THE ABSOLUTE MEASUREMENT OF INDUCTANCE.[a]

By Edward B. Rosa and Frederick W. Grover.

1. Methods of Measuring Inductance.

Self inductance may be determined in absolute measure (that is, in terms of resistance and time) by the methods of Maxwell, Wien, or Rowland. The first named is complicated and scarcely capable of giving results of high accuracy. The other two methods are probably capable of yielding results of satisfactory accuracy, but so far as we know few results by these methods have been published, and none of a degree of accuracy equal to the results which have been obtained in the absolute measurement of a capacity.

The most obvious method of directly determining the inductance of a coil, originally proposed by Joubert, consists in first determining the impedance of the coil and then calculating the inductance, after having found the ohmic resistance of the wire and the frequency of the current employed.

Brew[b] has given some determinations of inductance by this method, using a Cardew voltmeter, first in series with the inductive coil and second with the coil cut out. Knowing the resistance of the coil and of the instrument, and the frequency of the current, the inductance is calculated. The results on a single coil are given; they show considerable variations, as would be expected. Nothing is said of the wave form, although the formula employed presupposes a sine wave.

Several variations of this method are described by Gray[c] and Fleming.[d] According to Gray, a noninductive resistance is placed in series with the coil whose inductance is to be measured, and an alternating current passed through both. By means of an electrometer,

[a] A paper presented at the International Electrical Congress, St. Louis, 1904.

[b] Electrician: **25**, p. 206; 1890.

[c] Absolute measurements: II, pt. II, p. 488.

[d] Handbook for the Electrical Laboratory: Vol II, p. 205.

the differences of potential at the terminals of the noninductive resistance R_2, and of the inductive coil (resistance R_1 and inductance L) are measured. The inductance is then given by the expression

$$L=\frac{R_1}{p}\left(\frac{R_2^2 V_1^2}{R_1^2 V_2^2}-1\right)^{\frac{1}{2}}$$

p being 2π times the frequency of the current employed, which is to be as nearly simply harmonic as possible.

According to Fleming, we "first send through the coil a continuous current and observe the potential difference of the ends of the coil with an electrostatic voltmeter, and measure the current flowing through it. Then repeat the experiment, using the alternating e. m. f. The ammeter should be a Kelvin balance, or dynamometer, or hot wire ammeter, suitable for both continuous and alternating currents. Adjust the voltage so that the current is the same in both cases. Then if A is this current, and if V is the volt-fall down the coil with continuous current, and V' that with the alternating current, and if R is the resistance and L the inductance of the coil, we have

$$A=\frac{V}{R}$$

$$A'=\frac{V'}{\sqrt{R^2+p^2L^2}}$$

where $p = 2\pi$ times the frequency of the alternating current."

Therefore,

$$L=\frac{R}{p}\sqrt{\frac{V'^2-V^2}{V^2}}$$

or the volt-drop may be kept constant and the current measured in each case. Then

$$L=\frac{R}{p}\sqrt{\frac{A^2-A'^2}{A'^2}}$$

If the current is not of sine wave form, a correction must be applied.

2. The Method of this Paper.

It occurred to us that a modification of the method quoted above from Gray would be as well adapted to precision measurements as any other proposed. Instead of using the electrometer to measure the difference of potential at the terminals of the inductive coil and of a fixed resistance R, we vary the resistance R until the difference of potential at its terminals is equal to that at the terminals of the inductive coil, as shown by an electrometer. Then, since the alternating current I is the same in both, and a sine wave form is assumed,

$$IR=I\sqrt{r^2+p^2L^2},$$

r being the ohmic resistance of the inductive coil. Then

$$L=\frac{1}{p}\sqrt{R^2-r^2}.\quad \text{(See fig. 1.)}$$

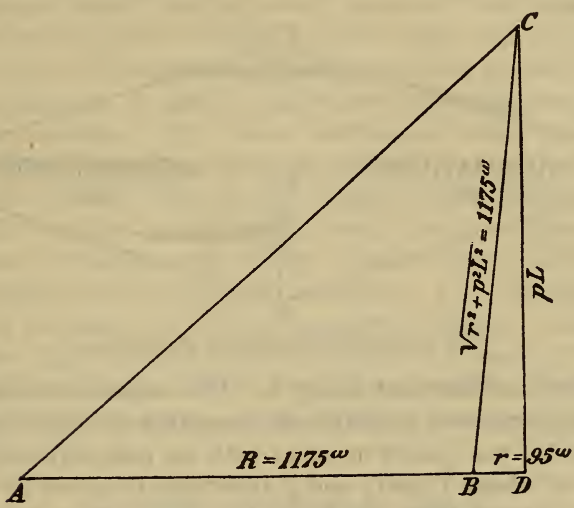

FIG. 1.—$A\ B$=noninductive resistance; $B\ C\ D$=impedance triangle for inductive coil.

This is an extremely simple formula, in which only two quantities, the resistance and the frequency, have to be determined accurately. The resistance r is usually so small that an approximate value for it is sufficient. In the simplicity and directness of the method and the small number of quantities to be determined lie the advantages of this over other methods.

The chief objection to this method is that it is necessary to have a perfect sine current, or to know the exact form of the current wave in order to calculate the correction due to any harmonics that may be present. So far as we know no accurate determination of inductance by this method has ever been published, and probably because of this requirement. Most alternating current generators yield currents having harmonics of considerable magnitude, and the wave form of course varies according to the load. It is necessary, therefore, to determine the wave form of the particular current used in the experiment in order to obtain the proper correction factor.

The Bureau of Standards possesses an alternating generator designed especially for testing purposes, which has a smooth core armature and pole pieces so shaped as to give a nearly sine wave. The correction to be applied to the measured inductance due to a small departure from a sine form is correspondingly small, and we therefore believed that by using this machine it would be possible to measure inductance in this manner with a high order of accuracy, provided, of course, that small harmonics in the current were carefully determined and allowed for.

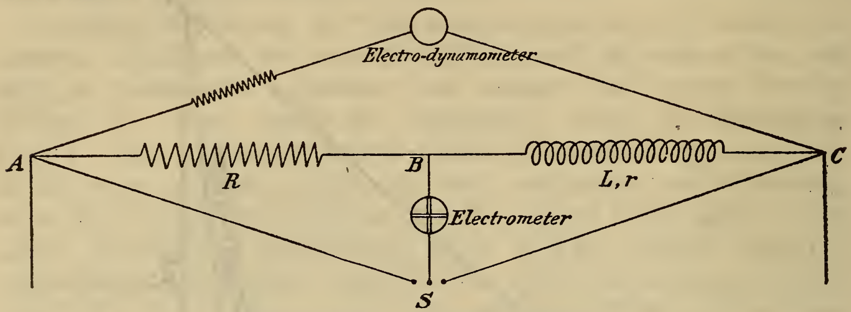

FIG. 2.—Diagram of connections of apparatus.

