THE SELF-INDUCTANCE OF A TOROIDAL COIL OF RECTANGULAR SECTION.

By Edward B. Rosa.

One of the most carefully constructed standards of self-inductance ever made is that described some years ago by Prof. Fröhlich, of Budapest.[1] It consists of a large marble ring of rectangular section, very accurately ground and measured, and wound with a single layer of fine silk-covered wire. Each turn of the wire lies in a plane passing through the axis of the ring, the wires being wound as closely as possible on the inner surface of the ring and uniformly spaced on the outer surface. Taking account of the magnetic field within the space occupied by the wire as well as the space within the marble core, Prof. Fröhlich obtained the following expression for the self-inductance of the coil:

$$L=2n^2h\left[\log\frac{r_2}{r_1}-0.28721\,\rho\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{2}{h}\log\frac{r_2}{r_1}\right)\right] \qquad (1)$$

in which r_1 and r_2, Fig. 1, are the inner and outer radii of the ring

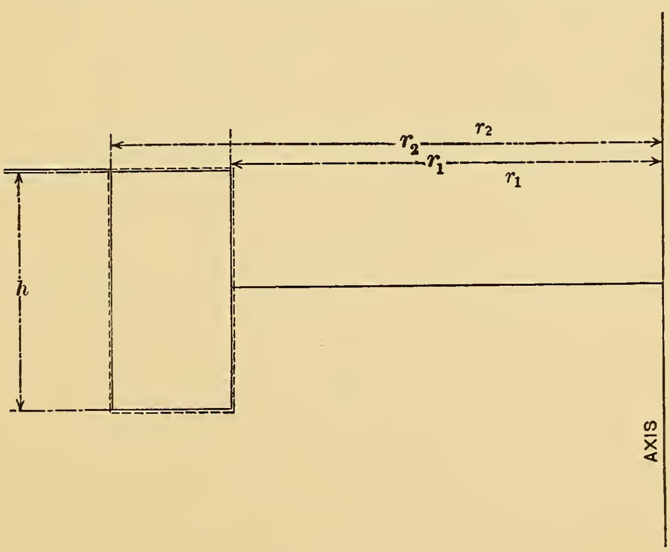

Fig. 1.

respectively, measuring to the axis of the wire in each case, ρ is the

[1] Wied. Ann., **63**, p. 142; 1897.

radius of the bare wire, h is the height of the section (center to center of wire), and n is the whole number of turns of wire on the coil.

These quantities as determined by Fröhlich's were as follows:

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\begin{aligned}
r_2 &= 35.05377 \text{ cm} \\
r_1 &= 24.97478 \text{ cm} \\
h &= 20.08455 \text{ cm} \\
\rho &= 0.011147 \text{ cm} \\
n &= 2{,}738
\end{aligned}
$$

The value found by substituting these quantities in equation (1) was

$$
\begin{aligned}
L &= (0.10208928 - .00009865)10^9 \text{ cm} \\
&= 0.10199063 \text{ henrys.}
\end{aligned}
$$

It was assumed in deriving formula (1) above that for a winding of fine wire one could consider the current as distributed in a thick current sheet, the thickness of the latter being 2ρ. That is, it was tacitly assumed that the flux of magnetic force, and therefore the self-inductance, were substantially the same as though the winding consisted of n turns of wire of rectangular section, the insulation being of infinitesimal thickness and the winding, therefore, completely enveloping the marble core. The wire in such a case would have a radial thickness of 2ρ and a breadth on the inner surface of the ring of $\frac{2\pi r_1}{n}$, on the outer surface of $\frac{2\pi r_2}{n}$, and on the upper and lower surfaces the breadth would increase gradually from the former to the latter value. Such a winding is of course impracticable, but was assumed as equivalent to the actual winding of round, insulated wire. I have elsewhere[2] pointed out that one can not assume a winding of round wires, no matter how fine, to be equivalent to a current sheet; and I wish now to show how serious is the error in the above case of a circular solenoid arising from this assumption.

