THE COMPLETE FORM OF FECHNER'S LAW.

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Underlying vision, audition, and other sense perception is a fundamental quantitative relation between stimulus and sensation, between the objective and the subjective. From this fundamental relation, once established, may be derived another relation giving the least perceptible increment to the stimulus in terms of the whole. Fechner's law is such a relation derived from experimental data. It states that the least perceptible increment is proportional to the whole stimulus over quite a wide range of moderate intensities. That is, the ratio of least perceptible increment to total stimulus is a constant. This constant experiment shows to be of the order of about two per cent.

Now, just at the threshold value of a stimulus, evidently the least perceptible increment is the whole, so that, calling S this stimulus and δS the least perceptible increment, $\delta S : S = 1$, while according to Fechner's law $\delta S : S =$ constant $=$ about 0.02, hence Fechner's law does not and can not hold in this form at low intensities for any sense organ. The general law must be such that $\delta S : S = 1$ for $S = S_0$, the threshold value, while for large values of S, $\delta S : S =$ a small constant. From the complete form of Fechner's law may be obtained the general relation between sensibility and intensity, and from the sensibility the desired general relation between stimulus and sensation. If this general relation is expressed in the proper mathematical form, it may be expected to hold with different constants for all sense organs.

Of the various sense perceptions, vision permits of far the most accurate quantitative determination. In this field, König and Brodhun[1] have published some excellent data bearing on Fechner's

[1] A. König and E. Brodhun. Sitzungsberichte der Berliner Akad. 1888, pp. 917–932.

law. They determined the least perceptible increment for light ranging in intensity from just above the threshold value to about 100,000 meter-candles and in color from violet to deep red. The data relating to König's own eye—a normal trichromat—are reproduced in the accompanying table and figure.

TABLE I.

λ =	670	605	575	505	470	430
L =	0.060	0.0056	0.0029	0.00017	0.00012	0.00012
L			$\delta L : L$			
200,000		0.0425				
100,000		0.0241	0.0325			
50,000	0.0210	0.0255	0.0260			
20,000	0.0160	0.0183	0.0205	0.0195		
10,000	0.0156	0.0163	0.0179	0.0181		
5,000	0.0176	0.0158	0.0166	0.0160		
2,000	0.0165	0.0180	0.0180	0.0175	0.0180	
1,000	0.0169	0.0198	0.0185	0.0184	0.0167	0.0178
500	0.0202	0.0235	0.0180	0.0194	0.0184	0.0214
200	0.0220	0.0225	0.0225	0.0220	0.0215	0.0245
100	0.0292	0.0278	0.0269	0.0244	0.0225	0.0246
50	0.0376	0.0378	0.0320	0.0252	0.0250	0.0272
20	0.0445	0.0460	0.0385	0.0295	0.0320	0.0345
10	0.0655	0.0610	0.0582	0.0362	0.0372	0.0396
5	0.0918	0.103	0.0888	0.0488	0.0464	0.0494
2	0.1710	0.167	0.136	0.0655	0.0715	0.0600
1	0.258	0.212	0.170	0.0804	0.0881	0.0740
0.5	0.376	0.276	0.208	0.0910	0.096	0.0966
0.2		0.332	0.268	0.110	0.127	0.116
0.10			0.396	0.133	0.138	0.137
0.05				0.183	0.185	0.154
0.02				0.251	0.209	0.223
0.01				0.271	0.189	0.249
0.005				0.325	0.300	0.312
0 002						0.369

König and Brodhun did not include the increment (δL) in the total light (L) in calculating the values of $\delta L : L$. Logically the increment should be included, otherwise the ratio $\delta L : L$ becomes meaningless as the threshold value is reached. The values quoted have been recalculated with increment included. In the figure $\delta L : L$ is plotted as a function of the natural logarithm of L, the curves being extended from the data of König and Brodhun to the point $\delta L : L = 1$ for $L = L_0$ the threshold value, shown as a dotted ordinate in the figure.