The method is illustrated in fig. 2. The noninductive resistance R (being an oil-immersed resistance of manganin of relatively large carrying capacity) was placed in series with an inductive coil having an inductance of about 1 henry and a resistance of about 95 ohms. An alternating current from the generator passes from A to C through these resistances, in parallel with which is an electro-dynamometer D, which serves as a very sensitive voltmeter. An electrometer is joined to the point B and to a switch S, by means of which it can be connected to $A\ B$ and to $B\ C$ successively. By means of the rheostats R_2 and R_3 (fig. 7), the first in series with the main current and the second in the field circuit of the alternator, the electromotive force at the terminals $A\ C$ is kept constant. The resistance R is now varied until the deflection of the electrometer is the same on $A\ B$ as on $B\ C$. Then

$$R = \sqrt{r^2 + p^2 L^2}$$

The frequency of the current is determined by means of a chronometer and chronograph. An electric contact being made for every 50 revolutions of the dynamo, a record is secured on the chronograph along with the second beats of the chronometer. Thus the frequency may be determined to a very high order of accuracy, provided the speed is maintained sufficiently constant. To do this a rotating commutator is directly connected to the alternator, and a Wheatstone bridge and condenser are joined up exactly as in the absolute measurement of capacity. When the bridge has been balanced at the desired speed an assistant maintains the speed constant by means of the carbon rheostat R_3, the criterion of constant speed being that the galvanometer continues to give zero deflection. That this condition of constant speed is sufficiently met will appear in the following pages.

The requirement of constant voltage is satisfied by having the alternator directly coupled to its driving motor, running the latter from a storage battery having a very constant electromotive force, keeping the bearings and brushes in good order, and regulating the exciting current of the alternator by means of the manual adjustment of a carbon rheostat, so that the deflection of the volt-dynamometer D is maintained constant. The latter instrument, which gave a deflection proportional to the square of the voltage, was so arranged that a variation of 1 volt in 140 gave a change in the reading of about 25 mm. The readings of the electrometer were only made when the dynamometer deflection was within 0.2 mm. of the selected mark, hence variations in the electromotive force on $A\ C$ were less than one part in ten thousand for all the separate readings. A differential electrometer might be used, and so obviate the necessity of maintaining so constant a voltage. We have designed such an instrument and it has been constructed, but we have not yet had an opportunity to use it in this work.

3. Possible Sources of Error.

The current flowing into the electrometer would cause an error in the result if the electrometer had a sufficiently large capacity, since it shunts first $A\ B$ and then $B\ C$. This effect would be very small when the electrometer shunts $A\ B$, since its current would differ in phase by 90° from that in $A\ B$; but when in shunt with $B\ C$ the electrometer current is nearly opposite in phase to the main current. The capacity of the electrometer is, however, so small that the current flowing into it from B is wholly inappreciable in comparison with the main current through $A\ B\ C$, which was usually nearly 0.1 ampere.

Any small inductance or capacity in the resistance R will produce no error, for its effect if present would be to slightly alter the phase of the electromotive force, but could not alter the impedance to be measured to an appreciable extent. Thus if R is 1,175 ohms, and we suppose that there is sufficient inductance present to make the impedance a hundredth of one per cent greater, or 1,175.12, this would require at a frequency of 180 cycles an inductance of 15 millihenrys, or a capacity of about 0.01 microfarad. A careful measurement shows that the capacity effect of this resistance exceeds the inductance effect, but that the resultant effect is equivalent to a capacity of about 0.001 microfarad, and hence the error produced is wholly inappreciable.

The heating of the inductive coil causes a change in its resistance; in fact, its resistance serves as an excellent indication of its mean temperature. Any such change in the resistance, however, produces a very much smaller change in the impedance of the coil. Thus, if r changes from 95 to 95.5 ohms, the impedance will change only from, say 1,175 to 1,175.04. As the resistance r is determined at frequent intervals during a series of measurements, the uncertainty of the impedance due to uncertainty in this resistance need never be as much as one part in fifty thousand. The change in the inductance due directly to changing temperature is, however, appreciable and needs to be taken into account very carefully. Foucault currents in the wire of the coil may cause the inductance to vary appreciably with the frequency, when the wire is relatively coarse and the frequency is relatively high. To avoid any error due to this cause, the wire should be fine, or stranded if a low resistance is desired, and high frequencies avoided in the measurements.

The electrostatic capacity of the coil is also a source of error when the frequency is relatively high. Dolazalek[a] has shown that at a frequency of 2,500 per second the measured value of the inductance of a coil may be 3 or 4 per cent greater than its true value, due to this cause. The error is, however, proportional to the square of the frequency, so that at a frequency of 180 per second it amounts to only one or two parts in ten thousand, and, by properly designing the coil, we believe that this correction could be reduced to perhaps five parts in a hundred thousand and its value determined experimentally with a fair degree of accuracy. This experimentally determined correction would include also any effect due to eddy currents, which effect is of opposite sign to the effect due to capacity. Thus no correction need be applied to the result derived from the simple formula except those due to the electrostatic capacity of the coil and the wave form of the current. The former we have not attempted to determine experimentally; the latter we now proceed to ascertain.

4. Correction for Wave Form.

The correction factor to be applied when using a single coil of negligible resistance has been given by H. F. Weber.[b] Calling the har-

[a] Dolazalek: Ann. der Phys. 1903, p. 1142.
[b] Wied. Ann., **63**, p. 366; 1897.

monic components of the electromotive force at the terminals of the coil E_1, E_3, E_5, etc., the correction factor for this case is

$$f=\sqrt{\frac{E_1^2+\frac{1}{9}E_3^2+\frac{1}{25}E_5^2+\text{etc.}}{E_1^2+E_3^2+E_5^2+\text{etc.}}}$$

This expression is, however, not applicable to the present case.

To find the correction due to the wave form where the resistance r is not negligible and where a resistance R is joined in series with the inductive coil, we remember that the square of the effective value of an alternating current is given by the following expression:

$$I^2=I_1^2+I_3^2+I_5^2+\text{etc., where } I_1,\ I_3,\ I_5, \text{ etc.,}$$

are the values of the components of the current of which the relative frequencies are 1, 3, 5, etc.

The electromotive force E_a on $A\ B$ is made equal in the experiment to the electromotive force E_b on $B\ C$. Therefore,

$$E_a^2=E_b^2 \qquad (1)$$

$$E_a^2=R^2\ (I_1^2+I_3^2+I_5^2+\text{ etc.}) \quad (2)$$

$$E_b^2=E_1^2+E_3^2+E_5^2+\text{ etc.}=I_1^2(r^2+p^2L^2)+I_3^2(r^2+9p^2L^2)+I_5^2(r^2+25p^2L^2)+\text{ etc.}$$

$$=r^2(I_1^2+I_3^2+I_5^2+\text{etc.})+p^2L^2(I_1^2+9\,I_3^2+25\,I_5^2+\text{ etc.})$$

where $p=2\pi n$, and n is the frequency of the fundamental.

Therefore, from (1),

$$(R^2-r^2)\,(I_1^2+I_3^2+I_5^2+\text{ etc.})=p^2L^2\,(I_1^2+9\,I_3^2+25\,I_5^2+\text{ etc.})$$

Therefore,

$$p^2L^2=(R^2-r^2)\left\{\frac{I_1^2+I_3^2+I_5^2+\text{ etc.}}{I_1^2+9\,I_3^2+25\,I_5^2+\text{ etc.}}\right\}$$

$$\text{or, } L=\frac{1}{p}\sqrt{R^2-r^2}\sqrt{\frac{I_1^2+I_3^2+I_5^2+\text{ etc.}}{I_1^2+9\,I_3^2+25\,I_5^2+\text{ etc.}}}$$

$$\text{or, } L=\frac{f}{p}\sqrt{R^2-r^2}$$

$$\text{where } f=\sqrt{\frac{I_1^2+I_3^2+I_5^2+\text{ etc.}}{I_1^2+9\,I_3^2+25\,I_5^2+\text{ etc.}}},$$

and is the correction factor sought.

It will be seen from this expression for f that the presence of higher harmonics in the current causes the correction factor to depart from unity much more rapidly than the lower harmonics.

