THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

By Edward B. Rosa and Louis Cohen.

Various formulæ for the calculation of the mutual inductance of coaxial solenoids have been given from time to time. Although few of these formulæ are exact, several of the approximate formulæ permit inductances to be calculated with very great accuracy by using a sufficient number of terms of the series by which they are expressed. We have collected a number of these formulæ and propose in this paper to compare and test them, and to give their derivations in some cases where the proofs that have been given are incomplete or wanting altogether.

1. MAXWELL'S FORMULA.[1]

CONCENTRIC, COAXIAL SOLENOIDS OF EQUAL LENGTH.

Maxwell's demonstration[2] of this formula is very incomplete and difficult to understand. We shall give the derivation more fully, and extend the results. It is shown elsewhere[3] in this Bulletin that the mutual inductance of the circle S_1, of radius a Fig. 1, and the solenoid PQ of radius A, and which extends from P to infinity (OP being x), is

$$N_1 = 2\pi^2 a^2 n_1 \left[\left(1 - \frac{x}{(x^2+A^2)^{\frac{1}{2}}} \right) - \frac{3}{8} \frac{a^2A^2x}{(x^2+A^2)^{\frac{5}{2}}} - \frac{5}{64} \frac{a^4(3A^4x - 4A^2x^3)}{(x^2+A^2)^{\frac{9}{2}}} - \frac{35}{1024} \frac{a^6(8A^2x^5 - 20A^4x^3 + 5A^6x)}{(x^2+A^2)^{\frac{13}{2}}} \right] \tag{1}$$

[1] Electricity and Magnetism, II, § 678.

[2] Quoted by Coffin, this Bulletin, **2**, p. 128; 1906.

[3] Rosa, p. 215, eq. (7).

This is the number of lines of force passing through the circle S_1 due to unit current in the infinite solenoid PQ, the latter having a winding of n_1 turns per cm.

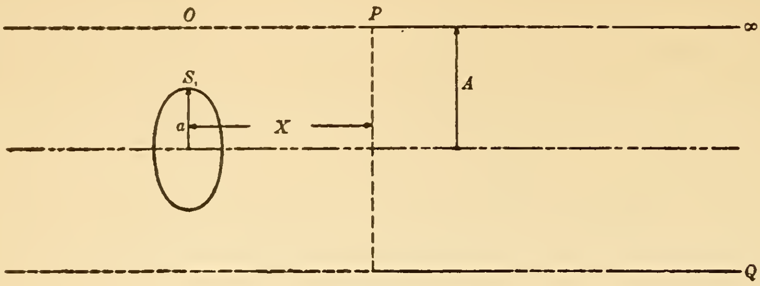

Fig. 1.

To find the number of lines of force due to PQ linked with all the turns of the solenoid RS, Fig. 2, the latter having n_2 turns per cm,

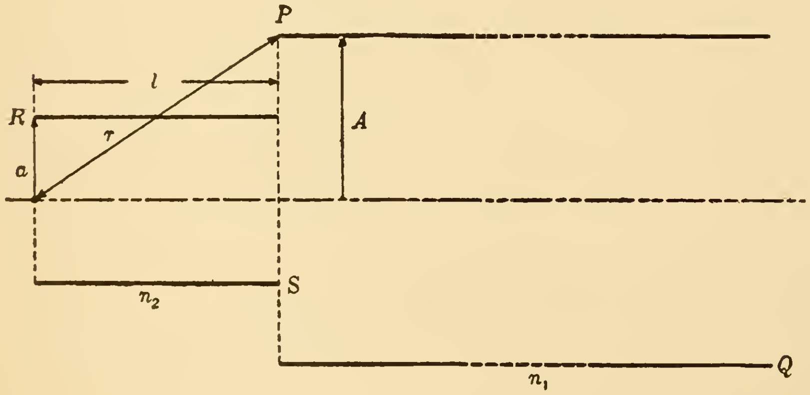

Fig. 2.

we must integrate equation (1) along the solenoid RS, from $x=0$ to $x=l$.

Thus, if

$$N=\int_0^l N_1 n_2 dx$$

$$N=2\pi^2 a^2 n_1 n_2 \Big[x-(x^2+A^2)^{\frac{1}{2}}$$

$$+\frac{1}{8}\frac{a^2A^2}{(x^2+A^2)^{\frac{3}{2}}}+\frac{5a^4}{64}\left(\frac{A^4}{(x^2+A^2)^{\frac{7}{2}}}-\frac{4}{5}\frac{A^2}{(x^2+A^2)^{\frac{5}{2}}}\right)$$

$$+\frac{35a^6}{1024}\left(\frac{8}{7}\frac{A^2}{(x^2+A^2)^{\frac{7}{2}}}-\frac{4A^4}{(x^2+A^2)^{\frac{9}{2}}}+\frac{3A^6}{(x^2+A^2)^{\frac{11}{2}}}\right)+\ \cdot\ \cdot\ \Big]_0^l \qquad (2)$$

Inserting the limits and putting $r=\sqrt{x^2+A^2}$

$$N=2\pi^2 a^2 n_1 n_2 \left[(l-r+A)+\frac{a^2A^2}{8}\left(\frac{1}{r^3}-\frac{1}{A^3}\right)+\frac{5a^4}{64}\left(\frac{A^4}{r^7}-\frac{4}{5}\frac{A^2}{r^5}-\frac{1}{5A^3}\right)\right.$$
$$\left.+\frac{35a^6}{1024}\left(\frac{8}{7}\frac{A^2}{r^7}-\frac{4A^4}{r^9}+\frac{3A^6}{r^{11}}-\frac{1}{7}\frac{1}{A^5}\right)+\ldots\right] \quad (3)$$

$$\therefore\ 2N=4\pi^2 a^2 n_1 n_2\left[(l-r+A)-\frac{a^2}{8A}\left(1-\frac{A^3}{r^3}\right)-\frac{a^4}{32A^3}\left(\frac{1}{2}+2\frac{A^5}{r^5}-\frac{5}{2}\frac{A^7}{r^7}\right)\right.$$
$$\left.-\frac{35a^6}{1024A^5}\left(\frac{1}{7}-\frac{8}{7}\frac{A^7}{r^7}+\frac{4A^9}{r^9}-\frac{3A^{11}}{r^{11}}\right)+\ldots\right] \quad (4)$$