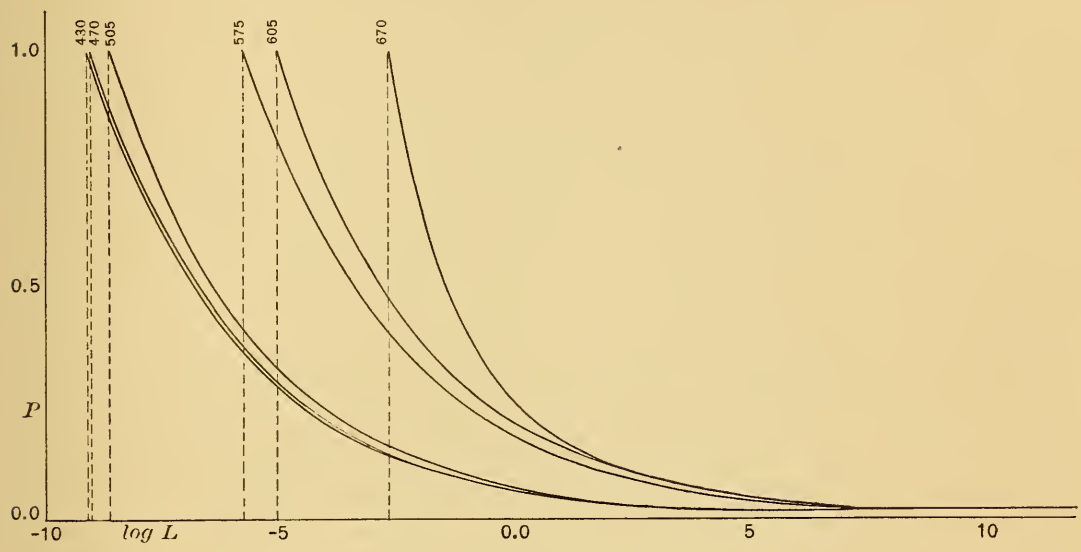

Fig. 1.—*Least Perceptible Increment.* $P(L)$

We have now to determine the mathematical form of these curves by the methods of synthetic function theory. For brevity, call $\delta L : L = P(L)$ or simply P, the photometric function. Denote the abscissa $\log L$ by x and the minimum value of P (about 0.016) by P_m.

All the curves have evidently the threshold value—the abscissa of the dotted ordinate—as natural origin of coordinates. That is, they would be strictly comparable if plotted as functions of $x - x_0$ or $\log (L : L_0)$ instead of x or $\log L$. Further, they are all of the general class of functions included in

$$P = a + b\, e^{-z}$$

when a and b are constants (independent of x) and z is a function of x. As a special case we may consider the particular function $z(x) = c(x - x_0)$ just as we should choose a particular integral of a differential equation as the practical working solution.

Taking then

$$P = a + b\, e^{-c(x-x_0)}$$

the parameter a is identified with P_m, being the value of P for x very large, b is $1 - P_m$ (since $a + b = 1$), c is from the plot (since $b\,c$ is the slope of the tangent at $x = x_0$) a number equal to about $\frac{1}{4}$ to $\frac{1}{2}$. Since $x - x_0$ is $\log L - \log L_0$ or $\log (L : L_0)$, P as a function of L is, according to the above,

$$P = P_m + (1 - P_m)\,(L_0 : L)^c$$

At very high intensities, instead of remaining constant, the values of $\delta L : L$ show a tendancy to increase again. To represent this would require an additional term in the photometric function P. But at these high intensities the light sensation becomes painful and we have every reason to believe that the mechanism of vision is altered if there is not an actual destruction of tissue. It would serve no useful purpose to extend the function to cover such a case of heterogeneity.

From the photometric function $P(L)$ may be derived the sensibility of the eye as a function of intensity. The sensibility of the eye as a physical instrument we know decreases steadily to accommodate itself to increasing intensity, otherwise its range would be very small. It can easily be used with illuminations varying by a factor of a million and that within a very few minutes. We have to determine the law of variation of sensibility $\sigma\,(L)$ with intensity L.