For example, suppose that the equation of the current is

$$I = I_1 \sin (pt - \phi_1) + I_3 \sin (3pt - \phi_3) + I_5 \sin (5pt - \phi_5)$$
$$+ I_7 \sin (7pt - \phi_7) + I_9 \sin (9pt - \phi_9),$$

and that $I_1 = 100$
$I_3 = 2$
$I_5 = 2$
$I_7 = 1$
$I_9 = 1$

Then $f = \sqrt{\dfrac{100^2+2^2+2^2+1^2+1^2}{100^2+9\times 2^2+25\times 2^2+49\times 1^2+81\times 1^2}} = \sqrt{\dfrac{10010}{10266}} = .9875.$

Thus the correction amounts to 1.25 per cent.

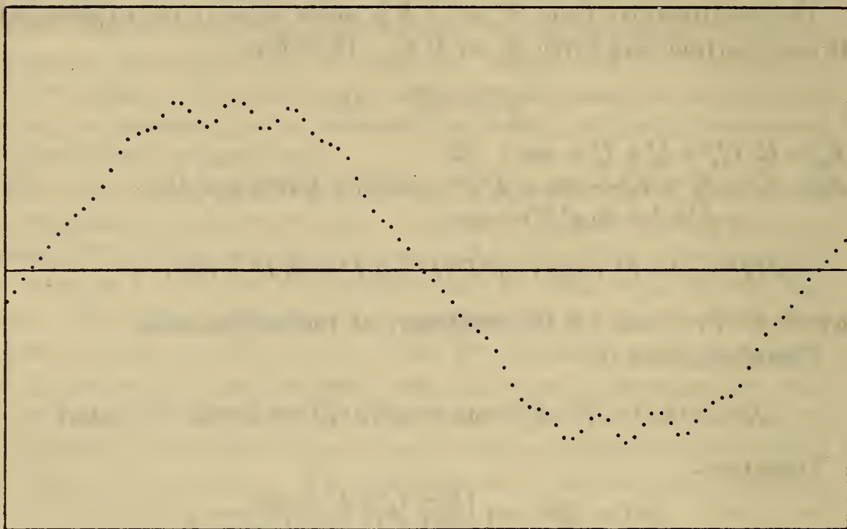

FIG. 3.—Curve of electromotive force of a Westinghouse alternator.

If the harmonics had been more pronounced, and I_7 and I_9 were each 5 instead of 1, then would

$$f = \sqrt{\frac{100^2+2^2+2^2+5^2+5^2}{100^2+9\times 2^2+25\times 2^2+49\times 5^2+81\times 5^2}} = \sqrt{\frac{10058}{13386}} = .8668.$$

The correction thus amounts to 13.32 per cent.

Professor Weber found the correction for a certain Ganz alternator with a greatly distorted wave to amount to 6.8 per cent.

The electromotive force shown in fig. 3, due to a small Westinghouse alternator with slotted armature, has the following equation:

$$E=30.66 \sin (x-0° 56')+.80 \sin (3x-6° 17')-1.65 \sin (5x-14° 22')$$
$$+.40 \sin (9x+7° 50')+1.11 \sin (13x-30° 45')-1.18 \sin (15x-38° 40').$$

The third, seventh, ninth, and eleventh harmonics are very small; the fifth, thirteenth, and fifteenth are relatively large. The correction factor for this curve is .9933, differing from unity by only .0067, and yet this is a larger correction than is desirable for precision measurements of inductance.

5. The Curves of Electromotive Force and Current.

The electromotive force given by the alternator in our experiments and the current flowing through the circuit *A B C* under this impressed electromotive force are shown in the curves of fig. 5. These curves were drawn under the same conditions as those of the experiment. The third harmonic of the electromotive force curve is in such phase as to slightly flatten the curve; this component being smaller in the current curve, the latter is less flattened and more nearly a sine curve.

Although these curves were drawn with great care and are reasonably accurate, the analysis of the current curve did not give as good results as were desired. This was because the harmonics are so small that very slight errors in the curves produce relatively large errors in the harmonics.

All the odd harmonics were looked for up to the fifteenth, and small values found in every case; above the seventh their average value was not more than a thousandth part of the fundamental, and yet the values obtained from different sets of ordinates of the same curve varied considerably. This suggested that these harmonics were really absent in the curve, and that the inevitable errors in drawing the curves gave rise to small residuals. The actual values of the small upper harmonics in question was on the average only one-twentieth of a millimeter measured on the plotted curve. Nevertheless, they made an appreciable error in the value of the correction factor sought, and hence it was desirable to eliminate all such residual errors.

The harmonics of the current are readily computed from those of the electromotive force which produces the current, and since in an inductive circuit the latter are larger than the former, the errors of the curve and of the analysis are divided down in the calculation and a more accurate result may therefore be obtained.

Thus, for the third harmonic the impedance is

$$\sqrt{(R+r)^2+9p^2L^2} \text{ instead of } \sqrt{(R+r)^2+p^2L^2}$$

and, in a particular case, its numerical value is as follows:

$$r=97.2$$
$$R=1176.1$$
$$L=1.0017$$
$$p=2\pi n=2\pi\times 186.23=1172.1.$$

Therefore,

$$Imp_1=\sqrt{(1273.3)^2+(1172.1)^2}=1730.6.$$

$$Imp_3=\sqrt{(1273.3)^2+9(1172.1)^2}=3739.7.$$

$$Imp_5=\sqrt{(1273.3)^2+25(1172.1)^2}=5997.2.$$

$$Imp_7=\sqrt{(1273.3)^2+49(1172.1)^2}=8302.8.$$

Hence, $\frac{Imp_3}{Imp_1}=2.161.\quad \frac{Imp_5}{Imp_1}=3.465.\quad \frac{Imp_7}{Imp_1}=4.798.$

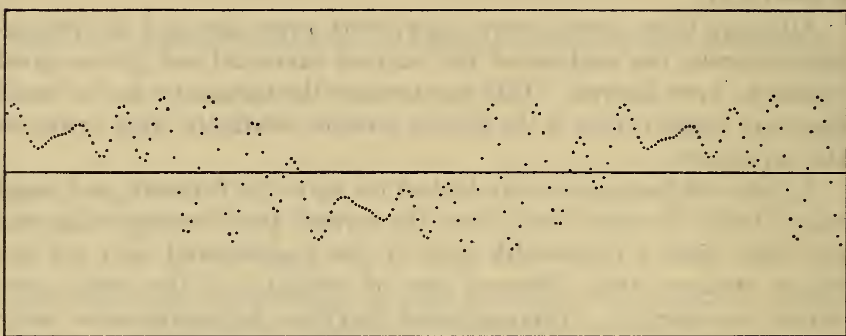

FIG. 4.—Current in condenser due to electromotive force of fig. 3.

These quotients show how much smaller relatively the third, fifth, and seventh harmonics are in the inductive current curve than in the electromotive force curve. Therefore the third, fifth, and seventh harmonics in the current curve may be found by dividing the corresponding harmonics in the electromotive force curve by the numbers 2.16, 3.46, 4.80, respectively. Small errors due to the curve itself or to the analysis are thus divided by these factors, and the values of the harmonics found are therefore more accurate and more consistent than if determined directly by analyzing the current curve.