$2N$ is the number of lines of force (due to unit current in the two infinite ends PQ and P′ Q′ of the larger solenoid, Fig. 3) passing through the short solenoid RS. If, however, the outer solenoid

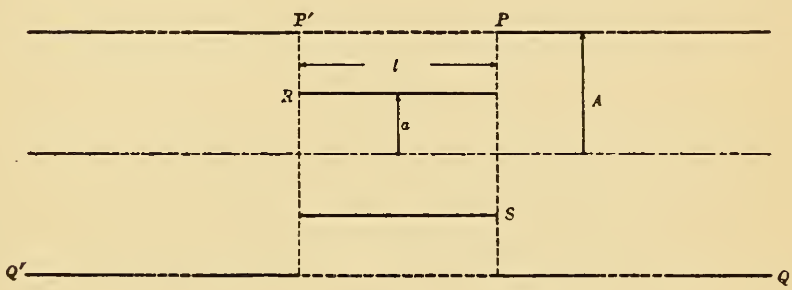

Fig. 3

were continuous and wound uniformly with n_1 turns of wire throughout, it would produce a uniform force within of $4\pi n_1$ when the current is one c. g. s. unit; and the number of lines linked with the $n_2 l$ turns of the inner solenoid would therefore be

$$N_1=4\pi n_1\times\pi a^2\times n_2 l=4\pi^2 a^2 n_1 n_2 l. \quad (5)$$

The number due to the short solenoid PP′ alone, that is, the mutual inductance of the two coaxial finite solenoids of length l, is N_1-2N, or

$$M=4\pi^2 a^2 n_1 n_2\ [l-2A\alpha] \quad (6)$$

where

$$\alpha=\frac{l-r+A}{2A}-\frac{a^2}{16A^2}\left(1-\frac{A^3}{r^3}\right)-\frac{a^4}{64A^4}\left(\frac{1}{2}+2\frac{A^5}{r^5}-\frac{5}{2}\frac{A^7}{r^7}\right)$$
$$-\frac{35a^6}{2048A^6}\left(\frac{1}{7}-\frac{8}{7}\frac{A^7}{r^7}+\frac{4A^9}{r^9}-\frac{3A^{11}}{r^{11}}\right)+\ldots \quad (7)$$

Putting

$$M = M_0 - \Delta M$$

M_0 is the mutual inductance of the infinite outer solenoid and the finite inner solenoid, while ΔM is the correction due to the ends. The number N given by equation (3) is $\Delta M \div 2$, the correction for one end.

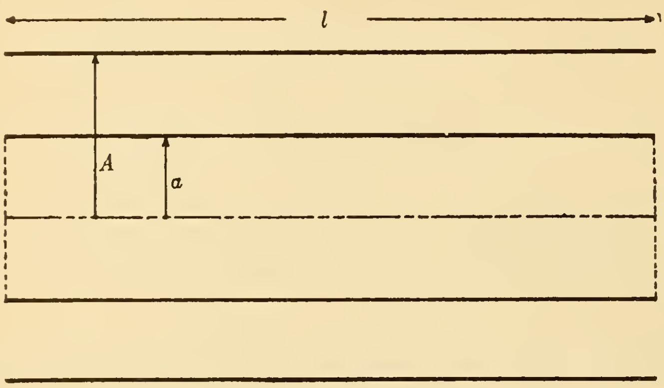

Fig. 4.

Equations (6) and (7) are Maxwell's expressions, except that (7) is here carried out further than the corresponding expression of Maxwell. This expression for M is rapidly convergent when a is considerably smaller than A, Fig. 4. Equation (6) shows that the mutual inductance is proportional to $l - 2A\alpha$; or the length l must be reduced by $A\alpha$ on each end. When a is small and l is large, α is 1/2 approximately. That is, the length l is reduced by A, the radius of the outer solenoid.

When the solenoids are very long in comparison with the radii, formula (7) may be simplified by omitting the terms in A/l, A^3/r^3, A^5/r^5, etc. Equation (7) then becomes

$$\alpha = \frac{1}{2} - \frac{a^2}{16A^2} - \frac{a^4}{128A^4} - \frac{5\,a^6}{2048A^6} - \dots \qquad (7a)$$

HEAVISIDE'S VARIATION OF MAXWELL'S FORMULA.

Heaviside gives a formula for the mutual inductance of two coaxial solenoids of equal length which differs in form from the above formula of Maxwell, and does not agree closely with it when applied to a particular case. Heaviside speaks of his formula as an extension of Maxwell's, but it is evidently derived in a somewhat different manner.[4] The main formula is the same as Maxwell's (6), the difference coming in the expression for α which, using A and a as the larger and smaller radii and $\rho = a/A$, is as follows:

$$2\alpha = 1 - \frac{\rho}{8}\left(1 + \frac{\rho}{8}\left(1 + \frac{5\rho}{16}\left(1 + \frac{7\rho}{16}\left(1 + \frac{21\rho}{40}\left(1 + \frac{33\rho}{56} + \ldots\right.\right.\right.\right.\right.$$

$$= 1 - \frac{\rho}{8} - \frac{\rho^2}{64} - \frac{5\rho^3}{1024} - \frac{35\rho^4}{16384} - \frac{147\rho^5}{131072} - \frac{693\rho^6}{1048576} - \ldots$$

This formula for α is less exact than (7) but better than (7α). A numerical example is given below to test its accuracy.

2. RÒITI'S FORMULA.

If the inner solenoid is shorter than the outer the limits of integration will be l_1 and l_2, Fig. 5, when we integrate over the length l of the inner solenoid.