The expression for the self-inductance of the circular solenoid of rectangular section enveloped by a *thin* current sheet is

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[2] This Bulletin, 2, p. 161; 1906.

$$L_0 = 2n^2h \log \frac{r_2}{r_1} \tag{2}$$

The correction term ΔL as obtained by Fröhlich *was negative, and proportional to* ρ, *the radius of the wire. For extremely fine wire* ΔL *would therefore become vanishingly small.*

A thin current sheet would be realized by a winding of infinitely thin tape, covering the core completely. The problem is to find the difference ΔL in the self-inductance of such a winding and the actual winding of round insulated wire. I have done that[2] for a straight solenoid and the results obtained are approximately correct in this case. It may be worth while, however, to derive the correction formulæ anew for this particular case.

The self-inductance of a square of round wire is

$$L = 2\left[l \log_e \frac{l+\sqrt{l^2+R^2}}{R} - \sqrt{l^2+R^2} + R\right]$$

where l is the length of wire in the square ($=4a$) and R is the geometrical mean distance of the section of the wire. For wires of which the diameter is small in comparison with the side of the square this is very approximately

$$L_w = 2l\left[\log \frac{2l}{R_w} - 1\right]$$

For a narrow tape bent into a square the expression for L is

$$L_t = 2l\left[\log \frac{2l}{R_t} - 1\right]$$

where R_t is the geometric mean distance of the section of the tape. The difference is

$$L_t - L_w = 2l\left[\log_e \frac{R_w}{R_t}\right]$$

For a wire of circular section $R_w = 0.3894d$, d being the diameter. For a tape of infinitesimal thickness and width D, $R_t = 0.22313\ D$.

Thus $\frac{R_w}{R_t} = 1.745 \frac{d}{D}$ and

$$L_t - L_w = 2l \log_e\left(1.745 \frac{d}{D}\right)$$

This is the excess of the self-inductance of one turn of thin tape over the self-inductance of one turn of round wire. For the entire winding of n turns we should therefore have

$$\Delta L_1 = 2nl \,.\, \log_e\left(1.745\frac{d}{D}\right)$$

This is equivalent to the expression previously found[2] for the case of a winding on a circular cylinder, $4\pi na$ being twice the whole length of the wire as $2nl$ is here. The same result would of course follow for a rectangular section not a square.

In addition to the correction for the difference in the self-inductances of the single turns is the correction for the differences in the mutual inductances upon each turn of all the others.

The mutual inductances of two parallel squares[3] is

$$M = 8\left[a \log_e\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{2a^2+d^2}}\frac{\sqrt{a^2+d^2}}{d}\right)\right] + 8\left[\sqrt{2a^2+d^2} - 2\sqrt{a^2+d^2} + d\right]$$

where a is the side of either square and d is the perpendicular distance between them. If each square is made up of round wire, d is the distance between the centers or axes of the wires. If, however, the squares are each made up of flat strip, d is the geometric mean distance of the two parallel strips, which may be written kd, where k is always a little less than unity,[4] but nearer to unity as the distance d is greater. Putting M_w for the mutual inductance of the two rectangles (or squares) of round wire and M_t for that of the rectangles (or squares) of thin tape, which will be the same expression with d replaced by kd, it can easily be shown that, neglecting small quantities of the second order,

$$\Delta M = M_t - M_w = 8a \log_e \frac{1}{k} = 2l\delta,$$

which is equivalent to the expression (23) previously found[2] for parallel circles, $4\pi a$ being twice the length of the wire, as is $8a$

[2] This Bulletin, 2, p. 161; 1906.

[3] Webster, Elect. and Mag., p. 456.

[4] This Bulletin, 2, p. 175; 1906.

here. As before, it is necessary to compute the value of this correction term $\varDelta M$ for all pairs of wires for which it is appreciable. It is greatest for two adjacent wires, and rapidly becomes inappreciable as the distance increases, so that the fact that the several turns are not quite parallel can not make any appreciable difference. It is only the small differences in the mutual inductances we have to consider, and these would not be changed appreciably by inclining two rectangles at a small angle.