This process may indeed be carried another step by placing a condenser in parallel with the circuit $A\ B\ C$, so that the alternating current flowing into it will be due to the same electromotive force that causes

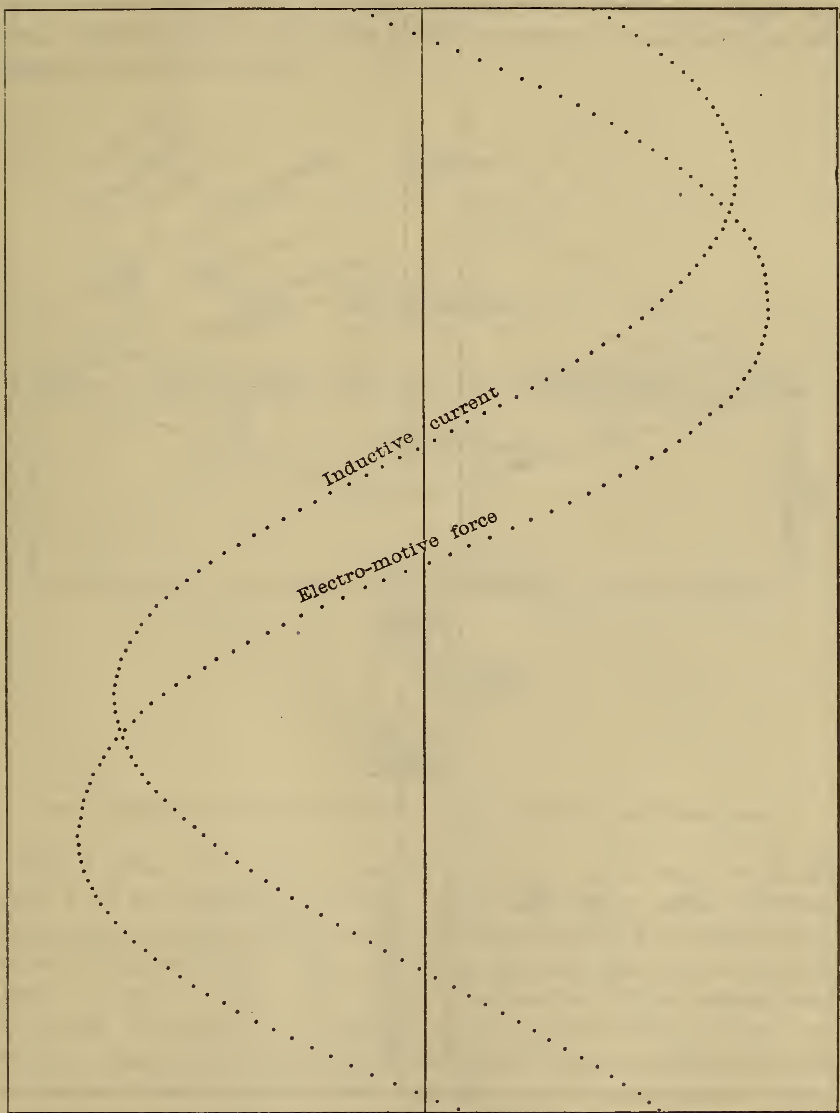

FIG. 5.

the current in the circuit $A\ B\ C$. In the condenser current, however, the harmonics are magnified in proportion to their order, the seventh harmonic in the condenser current being seven times as great, in proportion to the fundamental, as in the electromotive force, and hence 7×4.798 or 33.58 times as great as in the current through *A B C*.

Fig. 4 shows the current in a condenser due to such an electromotive

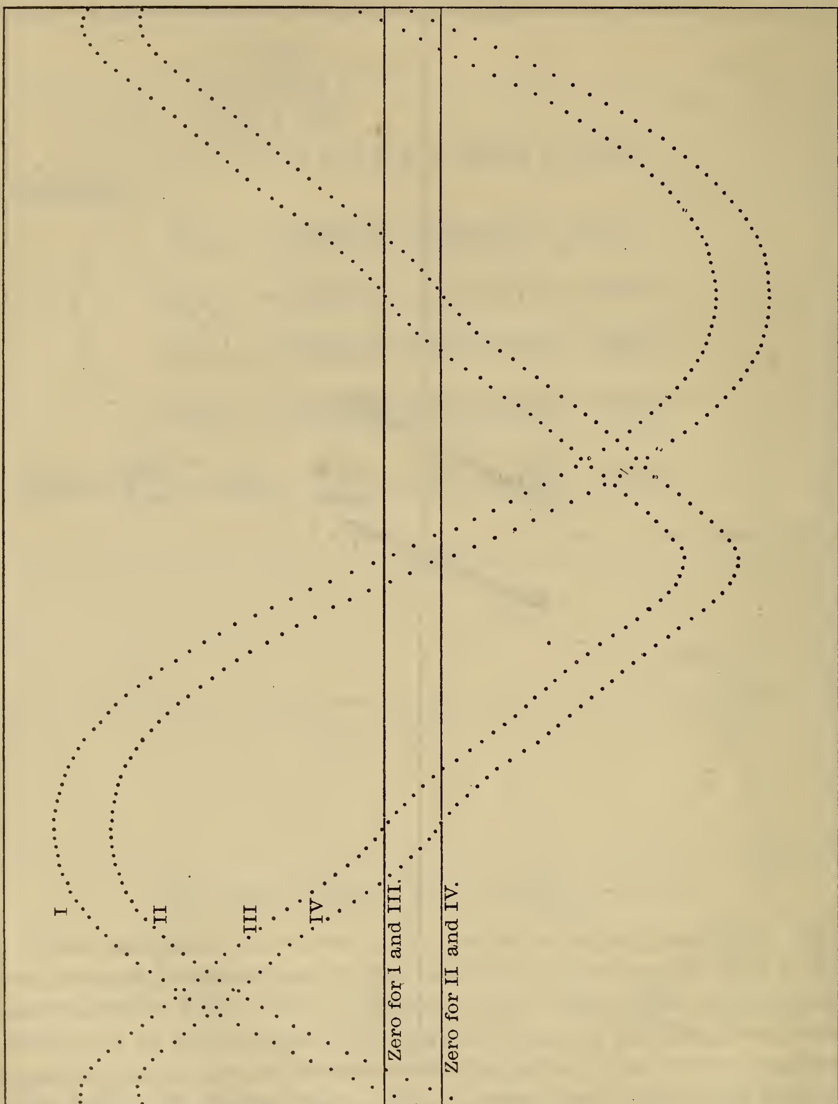

FIG. 6.

force as that of fig. 3, showing the prominence of the higher harmonics, especially those of thirteen and fifteen times the fundamental. In a similar manner we should expect that if there are any small harmonics of these frequencies in the current through the circuit *A B C*, they would be multiplied by 114 and 153, respectively, in such a condenser current in parallel with *A B C*. On the other hand, if no such harmonics are found in the condenser current we may be sure that the small values found by analysis in the inductive current were, as we supposed, residual errors.