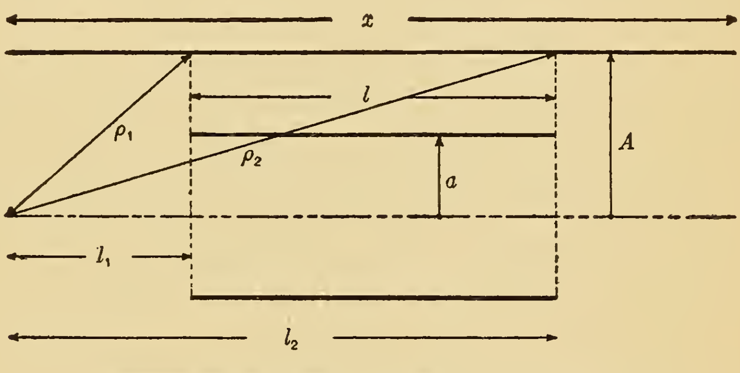

Fig. 5.

Put

$$\sqrt{l_1^2 + A^2} = \rho_1 \qquad \text{where } l_1 = \frac{x - l}{2}$$

$$\sqrt{l_2^2 + A^2} = \rho_2 \qquad l_2 = \frac{x + l}{2}$$

$$l_2 - l_1 = l$$

[4] There is a misprint in Heaviside, 2, p. 277. The radius of the *inner* solenoid should be c_2, of the *outer* c_1, and ρ is c_2/c_1.

Instead of (3) we shall have

$$N=2\pi^2a^2n_1n_2\Bigg[l-(\rho_2-\rho_1)+\frac{a^2A^2}{8}\left(\frac{1}{\rho_2^{\,3}}-\frac{1}{\rho_1^{\,3}}\right)$$
$$+\frac{5a^4A^2}{64}\left(\frac{A^2}{\rho_2^{\,7}}-\frac{A^2}{\rho_1^{\,7}}-\frac{4}{5\rho_2^{\,5}}+\frac{4}{5\rho_1^{\,5}}\right)+\frac{35a^8}{1024}\left\{\frac{8A^2}{7}\left(\frac{1}{\rho_2^{\,7}}-\frac{1}{\rho_1^{\,7}}\right)\right.$$
$$\left.-4A^4\left(\frac{1}{\rho_2^{\,9}}-\frac{1}{\rho_1^{\,9}}\right)+3A^6\left(\frac{1}{\rho_2^{\,11}}-\frac{1}{\rho_1^{\,11}}\right)\right\}+\ \ldots\Bigg] \tag{8}$$

For both ends,

$$\varDelta M=2N=4\pi^2a^2n_1n_2\Bigg[l-(\rho_2-\rho_1)+\frac{a^2A^2}{8}\left(\frac{1}{\rho_2^{\,3}}-\frac{1}{\rho_1^{\,3}}\right)$$
$$+\frac{5a^4A^4}{64}\left(\frac{1}{\rho_2^{\,7}}-\frac{1}{\rho_1^{\,7}}\right)-\frac{a^4A^2}{16}\left(\frac{1}{\rho_2^{\,5}}-\frac{1}{\rho_1^{\,5}}\right)+\frac{5a^6A^2}{128}\left(\frac{1}{\rho_2^{\,7}}-\frac{1}{\rho_1^{\,7}}\right)$$
$$-\frac{35a^6A^4}{256}\left(\frac{1}{\rho_2^{\,9}}-\frac{1}{\rho_1^{\,9}}\right)+\frac{105a^6A^6}{1024}\left(\frac{1}{\rho_2^{\,11}}-\frac{1}{\rho_1^{\,11}}\right)+\ \ldots\Bigg] \tag{9}$$

For an infinite outer cylinder and the given inner cylinder

$$M_0=4\pi^2a^2n_1n_2l=M+\varDelta M$$

Subtracting $\varDelta M$ as given by equation (9) from M_0 we have

$$M=4\pi^2a^2n_1n_2\Bigg[\rho_2-\rho_1+\frac{a^2A^2}{8}\left(\frac{1}{\rho_1^{\,3}}-\frac{1}{\rho_2^{\,3}}\right)-\frac{a^4A^2}{16}\left(\frac{1}{\rho_1^{\,5}}-\frac{1}{\rho_2^{\,5}}\right)$$
$$+\frac{5a^4A^4}{64}\left(\frac{1}{\rho_1^{\,7}}-\frac{1}{\rho_2^{\,7}}\right)+\frac{5a^6A^2}{128}\left(\frac{1}{\rho_1^{\,7}}-\frac{1}{\rho_2^{\,7}}\right)$$
$$-\frac{35a^6A^4}{256}\left(\frac{1}{\rho_1^{\,9}}-\frac{1}{\rho_2^{\,9}}\right)+\frac{105a^6A^6}{1024}\left(\frac{1}{\rho_1^{\,11}}-\frac{1}{\rho_2^{\,11}}\right)+\ \ldots\Bigg] \tag{10}$$

The first part of this expression is Ròiti's formula given by Coffin[5] (without demonstration). As the proof of this formula has never been published, we have given the derivation above and have extended the formula so that it is now very accurate as well as very convenient. We give below an example to test and illustrate the formula.

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[5] This Bulletin, 2, p. 130; 1906. There is an error in one term of the formula in Coffin's paper. The second coefficient within the brackets should be $-\frac{a^4A^2}{16}$ instead of $-\frac{a^4A^4}{16}$

3. GRAY'S FORMULA.

Gray[6] gives a general expression for the mutual kinetic energy of two solenoidal coils which may or may not be concentric, and their axes may be at any angle ϕ. The most important case in practice is when the two coils are concentric and coaxial. In that case the zonal harmonic factors in each term reduce to unity, and half the terms become zero. Putting the current in each equal to unity the mutual kinetic energy becomes the mutual inductance M.