For two adjacent wires we have

$$\delta M_1 = 8a \log \frac{1}{k_1} = 2l\delta_1$$

For two wires separated by one turn

$$\delta M_2 = 8a \log \frac{1}{k_2} = 2l\delta_2$$

and so for each succeeding pair. In an endless ring every wire is affected by the same corrections as every other, so that the summation will be

$$\varDelta M = 2n[\delta M_1 + \delta M_2 + \delta M_3 + \ldots\ldots]$$
$$= 2nl[2\Sigma\delta].$$

The values of the δs are given in the first column of Table IV.[5] $\Sigma\delta$ is about 0.332 for 24 terms, being a little greater than in the case of a straight solenoid of 25 turns of wire. We may neglect the effect beyond 25 turns, and use Table VII for the values of the constant A and take B as equal to $+0.332$ in the expression

$$\varDelta L = -(\varDelta L_1 + \varDelta M) = -2nl\,(A+B)$$

corresponding to (27) in the previous article. This expression for *$\varDelta L$ is always to be added;* when $(A+B)$ is negative $\varDelta L$ will be positive and will therefore tend to increase the self-inductance.

In the case of Fröhlich's circular solenoid the diameter of the bare wire is

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[5] This Bulletin, 2, p. 178; 1906.

$$d=0.0223.$$

The mean distance between centers of successive turns is

$$D=2\pi\left(\frac{r_1+r_2}{2\pi}\right)=0.0689$$

$$\therefore \frac{d}{D}=0.324$$

From Table VII,

$$\begin{aligned} A&=-0.57\\ B&=+0.33\\ A+B&=-0.24\\ 2nl&=2\times 2738\times 60.327=330300 \text{ cm}\\ \Delta L&=+79300 \text{ cm}\\ &=+0.0000793 \text{ henry}\\ L_0&=0.1020893 \quad \text{"}\\ L=L_0+\Delta L&=0.1021686 \quad \text{"} \end{aligned}$$

instead of 0.1019906 as found by Fröhlich.

The value of A is positive when the insulation is thin, that is when d is nearly equal to D. When

$$1.745\frac{d}{D}=1,$$

the correction term A is zero. That is, for $\frac{d}{D}=0.57$ approximately. For $\frac{d}{D}$ less than 0.57, A is negative, and the corresponding correction tends to increase L_0. The term B, being positive, always tends to decrease L_0. Hence when A is negative and greater than 0.33 the combined correction increases L_0, as in the present case.

The meaning of this is that the self-inductance of a circle or rectangle of wire is less than that of a flat strip of width equal to the diameter of the wire. As the diameter of the wire is reduced, however, its self-inductance increases, and becomes greater than that of the strip. So if the solenoid is wound with round wire of very thin insulation its self-inductance is less than L_0, the current sheet value; but if the wire is finer and covered with thick insulation (its outer diameter remaining the same), the total self-inductance increases and may be *more than* L_0.

To further show the marked effect of spacing the wire, suppose the above solenoid has been wound with a double winding of 1,369 turns each, so that either could be employed, or both, in series or in parallel. If both were in parallel and half the current flowing in each, the self-inductance would be one-fourth of that just calculated, or $L = 0.0255223 + .0000198 = .0255421$.

By Fröhlich's formula it would make no difference whether one winding or both in parallel were employed as to the value of L. But if one winding only is used D is twice as great as before, and therefore $\frac{d}{D} = 0.162$.

By Table VII,

$$\begin{aligned} A &= -1.42 \text{ approx.} \\ B &= +0.33 \\ \hline A+B &= -1.09 \\ -2nl\,(A+B) &= 180{,}013. \\ \Delta L &= +0.0001801 \text{ henry} \\ L_0 &= 0.0255223 \quad \text{"} \\ \hline L &= 0.0257024 \quad \text{"} \end{aligned}$$

This is greater than the preceding value by *nearly two-thirds of 1 per cent.* The meaning of this is that when only one winding is used there are a great many lines of force immediately surrounding the wires, which, carrying the whole current and having more space between the wires, give a larger correction term than when the current is divided between the two wires.

If the wires were very fine indeed (the whole number of wires remaining constant), the correction ΔL would be very great, easily *several per cent* of the value of L_0, whereas by Fröhlich's formula the correction ΔL would be approaching zero and the self-inductance would be simply L_0.

I have given these examples rather fully to emphasize the serious error that may result from assuming a winding of fine wires to be equivalent to a current sheet, and to show that for a given spacing of the wires *the finer they are* (*beyond a certain point*) *the greater is the error* arising from this assumption.

The straight solenoid offers many advantages over the circular solenoid as a standard. It is easier to construct and measure the core and immensely easier to wind, and the calculation of L from the dimensions is just as easy and just as certain.

WASHINGTON, August 10, 1907.