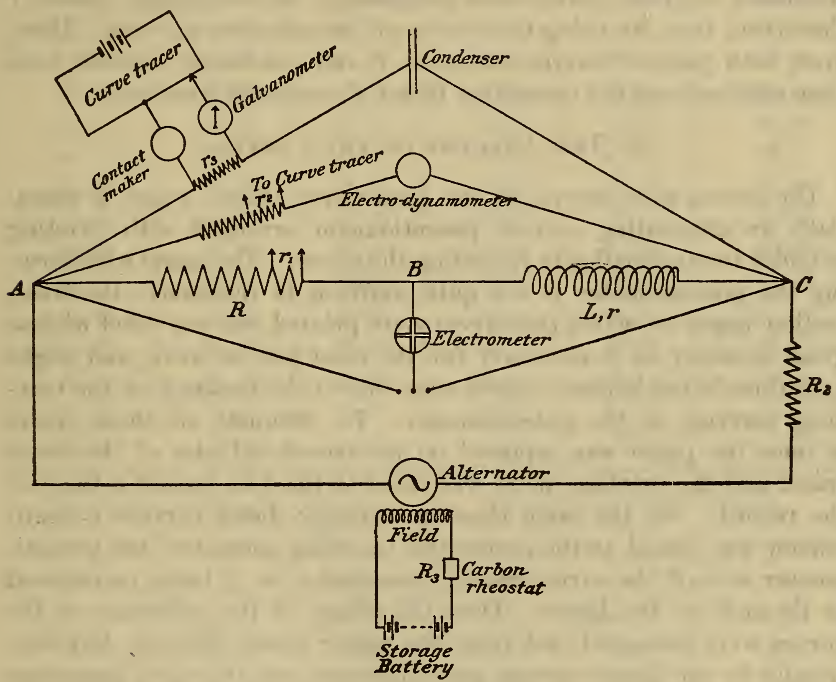

FIG. 7.—Diagram of connections for drawing curves of current and electromotive force.

Fig. 6 gives two curves *I* and *II* of e. m. f., and two curves *III* and *IV* of the condenser current. They were drawn under the same circumstances as prevailed during the measurement of the inductance, to be described later. The method of drawing the curves is illustrated in fig. 7. The curve tracer is connected to the terminals of the small noninductive resistances r_1, r_2, r_3, successively, these terminals being joined to the potentiometer of the curve tracer through the galvanometer on the one side and the contact maker on the other. Evidently a curve drawn with the connecting wires at the terminals of r_1 will represent the inductive current through *A B C;* similarly joining to r_2 gives the e. m. f. on *A C*, and joining to r_3 gives the condenser current. The curves of fig. 6, which are far from being sine curves, but which evidently have no appreciable high harmonics, show the magnification of the harmonics in the condenser current. Since these condenser currents can be drawn with the same precision as the electromotive force and current through the inductive circuit, it is evident that greater accuracy will be secured by analyzing the condenser current curves and calculating the correction factor f therefrom than by using the curves of the inductive current. However, both pairs of curves, the e. m. f. and condenser current, have been analyzed and the correction factor f computed from both.

6. The Analysis of the Curves.

The curves were drawn on the Rosa curve tracer, which is essentially an alternating current potentiometer arranged with printing cylinder for automatically recording the curve. The spiral wire forming the potentiometer is not quite uniform in resistance; the cross-section paper on which the curves were printed was not ruled with as great accuracy as is necessary for the most precise work, and slight variations in the printed record were due to the backlash of the traveling carriage on the potentiometer. To eliminate all these errors at once the paper was replaced on the record cylinder of the curve tracer and the printing point was re-set in the dots (one at a time) of the record. At the same time an ordinary direct current potentiometer was joined to the center and traveling contact of the potentiometer wire of the curve tracer, a constant e. m. f. being maintained at the ends of the latter. Thus the values of the ordinates of the curves were measured, not from the paper record directly, but electrically by the direct current potentiometer, and the most important errors in the curves were thereby eliminated. These errors were, of course, all relatively small, but important enough to be avoided in the present work.

The curves of fig. 6 were drawn after those of fig. 5, and in virtue of a better adjustment of the brushes of the alternator the current was somewhat steadier and the curves a little more accurate. Hence the results of the analysis of the curves of fig. 6 will be given. The alternator has 12 poles, thus giving 6 complete periods in one revolution. The contact maker has 3 contact points 120° apart, and thus makes a contact on every other wave. The first curve of each pair of fig. 4 was drawn while the contact maker turned, step by step, through 60°, and the second curve of each pair was drawn while the contact maker turned through another 60°. Thus one curve may be regarded as a composite made up of every other wave of the train superposed,

TABLE I.—*Showing the values of the three sets of ordinates of fifteen each (at angular intervals of 12°) for each of the four curves analyzed.*

[The values of the ordinates are expressed in arbitrary units.]

Curve I. Electromotive force.			Curve II. Electromotive force.		
a	*b*	*c*	*a*	*b*	*c*
− 4.72	− 1.90	+ 0.96	+ 1.30	+ 4.28	+ 7.05
+ 6.61	+ 9.51	12.17	12.41	14.96	17.56
17.05	19.23	21.76	22.12	24.16	26.12
26.13	27.95	29.96	29.90	31.63	33.28
33.38	34.82	36.43	36.32	37.63	38.86
39.02	40.09	41.07	40.96	41.82	42.57
42.64	43.36	44.01	43.70	44.18	44.44
44.50	44.77	44.92	44.70	44.74	44.72
44.96	44.94	44.76	44.56	44.22	43.88
44.12	43.70	43.09	42.92	42.43	41.55
41.86	41.02	39.99	39.53	38.57	37.49
37.75	36.46	35.08	34.79	33.06	31.50
31.98	30.18	28.36	28.10	26.13	23.92
24.60	22.56	20.45	19.87	17.47	14.94
15.27	12.90	10.58	9.88	7.12	4.52

Curve III. Condenser current.			Curve IV. Condenser current.		
a	*b*	*c*	*a*	*b*	*c*
+49.26	+49.49	+49.11	+49.06	+49.26	+48.87
46.62	45.03	43.46	46.60	45.15	43.41
39.63	37.91	36.21	39.85	37.70	36.04
32.46	30.96	29.20	32.61	31.00	29.14
25.88	24.11	22.02	25.86	24.22	22.42
17.80	16.10	13.64	17.96	16.06	13.86
9.46	7.37	5.54	9.69	7.78	5.98
+ 2.54	+ 1.21	+ 0.11	+ 2.98	+ 1.75	+ 0.67
− 2.24	− 3.44	− 5.08	− 1.76	− 3.03	− 4.76
− 8.49	−10.31	−12.54	− 8.08	− 9.83	−11.63
−16.49	−18.30	−20.18	−16.10	−17.99	−19.73
−23.68	−25.28	−27.08	−23.33	−24.92	−26.67
−30.46	−32.24	−33.88	−30.16	−32.20	−33.95
−38.01	−39.86	−41.87	−37.50	−39.53	−41.33
−45.70	−46.94	−48.25	−45.08	−46.64	−47.68

TABLE II.—*Results of analyses of the two e. m. f. curves (I and II) of fig. 6, and calculation of components of current in the inductive circuit.*

[The values of E and I are expressed in arbitrary units.]

Components of electromotive force found by analysis.							
	Fundamental $=E_1$	Harmonics.					
		E_3	E_5	E_7	E_9	E_{11}	E_{13}
Curve I	a=46.880	1.898	0.122	0.127	0.012	0.021	0.041
	b=46.838	1.840	0.186	0.200	0.051	0.024	0.075
	c=46.882	1.939	0.242	0.222	0.082	0.062	0.046
Curve II	a=46.620	1.889	0.223	0.154	0.066	0.022	0.055
	b=46.608	1.833	0.206	0.220	0.036	0.031	0.031
	c=46.601	1.833	0.185	0.241	0.075	0.041	0.009
Divisors=1		2.16	3.46	4.80	6.14	7.48	8.84

Components of inductive current calculated from above values.							
	Fundamental $=I_1$	Harmonics.					
		I_3	I_5	I_7	I_9	I_{11}	I_{13}
Curve I	a=46.880	0.878	0.035	0.027	0.002	0.003	0.005
	b=46.838	0.851	0.054	0.042	0.008	0.003	0.008
	c=46.882	0.897	0.070	0.046	0.013	0.008	0.005
Curve II	a=46.620	0.874	0.064	0.032	0.011	0.003	0.006
	b=46.608	0.848	0.059	0.046	0.006	0.004	0.003
	c=46.601	0.848	0.053	0.050	0.012	0.005	0.001

and the second curve similarly consists of the alternate waves (not included in the first) superposed. We have analyzed both pairs of curves.