Let $2x$ = the length of outer solenoid
$2l$ = " " " inner "
A = radius of outer "
a = " " inner "
n_1 = Number of turns per cm on outer solenoid
n_2 = " " " " " " inner "

Gray's expression with these changes becomes

$$M=\pi^2a^2A^2n_1n_2[K_1k_1+K_3k_3+K_5k_5+\ .\ .\ .\ ] \tag{11}$$

where K_1, K_3, etc., are functions of x and A, and k_1, k_3, etc., are functions of l and a.[7] When the ratio of the length of the winding of the outer coil to the radius is $\sqrt{3}$ to 1, $K_5=0$, and if the same condition holds for the inner coil, $k_3=0$. If in addition a is considerably smaller than A, the terms of higher order become negligible and (11) reduces to

$$M=\frac{2\pi^2a^2N_1N_2}{d}, \tag{12}$$

where d is half the diagonal of the outer coil, $=\sqrt{x^2+A^2}$, and the other letters have the meanings given above.

4. SEARLE AND AIREY'S FORMULA.

The following expression for the mutual inductance of two concentric and coaxial coils has been given by Searle and Airey.[8]

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[6] Absolute Measurements, **2**, Part I, p. 274, equation 53.
[7] Rosa, this Bulletin, **3**, p. 221.
[8] The Electrician (London), **56**, p. 318; 1905.

$$M=g_1G_1+g_3G_3+g_5G_5+g_7G_7+$$

$$=\frac{2\pi^2a^2N_1N_2}{d}\Big[1-\frac{A^2}{2d^4}\cdot\frac{4l^2-3a^2}{4}$$

$$-\frac{A^2(4x^2-3A^2)}{8d^8}\cdot\frac{8l^4-20l^2a^2+5a^4}{8}$$

$$-\frac{A^2(8x^4-20x^2A^2+5A^4)}{16d^{12}}\cdot$$

$$\frac{(64l^6-336l^4a^2+280l^2a^4-35a^6)}{64}-\ .\ .\ .\Big] \qquad (13)$$

The notation of (13) differs slightly from that used by Searle and Airey, being the same as used above, Fig. 6. This is the same

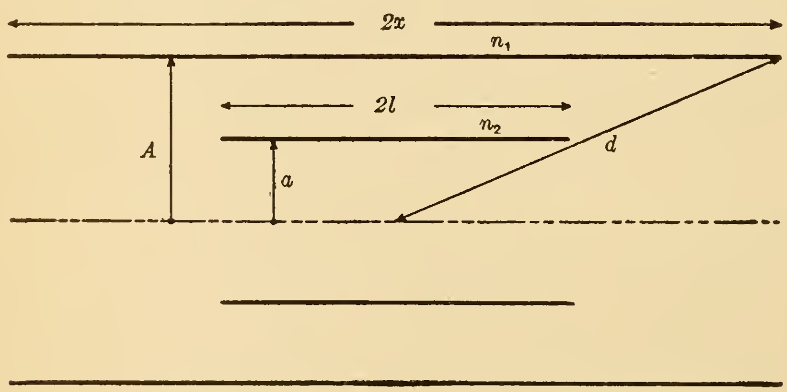

Fig. 6.

result that is obtained by substituting in (11) the values of K_1, K_3, K_5, . . . and k_1, k_3, k_5 . . . as given by Gray for the special case of concentric coils, and putting Z_1, Z_3, Z_5, etc., each equal to unity.[9] Equation (13) may be put for greater convenience in calculation in the following form:[9]

$$M=\frac{2\pi^2a^2N_1N_2}{d}\Big[1+\frac{A^2a^2}{8d^4}L_2+\frac{A^4a^4}{32d^8}X_2L_4$$

$$+\frac{A^6a^6}{32d^{12}}X_4L_6+\frac{A^8a^8}{32d^{16}}X_6L_8+\ .\ .\ .\Big] \qquad (14)$$

[9] Rosa, this Bulletin, 3, p. 224.

where

$$\left.\begin{aligned}
X_2 &= 3-4\frac{x^2}{A^2}\\
X_4 &= \frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}\\
X_6 &= \frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6}\\
L_2 &= 3-4\frac{l^2}{a^2}\\
L_4 &= \frac{5}{2}-10\frac{l^2}{a^2}+4\frac{l^4}{a^4}\\
L_6 &= \frac{35}{16}-\frac{35}{2}\frac{l^2}{a^2}+21\frac{l^4}{a^4}-4\frac{l^6}{a^6}\\
L_8 &= \frac{63}{32}-\frac{105}{4}\frac{l^2}{a^2}+63\frac{l^4}{a^4}-36\frac{l^6}{a^6}+4\frac{l^8}{a^8}
\end{aligned}\right\} \quad (15)$$

Examples are given below to test and illustrate these formulæ.

5. HIMSTEDT'S FORMULÆ.

Himstedt has given several formulæ for different cases of coaxial solenoids. The first[10] is for the case of a short secondary on the outside of a long primary, Fig. 7. The formula is very complicated, and tedious to calculate. By putting the shorter coil inside, the formula of Ròiti or of Searle or Airey may be used to much better advantage.

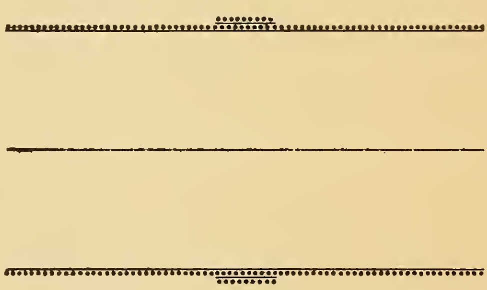

Fig. 7.

