction terms, C , F , and E . The determination of these quantities, which depend on the shape of the section of the wire, the thickness of the insulation, the number of turns of wire in the coil, and the shape of the section of the coil, has been fully discussed elsewhere⁵ and their values given. The correction $\Delta_2 L$ is much smaller than $\Delta_1 L$, and can be neglected except when the highest accuracy is sought. The value of L_s and $\Delta_1 L$ can be calculated with accuracy if the dimensions are accurately known, and this is possible if one uses enameled wire of uniform section and takes proper care in winding and measuring the coil. However, such a coil can not be recommended for a standard of the highest precision, and I have given the full theory for the sake of completeness and to show the magnitude of the smaller corrections, rather than because all the corrections are likely to be generally needed in practice.

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⁵ This Bulletin, 3, pp. 3, 4, 28-37.