In each wave of current and electromotive force 120 points are determined and printed at distances of 3° each in the wave, or 30′ each in the generator, since one revolution gives 6 complete waves. We have used 15 points in each half wave in each analysis for the fundamental and harmonics, the mean value of the ordinates of the positive and negative half waves being taken in each case. Three

TABLE III.—*Results of analyses of two condenser currents, and calculation of components of current in the inductive circuit.*

[The values of I are expressed in arbitrary units.]

Components of condenser current found by analysis.							
	Fundamental $=I_1$	Harmonics.					
		I_3	I_5	I_7	I_9	I_{11}	I_{13}
Curve III	42.374	5.119	0.854	1.095	0.101	0.052	0.090
	42.427	5.170	0.900	1.138	0.069	0.107	0.053
	42.393	5.110	0.847	1.172	0.108	0.150	0.032
Curve IV	42.162	5.117	0.809	1.067	0.088	0.066	0.058
	42.160	5.132	0.804	1.101	0.135	0.033	0.135
	42.218	5.116	0.809	1.145	0.119	0.125	0.060
Divisors	=1	6.48	17.33	33.58	55.24	82.22	114.7
Components of inductive current calculated from above values.							
	Fundamental $=I_1$	Harmonics.					
		I_3	I_5	I_7	I_9	I_{11}	I_{13}
Curve III	42.374	0.790	0.050	0.031	0.002	0.001	0.001
	42.427	0.798	0.052	0.034	0.001	0.001	0.000
	42.393	0.788	0.049	0.035	0.002	0.002	0.000
Curve IV	42.162	0.790	0.047	0.032	0.002	0.001	0.001
	42.160	0.792	0.046	0.033	0.002	0.000	0.001
	42.218	0.789	0.047	0.034	0.002	0.001	0.001

separate sets of ordinates were in this way prepared for each curve, and the analysis carried through for each set to the thirteenth harmonic—that is, the harmonic having a frequency of thirteen times the fundamental. The ordinates used in these analyses are given in Table I.

The results of these analyses are given in Tables II and III, where in the first part of each table the six values of the fundamental and of each harmonic are given in the proper columns, the first three as found for one curve, and the second group of three for the other.

TABLE IV.—*Calculation of correction factor f from electromotive force curves.*

	Curve I.			Curve II.		
	a	*b*	*c*	*a*	*b*	*c*
$I_1^2=$	2197.6846	2193.4475	2197.8827	2173.3750	2172.3367	2171.6776
$I_3^2=$	0.7714	0.7250	0.8051	0.7640	0.7195	0.7195
$I_5^2=$	0.0012	0.0029	0.0049	0.0041	0.0035	0.0028
$I_7^2=$	0.0007	0.0017	0.0021	0.0010	0.0022	0.0025
Sum ..	2198.4579	2194.1771	2198.6948	2174.1441	2173.0619	2172.4024
$I_1^2=$	2197.6846	2193.4475	2197.8827	2173.3750	2172.3367	2171.6776
$9\ I_3^2=$	6.9426	6.5250	7.2459	6.8760	6.4755	6.4755
$25\ I_5^2=$	0.0310	0.0720	0.1220	0.1025	0.0862	0.0712
$49\ I_7^2=$	0.0343	0.0853	0.1049	0.0490	0.1054	0.1235
Sum ..	2204.6925	2200.1298	2205.3555	2180.4025	2179.0038	2178.3478
f	0.99859	0.99865	0.99849	0.99856	0.99863	0.99863
	Mean from curve I=0.998577			Mean from curve II=0.998607		

TABLE V.—*Calculation of correction factor f from condenser current curves.*

	Curve III.			Curve IV.		
	a	*b*	*c*	*a*	*b*	*c*
$I_1^2=$	1795.6048	1800.0458	1797.1532	1777.6218	1777.4734	1782.3466
$I_3^2=$	0.6240	0.6360	0.6213	0.6230	0.6267	0.6227
$I_5^2=$	0.0024	0.0027	0.0024	0.0022	0.0022	0.0022
$I_7^2=$	0.0011	0.0012	0.0012	0.0010	0.0011	0.0012
Sum ..	1796.2323	1800.6857	1797.7781	1778.2480	1778.1034	1782.9727
$I_1^2=$	1795.6048	1800.0458	1797.1532	1777.6218	1777.4734	1782.3466
$9\ I_3^2=$	5.6160	5.7240	5.5917	5.6070	5.6403	5.6043
$25\ I_5^2=$	0.0607	0.0675	0.0598	0.0550	0.0540	0.0545
$49\ I_7^2=$	0.0529	0.0564	0.0598	0.0490	0.0524	0.0568
Sum ..	1801.3344	1805.8937	1802.8645	1783.3328	1783.2201	1788.0622
f	0.99858	0.99856	0.99859	0.99857	0.99856	0.99857
	Mean from curve III=0.998577			Mean from curve IV=0.998567		

In the middle of the table are given the divisors by which we determine any harmonic of the current in the inductive circuit $A\ B\ C$ from the harmonic of corresponding order in the e. m. f. or condenser circuit in parallel with it. The calculation of these divisors has already been given (p. 134). The second part of the tables gives the values of the harmonics of the current curves so calculated.

There is a very noticeable drop in the values of the harmonics after the seventh; these small quantities are without doubt residual errors arising in drawing the curve and getting the values of the ordinates. When they are divided by the corresponding divisors to get the harmonics of the current they become insignificant, amounting on the average to about one part in 8,000 of the fundamental for the e. m. f. curves and one part in 40,000 of the fundamental for the condenser current, and thus having no appreciable effect whatever in the correction factor to be determined.

7. The Calculation of the Correction Factor f.

In the first part of Tables IV and V are given the values of the squares of the harmonics as found from the analyses of the twelve sets of ordinates of the four curves, and in the second part of the tables the same squares multiplied by 1, 9, 25, 49, the coefficients occurring in the expression for the correction factor f. In the last line the values of f are given.

The mean value of f found from curve I differs from that found from curve II by only 3 parts in 100,000, whereas the mean value of f calculated from curve III differs from that of curve IV by only 1 part in 100,000.

If we had omitted the small fifth and seventh harmonics in the calculation of f, using only the fundamental and the third harmonic, the result would have differed by only 3 parts in 100,000. This illustrates how unimportant are all the harmonics above the third. But the residual errors in the current curve determined directly, instead of indirectly through the condenser current, are much greater. That the values obtained from different curves and from different sets of ordinates in the same curve agree so closely shows that the curve tracer gives a very accurate reproduction of the waves. It also proves that the six waves produced by one revolution of the twelve-pole alternator are very accurately the same, inasmuch as curves I and II were produced in different parts of the revolution, and III and IV also.

Summary of values of f.

Mean value from curve I of e. m. f	=0.998577	
Mean value from curve II of e. m. f	=0.998607	
Mean value from both curves		=0.998592(a)
Mean value from curve III, condenser current	=0.998577	
Mean value from curve IV, condenser current	=0.998567	
Mean value from both curves		=0.998572(b)
Weighted mean of (a) and (b), giving the result from the condenser current curves three times the weight of that of the e. m. f. curve		=0.998577

In what follows we shall use **0.99858** as the most probable value.