Himstedt's second expression is for the case of two coaxial solenoids, the distance b, Fig. 8, between their mean planes having any value; but the radius of one is supposed to be considerably smaller than the other. This also is a very complicated formula, involving second and fourth derivatives of expressions containing the elliptic integrals F and E. Gray's general equation is much simpler to calculate. This is not, however, an important case in practice, and

[10] Wied. Annalen, **26**, p. 551; 1885.

we do not therefore give Himstedt's equation. Himstedt's third equation is general and applies to two coaxial solenoids of nearly equal or very different radii, and they may be concentric or not. A limiting case is when the two solenoids coincide, and the expression for M then becomes the self-inductance of a solenoid. This

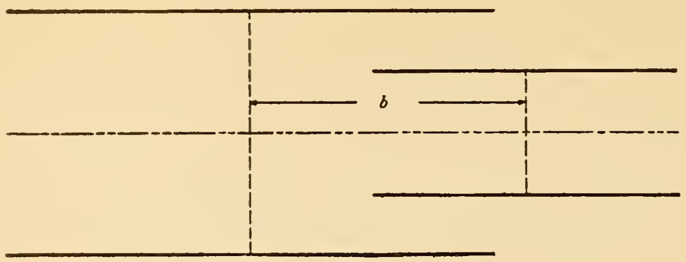

Fig. 8.

expression of Himstedt's consists of four terms, each of which is a somewhat complicated expression involving both complete and incomplete elliptic integrals. A less inconvenient general expression for M in elliptic integrals is given below.

6. COHEN'S FORMULA.

The general formula given by Coffin[11] as Kirchhoff's is seriously in error. Cohen has, however, deduced a general expression in elliptic integrals which gives correct results when tested by the formulæ of Ròiti and Searle and Airey; but as it is much more difficult to use in numerical calculations than the formulæ given above, it should only be employed where other formulæ are not sufficiently convergent, or to test other formulæ. Cohen's formula is as follows:

$$\left.\begin{aligned} M &= 4\pi n_1 n_2 (V - V_1) \\ V &= -(A^2-a^2)\,c\,[F\{F(k',\theta)-E(k',\theta)\} - E\,F(k',\theta)] \\ &\quad + \frac{c^4-(A^2-6Aa+a^2)c^2-2(A^2-a^2)^2}{3\sqrt{(A+a)^2+c^2}}\cdot F \\ &\quad + \frac{2(A^2+a^2)-c^2}{3}\sqrt{(A+a)^2+c^2}\cdot E - c(A^2-a^2)\frac{\pi}{2} \end{aligned}\right\} \qquad (16)$$

V_1 is obtained from V by replacing c by c_1

$$c = l + l_1 \qquad c_1 = l - l_1$$

[11] This Bulletin, 2, p. 125, equations (33) and (34); 1906.

F and E are the complete elliptic integrals of the first and second kind to modulus k, where $k^2 = \frac{4Aa}{(A+a)^2+c^2}$

$F(k', \theta)$ and $E(k', \theta)$ are the incomplete elliptic integrals of modulus k' and amplitude θ

$$k'^2 = 1 - k^2 = 1 - \frac{4Aa}{(A+a)^2+c^2} = \frac{(A-a)^2+c^2}{(A+a)^2+c^2}$$

$$\sin^2\theta = \frac{(A^2-a^2)^2+c^2(A-a)^2}{(A^2-a^2)^2+c^2(A+a)^2}$$

7. OTHER FORMULÆ.

The case of the mutual inductance of a solenoid and a coaxial circle within (the limiting case of coaxial solenoids when $l=0$) is discussed elsewhere in this Bulletin.[12] This case is important from its application in the Lorenz experiment and in the Ayrton-Jones absolute electrodynamometer rather than as a method of obtaining a standard of mutual inductance. Formulæ have been proposed for coils of several layers; but as such coils can not be constructed so as to give very accurate results from their dimensions, the formulæ will not be given here.

8. NUMERICAL TESTS OF THE FORMULÆ.

EXAMPLE 1. MAXWELL'S FORMULÆ, (6) AND (7).

Two solenoids, Fig. 9, of equal length, 200 cm, each wound with a single layer coil.

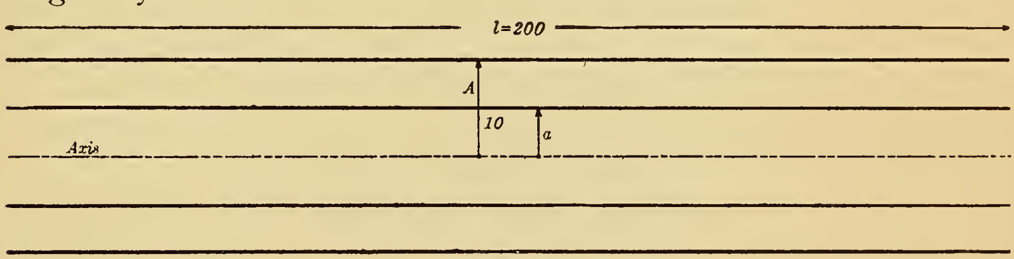

Fig. 9.

$A = 10 =$ radius of outer.
$a = 5 =$ " " inner.

[12] Rosa, this Bulletin, **3**, p. 222.

Substituting in (7) for a we have the following:

$$a=0.487508-\frac{1}{16}\frac{a^2}{A^2}(0.999875)-\frac{1}{64}\frac{a^4}{A^4}(0.500001)-\frac{35}{2048}\frac{a^6}{A^6}\left(\frac{1}{7}\right)$$

$$=0.487508-.015610-.000488-.000038$$

$$=0.471372$$

$$\therefore\ M=4\pi^2a^2n^2(200-9.42744)$$

$$\frac{M}{\pi^2n^2}=19057.25$$

If $n=10$ turns per cm, $M=\dfrac{100\ \pi^2\times 19057.25}{10^9}$ henry

$$=0.018809 \text{ henry}$$

By the approximate formula of Maxwell (7*a*)

$$2a=1-\frac{1}{8.4}-\frac{1}{64.16}-\frac{1}{1024.64}-\ .\ .\ .$$

$$=0.96773$$

$$\therefore\ \frac{M}{\pi^2n^2}=19032.27$$

This example by Heaviside's formula is as follows:

$$\rho=\frac{5}{10}=\frac{1}{2}$$

$$2a=1-\frac{1}{16}-\frac{1}{64\times 4}-\frac{5}{1024\times 8}-\frac{35}{16384\times 16}-\frac{147}{131072\times 32}$$

$$-\frac{693}{1048576\times 64}-\ .\ .\ .$$

$$=0.932805$$

$$\therefore\ \frac{M}{\pi^2n^2}=19067.08$$

It is thus seen that Maxwell's shorter formula for a gives M too small by 25 in 19000, whereas Heaviside's gives M too large by 10 in 19000.