Consider any physical instrument—a galvanometer for instance—capable of indicating on a scale the amount of a stimulus affecting it. The scale reading will be some function $\rho\,(S)$ of this stimulus. The sensibility of the instrument is also in general a function of the stimulus. Call this function $\sigma\,(S)$. This sensibility function is the rate of change of scale reading with the intensity of the stimulus.

$$\sigma = \frac{\partial \rho}{\partial s}, \qquad \rho = \int \sigma\, ds.$$

Now, in the case of vision, the sensibility to variations of intensity varies inversely as the least perceptible increment, the δL above. But $\delta L : L$ has been expressed as a function $P(L)$ of intensity, and δL is $\delta L : L$ multiplied by L, or $\delta L = LP$. Sensibility as a function $\sigma(L)$ of intensity is then

$$\sigma = {}^1/LP = \frac{1}{L\,(P_m + (1 - P_m)\,(L_o : L^c)}$$

For low intensities σ is approximately proportional to L^{c-1}, for high intensities $\sigma = {}^1/P_m L$; the sensibility varies inversely as the intensity at high intensities exactly at low intensities σ varies from $L^{-\frac{1}{2}}$ to $L^{-\frac{3}{4}}$.

Having the sensibility function, the scale reading is at once obtained by direct integration. In the visual case, the scale reading is the sensation of visual brightness. This is to be expressed as a function of the stimulus light or luminosity of radiation. For the brightness $B(L)$ we have

$$B = \int \sigma dL = \int \frac{dL}{L\left(P_m + (1 - P_m)\,(L_o : L)^c\right)}$$

$$= \frac{1}{P_m c} \log\left(1 + P_m\,(L^c L_o^{-c} - 1)\right) + \text{constant.}$$

The integration constant is zero, since for $L = L_o$ at the threshold value the visual sensation must be zero. The coefficient $1/CP_m$ may be included in a more general one which we may call B_0. Thus the general relation between visual sensation and the stimulus is

$$B = B_0 \log\left(1 + P_m\,(L^c L_o^{-c} - 1)\right)$$

It is of interest here to note what was accomplished by Fechner and subsequent workers in this field. Fechner himself set up the form (using the notation here employed)

$$\frac{\delta L}{L + L_0}$$

to represent his own observations. He proceeded further to treat δL as a true differential, placing the above expression equal to a differential brightness dB and integrating, thus obtaining

$$B = K \log (L + L_0)$$

But δL and dB are fixed finite quantities and by no means to be regarded as differentials vanishingly small. At low intensities for instance, δL is as large as L itself. Nor does there appear to be any possible interpretation of the integration performed.

SUMMARY OF CONCLUSIONS.

1. The least perceptible increment δS of a stimulus S is not strictly proportional to that stimulus ($\delta S : S =$ constant) but a function of the form

$$\frac{\delta S}{S} = a + b\, e^{-T}$$

in which a and b are constants and T is a function of S.

2. A special form of the above function closely representing visual sensation is

$$\frac{\delta L}{L} = P = P_m + (1 - P_m)\,(L_0 : L)^c$$

when P_m is a number equal to about 0.016 and c varies between one-fourth and one-half.

3. Sensibility as a function of stimulus S has been derived in the form $\sigma :: 1 : \delta S$, or

$$\sigma = \frac{1}{S\left(S_m + (1 - S_m)\,(S_0 : S_0)^c\right)}$$

4. Taking sensibility, by analogy with physical instruments, as the derivative of sensation (scale reading) with respect to stimulus, the integration of the above expression for sensibility gives for sensation in terms of stimulus the relation

$$B = B_0 \log\left(1 + S_m(S^c\, S_0^{-c} - 1)\right).$$

While the derivation of these relations is such that they are not to be regarded as more than particular solutions of the general problem, yet they are put forward with some confidence that more extensive experimental investigation will more fully substantiate them.