Having now determined the correction factor to be applied to the values of L found in the use of these alternating currents, we may proceed to an account of the determination of frequency.

8. Measurement of Frequency.

As already stated, the speed of the alternator was maintained as nearly constant as possible by hand control, using a carbon rheostat in the main circuit of the driving motor, the criterion of constant speed being that the deflection of a galvanometer, observed with a telescope and scale, should be kept zero. The galvanometer was joined to a Wheatstone bridge in which a condenser and a rotating commutator form one arm, precisely as though the object of the experiment was to determine the capacity of the condenser. This arrangement we have found to be a sensitive and satisfactory method of controlling the speed. The galvanometer is quick and nearly dead beat, and instantly shows any tendency of the motor to change its speed. Such tendency can be quickly checked by a slight change in the pressure on the carbon rheostat. Slight variations in speed above or below the normal cause corresponding deflections of the galvanometer to the right or left, and the operator balances these small deflections as nearly as possible during the period of the run.

To illustrate, we give the determinations of frequency for the first four runs of the observations of May 28, which are fair samples of all the runs. Contact is made on the chronograph every 50 revolutions of the alternator, and as the latter is a 12-pole machine there are 6 waves in every revolution, or 300 in every period of 50 revolutions. Ten contacts are read off at the beginning of each run and ten at the end, and from these the mean interval for the whole number of periods is determined. The first two runs were divided into two parts each, and the frequency found for each, in order to see what the change was. After that the mean speed was found for the whole period of the run.

TABLE VI.—*Run 1.*

Chronograph record—First part.			Chronograph record—Second part.		
Beginning.	End.	Interval.	Beginning.	End.	Interval.
Min. Sec.	*Min. Sec.*	*Seconds.*	*Min. Sec.*	*Min. Sec.*	*Seconds.*
10 11.16	13 11.60	180.44	13 31.65	15 20.27	108.62
12.82	13.26	.44	33.33	21.94	.61
14.49	14.94	.45	35.03	23.60	.57
16.15	16.61	.46	36.67	25.28	.61
17.84	18.28	.44	38.33	26.95	.62
19.51	19.94	.43	40.02	28.62	.60
21.18	21.62	.44	41.68	30.29	.61
22.85	23.30	.45	43.34	31.96	.62
24.52	24.97	.45	45.02	33.63	.61
26.20	26.63	.43	46.69	35.30	.61
108 periods =		180.443	65 periods =		108.608
$108 \times 300 = 32400$ waves.			$65 \times 300 = 19500$ waves.		
$n = \frac{32400}{180.443} = 179.557.$			$n = \frac{19500}{108.608} = 179.546.$		
Mean value of $n = 179.552.$					

TABLE VII.—*Run 2.*

Chronograph record—First part.			Chronograph record—Second part.		
Beginning.	End.	Interval.	Beginning.	End.	Interval.
Min. Sec.	*Min. Sec.*	*Seconds.*	*Min. Sec.*	*Min. Sec.*	*Seconds.*
18 2.29	22 9.59	247.30	24 53.34	29 59.09	305.75
3.97	11.26	.29	55.02	30 0.77	.75
5.64	12.93	.29	56.66	2.43	.77
7.32	14.61	.29	58.35	4.06	.71
8.99	16.27	.28	25 0.00	5.77	.77
10.66	17.95	.29	1.68	7.44	.76
12.34	19.62	.28	3.34	9.07	.73
14.02	21.30	.28	5.01	10.78	.77
15.67	22.96	.29	6.70	12.46	.76
17.34	24.64	.30	8.36	14.10	.74
148 periods =		247.289	183 periods =		305.751
$148 \times 300 = 44400$ waves.			$183 \times 300 = 54900$ waves.		
$n = \frac{44400}{247.289} = 179.542$			$n = \frac{54900}{305.751} = 179.558$		
Mean value of $n = 179.550$					

TABLE VIII.

Run 3. Chronograph record. Beginning.	Run 3. Chronograph record. End.	Interval.	Run 4. Chronograph record. Beginning.	Run 4. Chronograph record. End.	Interval.
Min. Sec.	*Min. Sec.*	*Seconds.*	*Min. Sec.*	*Min. Sec.*	*Seconds.*
5 51.69	12 59.11	427.42	14 21.55	20 19.97	358.42
53.37	13 0.81	.44	23.24	21.65	.41
55.06	2.50	.44	24.92	23.34	.42
56.75	4.18	.43	26.61	25.00	.39
58.43	5.87	.44	28.28	26.71	.43
6 0.10	7.53	.43	29.94	28.39	.45
1.78	9.23	.45	31.65	30.06	.41
3.47	10.90	.43	33.33	31.74	.41
5.16	12.60	.44	35.00	33.43	.43
6.85	14.28	.43	36.70	35.11	.41
254 periods	=	427.435	213 periods	=	358.418
254 × 300	=	76200 waves.	213 × 300	=	63900 waves.
$n=\frac{76200}{427.435}=178.272.$			$n=\frac{63900}{358.418}=178.284.$		

The examples given being the first runs of the first of the three days' work are not as good as the best, but show that the frequency can be maintained very nearly constant. The first two runs were made with the same resistances in the Wheatstone bridge, and the speed should therefore be the same. The mean frequencies found, 179.552 and 179.550, are practically identical. The resistance in the third arm of the bridge was now changed from 41,600 to 41,900, and the speed decreased until a balance was obtained. The frequencies for runs 3 and 4 were found to be, as shown in the table, 178.272 and 178.284. To show how nearly the mean speed remains constant, while the bridge resistance is unaltered, Table IX is given.

TABLE IX.—*Frequencies of alternating current in ten runs of May 28.*

[R=resistance in the third arm of the auxiliary Wheatstone bridge.]

R=41,600 ohms.	Run 1, n=179.552	Run 2, n=179.550
R=41,900 ohms.	Run 3, n=178.272	Run 4, n=178.284
R=42,200 ohms.	Run 5, n=177.009	Run 6, n=177.009
R=42,250 ohms.	Run 7, n=176.804	Run 8, n=176.793
R=41,250 ohms.	Run 9, n=181.083	Run 10, n=181.081

9. The Determination of R.

Instead of attempting to make the electromotive force on the non-inductive resistance $A\ B$ exactly equal to that on $B\ C$, we adjusted $A\ B$ to the nearest ohm and found the readings of the electrometer when the latter was joined first to $A\ B$ and then to $B\ C$, three pairs of readings being taken. If the resistance $A\ B$ was found to be too small it was then increased by one ohm and three pairs of readings of the electrometer again taken. By interpolation the value of the resistance was then found which would exactly balance the impedance of the inductive coil. As examples, the first three runs of June 2 are given.