To show that the result by Maxwell's formulæ (6) and (7) is very accurate for this case we may now calculate M by Cohen's absolute formula:

$$M = 4\pi n^2 (V - V_1)$$

Substituting in (17) for V we have the following terms:

$$V = 7863.79 + 4200532.04 - 4169106.25 - 23561.95$$
$$= 15727.63$$
$$V_1 = 1392.18 - 632.16 = 760.02$$
$$\therefore\ M = 4\pi n^2 \,(15727.63 - 760.02)$$
$$\frac{M}{\pi^2 n^2} = 19{,}057.36$$

This agrees with the result by Maxwell's formula to within 1 part in 175000.

This shows that for such a case as this, Maxwell's formula (7) is much more accurate than Heaviside's extension of it. The latter is sufficiently convergent to give a more accurate result than it does, and does not agree with Maxwell's, no matter how long the solenoid be taken. Hence there is some assumption made or some quantity neglected in the derivation of Heaviside's formula that is not strictly correct, aside from the assumption of a long solenoid.

The example of Cohen's formula illustrates the disadvantage of that formula for numerical calculations. Aside from the fact that it is complicated, and involves the use of both complete and incomplete elliptic integrals, the value of M depends on the difference between very large positive and negative terms, so that in order to get a value of M correct to 1 part in 100000 it is necessary in the above example to calculate the large terms to 1 part in 25000000. As a means of testing other formulæ, however, this absolute formula is of great value.

EXAMPLE 2. RÒITI'S FORMULA (10) COMPARED WITH SEARLE AND AIREY'S (13).

We will now calculate the example, Fig. 10, given by Searle and Airey,[13] by Ròiti's formula.

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[13] Electrician (London), **56**, p. 319; 1905.

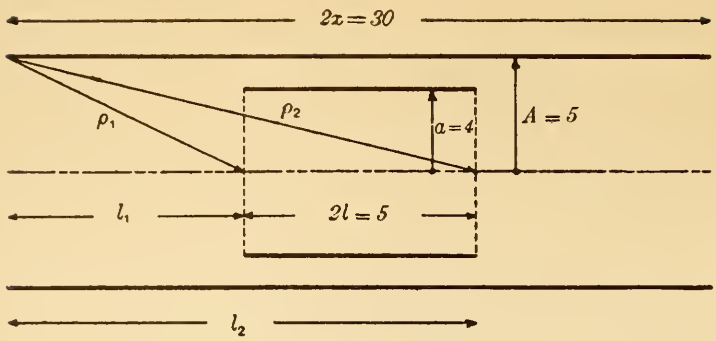

Fig. 10.

$2x = 30$ cm = length of outer solenoid.
$2l = 5$ " = " " inner "
$A = 5$ " = radius " outer "
$a = 4$ " = " " inner "

$N_1 = 300$ turns $\therefore n_1 = \frac{300}{30} = 10$ per cm

$N_2 = 200$ " $n_2 = \frac{200}{5} = 40$ per cm

$l_1 = 12.5 \qquad \rho_1 = \sqrt{12.5^2 + 25} = 13.462912$

$l_2 = 17.5 \qquad \rho_2 = \sqrt{17.5^2 + 25} = 18.200275$

$$\therefore \rho_2 - \rho_1 = 4.737363$$

$$\frac{A^2a^2}{8}\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right) = +\ .012200$$

$$-\frac{A^4a^2}{16}\left(\frac{1}{\rho_1^5} - \frac{1}{\rho_2^5}\right) = -\ .000704$$

$$\left[\frac{5A^4a^4}{64} + \frac{5A^2a^6}{128}\right]\left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7}\right) = +\ .000181$$

$$-\frac{35A^2a^6}{256}\left(\frac{1}{\rho_1^9} - \frac{1}{\rho_2^9}\right) = -\ .000022$$

$$+\frac{105A^6a^6}{1024}\left(\frac{1}{\rho_1^{11}} - \frac{1}{\rho_2^{11}}\right) = +\ .000002$$

$$\text{Sum} = 4.749020$$

$$4\ \pi^2a^2n_1n_2 = 25600\ \pi^2$$

$$\therefore M = \frac{25600\ \pi^2 \times 4.749020}{10^9} \text{ henry}$$

or $M = .001199896$ "

Searle and Airey give $M = .00119990$ henry.
The difference is inappreciable.

EXAMPLE 3. GRAY'S FORMULA (12) COMPARED WITH RÒITI'S (10).

Let the winding be 20 turns per cm on each coil; $n_1 = n_2 = 20$.

$A = 25$ cm $\qquad N_1 = n_1\, A\sqrt{3}$

$$\therefore\ N_1 N_2 = 3 n_1 n_2 A a$$

$a = 10$ cm $\qquad N_2 = n_1\, a\sqrt{3}$

$$d = \sqrt{x^2 + A^2} = \frac{A}{2}\sqrt{7}$$

$$\therefore\ M = \frac{2\pi^2 a^2 N_1 N_2}{d} = 4\pi^2 a^2 n_1 n_2 \left[\frac{3a}{\sqrt{7}}\right]$$

$$M = .0179057 \text{ henry.}$$

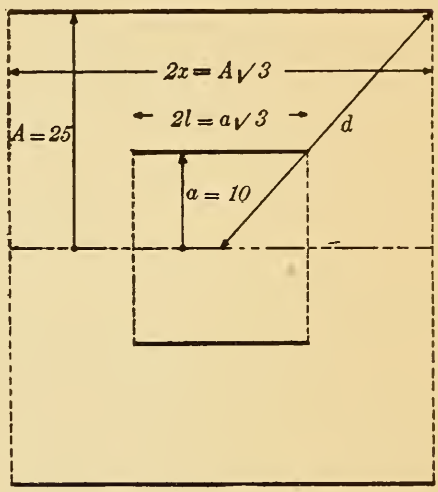

Fig. 11.