Table X.—*Electrometer readings.*

	On $A\ B$	On $B\ C$	Difference ×100.	Mean.	Interpolated value of R
Run 1.					
L_C					
R=1193	14.92	14.765	−15.5		
	.92	.76	−16.		
	.925	.77	−15.5	−15.7	
R=1194	14.85	14.865	+ 1.5		
	.85	.86	+ 1.0	+ 1.2	1193.92
Run 2.					
L_F					
R=1178	14.92	14.95	+ 3.		
	.925	.955	+ 3.		
	.925	.96	+ 3.5	+ 3.17	
R=1177	14.975	14.87	−10.5		
	.975	.86	−11.5		
	.975	.87	−10.5	−10.83	1177.77
Run 3.					
L_C					
R=1194	14.78	14.79	+ 1.		
	.77	.78	+ 1.	+ 1.0	
R=1193	14.85	14.69	−16.		
	.85	.71	−14.		
	.85	.70	−15.	−15.0	1193.94

A summary of the results of three sets of measurements, taken on three separate days, is given in Tables XI, XII, and XIII. In each of these sets measurements are made successively on an inductance standard by Carpentier, of Paris, having a nominal value of one henry, and another by Franke & Co., of Hannover, of the same nominal value. In the tables the results for each coil are grouped together, the numbers of the runs indicating the order in which the measurements are made. The several columns give (1) the corrected values of the noninductive resistance R, (2) the ohmic resistance r of the inductive coil, (3) the frequency of the current, (4) the computed value of L from the formula (p. 127) uncorrected for wave form, (5) the mean of these latter values, and finally (6) the mean after correction for wave form, using $f=0.99858$. The corrected values of the resistances were found by means of a carefully calibrated bridge, and were redetermined each day.

TABLE XI.—*Results of May 28.*

	1	2	3	4	5	6
	R	r	n	L_1 uncorrected for wave form.	Mean of L_1	L
L_C						
Run 1	1149.98	97.6	179.551	1.01567		
Run 4	1141.90	97.75	178.284	1.01564		
Run 5	1133.93	97.9	177.009	1.01575	1.01572	1.01428
Run 8	1132.69	98.05	176.793	1.01586	(D=7.2 in 100000)	
Run 9	1159.76	98.2	181.083	1.01566		
L_F						
Run 2	1134.01	97.4	179.550	1.00144		
Run 3	1125.93	97.6	178.272	1.00140		
Run 6	1118.19	97.7	177.009	1.00156	1.00144	1.00002
Run 7	1116.68	97.8	176.804	1.00135	(D=5.2 in 100000)	
Run 10	1143.62	97.9	181.081	1.00145		

D = average deviation of a single determination.

TABLE XII.—*Results of May 31.*

	1	2	3	4	5	6
	R	r	n	L_1 uncorrected for wave form.	Mean of L_1	L
L_C						
Run 1	1194.42	97.0	186.550	1.01561	....	....
Run 4	1193.44	97.3	186.437	1.01540	....	....
Run 5	1192.36	97.5	186.230	1.01560	1.01560	1.01416
Run 8	1192.54	97.7	186.225	1.01577	(D = 9.5 in 100000)	
L_F						
Run 2	1178.16	96.9	186.561	1.00168	....	....
Run 3	1177.10	97.1	186.440	1.00141	....	....
Run 6	1176.13	97.2	186.234	1.00166	1.00160	1.00018
Run 7	1176.16	97.3	186.237	1.00168	(D = 10.2 in 100000)	

D = average deviation of a single determination.

10. Discussion of the Results.

The results of June 2 are somewhat more uniform than those of the two preceding days, due largely to the fact that the electrometer was adjusted to a greater sensibility and readings were consequently more accurate. The average deviations of the separate determinations from the mean is less than 4 parts in 100,000 in this set.

The two determinations of L at lower frequency on June 2 are subject to a correction for wave form which was not separately determined, and probably the capacity correction, which would be smaller for the lower frequency, accounts for part of the difference. A larger number of determinations at the lower frequency should be made to determine what difference is due to the different frequency. We mean to resume the experiments and make a longer series of measurements at other frequencies.

Table XIV shows a progressive change in the values of L_C and L_F, but in opposite directions; in both cases the change between May 28 and June 2 is about 3 parts in 10,000. Referring to Tables XI, XII, and XIII, in which the values of r, the resistances of the coils are given, it will be observed that the resistances happened to be almost exactly equal for the two coils, and that both were lower on May 31

TABLE XIII.—*Results of June 2.*

	1	2	3	4	5	6
	R	r	n	L_1 uncorrected for waveform.	Mean of L_1	L
L_C						
Run 1	1194.32	96.9	186.579	1.01541		
Run 3	1194.35	96.9	186.584	1.01541		
Run 5	1194.67	96.9	186.621	1.01549	1.01542	1.01398
Run 8	1182.92	96.9	184.791	1.01539		
Run 10	1182.96	96.9	184.793	1.01541	(D=2.6 in 100000)	
Run 11	687.53	96.9	106.704	1.01525		
Run 13	687.60	96.9	106.704	1.01535	1.01530	
L_F						
Run 2	1178.18	96.7	186.578	1.00162		
Run 4	1178.56	96.7	186.619	1.00173		
Run 6	1178.56	96.7	186.621	1.00171	1.00173	1.00031
Run 7	1167.11	96.7	184.778	1.00183		
Run 9	1167.07	96.7	184.784	1.00174	(D=4.8 in 100000)	
Run 12	678.25	96.7	106.693	1.00142		
Run 14	678.35	96.7	106.698	1.00152	1.00147	

D=average deviation of a single determination.

TABLE XIV.—*Summary of results.*

	L_C	L_F	Ratio$=\frac{L_C}{L_F}$
May 28	1.01428	1.00002	1.01426
May 31	1.01416	1.00018	1.01398
June 2	1.01398	1.00031	1.01367

than on May 28, and still lower June 2, the total difference being almost exactly one ohm for each coil. This corresponds to about 2.5° C., and is due to the lower temperature of the laboratory on the later days. We were surprised to find evidence of a positive temperature coefficient in one coil and a negative coefficient in the other, and therefore made some direct comparisons of the two coils with each other with a view to testing this point. L_C, being maintained at a constant temperature of about 21.5° C., L_F was cooled about 3° by leaving it in a cooler room over night. The two coils being balanced against each other, with a variable inductance included with the smaller, L_F was warmed in an inclosed space and its inductance was observed to *decrease* about 3 parts in 10,000. On another day L_F was kept constant and L_C heated in a similar manner. The result was an *increase* in the value of L_C. L_F being again heated while L_C remained constant, its value *decreased* with respect to L_C. An exact measure of the change of temperature was not obtained, and hence no definite value of the temperature coefficient was found.

A possible explanation of the opposite sign of the temperature coefficients suggested itself when we removed the covering of L_F. This coil is wound on a spool of serpentine, and the wire is embedded in paraffin. The formula for the induction of such a coil is

$$L = 4\pi n^2 a \left(\log \frac{8a}{R} - 2 \right)$$

where a is the mean radius of the coil and R is the geometric mean distance of the wires in the cross section of the coil. When the paraffin (which has a temperature coefficient many times larger than copper) expands, it tends to increase the geometric mean distance of the wires and so decrease L, and this effect may be greater than the increase due to the expansion of the copper, which increases a. The other coil, however, is wound on a spool of mahogany with dry, silk-covered wire, and there is no such tendency to increase R. Whether part of the observed increase of L_C with increase of temperature is due to the spool itself we do not know.

It is evident that we must either keep these coils continuously at a constant temperature when measuring their inductances, or else get some new ones not subject to so large temperature coefficients. Whether this is possible we do not know, but hope soon to make some trials in this direction and also to study more carefully the magnitude of the temperature coefficients of these coils and their causes.

This method of measuring inductance is capable of yielding somewhat better results than those given above, when all possible refinements are introduced. It seems to us desirable to measure in this way some carefully constructed inductance standards whose values can be computed from their dimensions. The determination of such pairs of values of L would amount to an absolute determination of the international ohm.

We are indebted to Mr. C. E. Reid for assistance in making some of the observations recorded in this paper and to Dr. N. E. Dorsey for assistance in analyzing the curves.