We have also calculated the mutual inductance for these coils by Ròiti's formula (10).

The value is, $M = .0179058$ which is practically identical with the value by Gray's formula.

When $A = 25$ cm and $a = 10$ cm, $N_1 = 20a\sqrt{3} = 866.025$ and $N_2 = 20a\sqrt{3} = 346.4$. As there must be an integral number of turns, let us suppose the winding is exactly 20 turns per cm on each coil and the lengths therefore 43.3 cm and 17.3 cm, respectively. Then $d = \sqrt{x^2 + A^2} = \sqrt{625 + \left(\frac{43.3}{2}\right)^2} = 33.0715$ cm. This does not exactly conform to the condition imposed in deriving the simple formula (12) of Gray used above. Hence (12) will not be as exact with these slightly altered dimensions, and we must calculate at least one correction term to get an accurate value of M.

By Gray's formula (12), $M = \dfrac{2\pi^2 100 \times 866 \times 346}{33.0715 \times 10^9} = .0178842$

The first correction term in (14) increases this value to .0178854 henry.

We will now calculate the mutual inductance of these coils by Ròiti's formula (10)

$$A=25 \quad 2x=43.3 \quad l_1=13.0 \text{ cm} \qquad \rho_1=28.17800$$
$$a=10 \quad 2l=17.3 \quad l_2=30.3 \text{ ''} \qquad \rho_2=39.28218$$

$\rho_2-\rho_1$	$=11.10418$
2nd term	$=+.22030$
3rd	$=-.01781$
4th	$=+.01952$
5th	$=+.00156$
6th	$=-.00453$
7th	$=+.00274$
Sum	$=11.32596$

$$M=\frac{4\pi^2a^2n_1n_2\times 11.32596}{10^9} \text{ henry}$$
$$= .0178853 \text{ henry.}$$

This differs from the result by Gray's formula by only 1 part in 178000.

In taking the dimensions of coils where an accurate value of the mutual inductance is sought it should be borne in mind that the above formulæ have been derived on the supposition that the current is uniformly distributed over the length of the coaxial solenoids; in other words, these formulæ are all current-sheet formulæ. Hence for coils made up of many turns of wire we must meet the conditions imposed by current-sheet formulæ. In calculating self-inductances, this requires an accurate determination of the size of the wire and of the distance between the axes of successive wires, from which we can calculate two correction terms to be combined with the value of L given by the current-sheet formulæ.[14]

In the case of mutual inductances, however, there are no correction terms to calculate; but we must take the dimensions of the current sheets that are equivalent to the coils of wire. That is, the radius of each coil will be the mean distance to the center of the wire, and the length of each will be the over-all length, includ-

[14] Rosa, this Bulletin, p. 181; 1906.

ing the insulation, when the coil is wound of insulated wires in contact, or the length from center to center of a winding of $N+1$ turns, where N is the whole number of terms used.[15] It is also very important that the winding on both coils shall be uniform,[16] and that the leads shall be brought out so that there shall be no mutual inductance due to them.

The mutual inductance will of course be different at high frequencies from its value at low frequencies. We assume, however, that for all purposes for which an extremely accurate mutual inductance is desired the frequency of the current would be low, say not more than a few hundred per second. If the value at very high frequency is desired the coil should be wound with stranded wire, each strand of which is separately insulated.

9. RUSSELL'S FORMULÆ.[17]

Since the above has been in type we have received the April number of The Philosophical Magazine containing Russell's article on The Magnetic Field and Inductance Coefficients.

Russell gives a number of formulæ for self and mutual inductance derived by original methods, and includes the case of coaxial cylinders. He derives the axial magnetic force at any point due to a cylindrical current sheet, and integrates this to get the number of lines of force due to one solenoid passing through the various turns of wire on the other, thus obtaining the mutual inductance of the two cylindrical coils considered as current sheets. The formula (41) obtained is as follows:

$$M=\pi b^2\frac{N_1N_2}{h_1h_2}\left[R_1\left\{1-\frac{1}{2}q_2k_1^2-\frac{1}{2}\cdot\frac{1}{4}q_3k_1^4-\ldots\right\}\right.$$
$$\left.-R_2\left\{1-\frac{1}{2}q_2k_2^2-\frac{1}{2}\cdot\frac{1}{4}q_3k_2^4-\ldots\right\}\right] \qquad (17)$$

In the notation of this article, where a and A are the radii of the inner and outer cylinders, respectively, $2l_1$ = length of the outer cylinder instead of $2h_1$ and $2l_2$ = the length of the inner instead of

[15] Rosa, this Bulletin, p. 161, 1906; and p. 1, 1907.

[16] Searle and Airey, Electrician (London), **56**, p. 318; 1905.

[17] Alexander Russell, Phil. Mag. Apr. 1907, p. 420.

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$2h_2$, and n_1, n_2 are the number of turns of wire per unit of length on the two coils, respectively, we have

$$M = 4\pi^2 a^2 n_1 n_2 \left[R_1 \left\{ 1 - \frac{1}{2} q_2 k_1^2 - \frac{1}{2} \cdot \frac{1}{4} q_3 k_1^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} q_4 k_1^6 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} q_5 k_1^8 - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} q_6 k_1^{10} - \dots \right\} - R_2 \left\{ 1 - \frac{1}{2} q_2 k_2^2 - \frac{1}{2} \cdot \frac{1}{4} q_3 k_2^4 - \text{terms with above coefs.} \right\} \right] \quad (18)$$

where

$$R_1^2 = (A+a)^2 + (l_1 + l_2)^2 \qquad k_1^2 = \frac{4\,Aa}{R_1^2}$$

$$R_2^2 = (A+a)^2 + (l_1 - l_2)^2 \qquad k_2^2 = \frac{4\,Aa}{R_2^2}$$

$$q_n = \frac{(A+a)^2}{4\,Aa} q_{n-1} - \frac{1}{n} \cdot \frac{1}{2} \cdot \frac{3 \cdot 5 \dots 2n-3}{4 \cdot 6 \dots 2n-2} \cdot \frac{A}{a}$$

$$q_2 = \frac{(A+a)^2}{4\,Aa} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{A}{a}$$

$$q_3 = \frac{(A+a)^2}{4\,Aa} q_2 - \frac{1}{3} \cdot \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{A}{a}$$

etc.

As a test of the formula Russell calculates the value of the mutual inductance of the two coaxial cylinders taken by Searle and Airey, and calculated above also by Ròiti's formula. Russell obtained the value $M = .0011995$ henry, which he supposed correct, evidently thinking that Searle and Airey's formula is less accurate than this own. There is, however, an error in Russell's work (the term $-.00100$ in the second part should be $-.001106$), so that when this error is corrected and each term carried out one more decimal place and two additional terms computed the value obtained for M is .00119989 instead of .0011995, as given by Russell. This value agrees with the results by the formulæ of Searle and Airey and Ròiti (page 318 above) to within 1 part in 100,000.

Russell's formula is thus seen to give an accurate result for this particular case, but it has serious limitations in other cases. Thus, for two coils of equal length the second part of the formula is not convergent, and hence it must be replaced by an expression in elliptic integrals. The formula thus becomes (equation 42 in Russell's paper)

$$M=4\,\pi a^2 n_1 n_2\left[R_1\left\{1-\frac{1}{2}q_2 k_1^{\,2}-\frac{1}{8}q_3 k_1^{\,4}-\ldots \text{ as above}\right\}\right]$$
$$+\frac{8\pi A a}{3(A+a)}n_1 n_2\left[(A^2+a^2)(F-E)-2\,AaF\right] \qquad (19)$$

This formula gives an accurate result for equal solenoids of considerable length, but Maxwell's formula (6) is just as accurate and much more convenient.

For short coils neither (18) or (19) will apply, but for that case as well as other cases Russell's general formula (38) may be used. There is a serious error, however, in this formula (38) introduced by a wrong value of the integral (5, p. 423) which should be

$$\int_0^{\frac{\pi}{2}}\frac{\sin^2\phi\cos^2\phi}{\Delta}d\phi=\frac{2-k^2}{3k^4}E-\frac{2-2k^2}{3k^4}F \qquad (20)$$

Russell's formula (38) for the mutual inductance of two coaxial solenoids becomes, therefore, when corrected:

$$M=4\pi ab\frac{N_1N_2}{h_1h_2}\left[R_1\left(1-\frac{k_1^{\,2}}{c^2}\right)\left(\frac{1-c^2}{c^2}\right)\left(F_1-\Pi_1\right)\right.$$
$$+R_1\left(\frac{1}{3k_1^{\,2}}-\frac{1}{c^2}+\frac{1}{3}\right)\left(F_1-E_1\right)+\frac{1}{3}R_1F_1$$
$$-R_2\left(1-\frac{k_2^{\,2}}{c_2^{\,2}}\right)\left(\frac{1-c^2}{c^2}\right)\left(F_2-\Pi_2\right)$$
$$\left.-R_2\left(\frac{1}{3k_2^{\,2}}-\frac{1}{c^2}+\frac{1}{3}\right)\left(F_2-E_2\right)+\frac{1}{3}R_2F_2\right] \qquad (21)$$

$$R_1^{\,2}=(a+b)^2+(h_1+h_2)^2 \qquad c^2=\frac{4ab}{(a+b)^2}$$

$$R_2^{\,2}=(a+b)^2+(h_1-h_2)^2 \qquad k_1^{\,2}=\frac{4ab}{(a+b)^2+(h_1+h_2)^2}$$

$$k_2^{\,2}=\frac{4ab}{(a+b)^2+(h_1-h_2)^2}$$

F_1 and E_1 are the complete elliptic integrals to modulus k_1, F_2 and E_2 the same to modulus k_2, Π_1 and Π_2 are the complete elliptic integrals of the third kind, the values of which may be expressed in terms of complete and incomplete integrals of the first and second kind.[18]

N_1 and N_2 are the whole number of turns of wire on the two coils, respectively. $2h_1$ and $2h_2$ are the lengths of the two solenoids, of which a and b are the radii. We have given the equation in Russell's notation in order to avoid confusion in verifying the corrections we have made.

By substituting the values given above for c^2, k_1^2, k_2^2, and Π_1, Π_2, this equation may be reduced to Cohen's formula (16) above. It is perfectly general for concentric, coaxial cylinders; but as already stated of (16) is a very tedious formula to employ in numerical calculations.

When the inner cylinder is considerably shorter than the outer, so that R_2 is much larger than $2\sqrt{Aa}$, formula (18) is sufficiently convergent to give very accurate values, although it may sometimes be necessary to calculate a good many terms. Thus, even for so favorable a case as the above problem of Searle and Airey, it is necessary to compute twelve terms to obtain as accurate a result as Searle and Airy's formula gives from three terms, and while Ròiti's is perhaps no more rapidly convergent than Russell's it is in simpler form to calculate. Russell's approximate formulæ (31) and (32) may be useful, but it is doubtful if they are enough simpler than the formulæ of Searle and Airey or Ròiti to compensate for the decreased accuracy.

Russell's paper is, nevertheless, an able and valuable one, and his derivation of the above formulæ, as well as many well-known formulæ by independent methods, is interesting and instructive.

WASHINGTON, March 30, 1907.

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[18] Cayley, Elliptic functions, p. 139, second edition.