EFFECT OF CONCENTRATED LOADS ON THE LENGTH OF MEASURING TAPES

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ABSTRACT

In the standardization and use of geodetic base line tapes there is a need for a precise formula enabling one to compute the change in the length of a tape when concentrated loads, such as thermometers, are added to the tape. Three formulas differing slightly have either been published or communicated to the author. When reduced to the same notation it is seen that they have a great deal of similarity. The most general formula is that of W. D. Lambert, of the United States Coast and Geodetic Survey.

Experimental data confirming this formula and that for the distance between the terminal marks of a tape suspended in a series of catenaries are given, and the possibility is shown of using in many cases a more simplified formula derived by Dr. C. Snow, of the Bureau of Standards.

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I. INTRODUCTION

In the standardization of metal tapes the tapes are supported throughout their length on a suitable horizontal support or are suspended at intervals along their length. Although the treatment of the elementary cases is given in the textbooks on geodetic surveying, there are important cases arising in modern geodetic surveying and in the standardization of the tapes for the geodesist which are not discussed in the texts. One of these is the case of vertical forces concentrated at points along the tape and it is the purpose of this paper to state the formulas applying in such cases and to give the results of experiments which have been carried out to determine their validity. The special application is to thermometers attached to base line tapes.

II. DESCRIPTION OF APPARATUS

The 50-meter geodetic tape comparator at the United States Bureau of Standards was used in this experimental work. This apparatus, which is shown in Figure 1, is, in its general features, the field base-measuring apparatus described in the annual report of the United

States Coast and Geodetic Survey for 1892.[1] It has been considerably modified in the laboratory, which is an underground tape-testing room. The standard used is the 5 m Woodward iced bar described in the publication of the Coast Survey referred to above. This bar has been standardized at intervals of about five years and is known to have retained a most satisfactory constancy, especially in the recent years when it has not been used in the field but kept all of the time in the laboratory. A similar standard is in use in Japan, and its length is also reported to be constant.

During the past few years a number of modifications have been made in this comparator as a part of an endeavor to reach an increased accuracy in geodetic tape standardization. These modifications have included the installation of specially designed rollers with ball bearings for the terminal and intermediate supports, the substitution of calibrated weights attached to the tape by a flexible wire passing over a grooved pulley which is adjustable in height in place of the spring balance formerly used, and the installation of new and greatly improved cut-off cylinders by the use of which the length of the base can be determined rapidly and accurately. These improvements made possible the high precision obtained in our experimental work.

In some of the measurements undertaken, comparative rather than absolute lengths were required. In such instances there was no occasion to use the 5 m bar at all, the apparatus being used as one of the simplest type of comparator; that is, one in which the fixed distance between the microscopes is the standard with which the unknown lengths are compared.

III. MATHEMATICAL FORMULAS FOR THE EFFECT OF CONCENTRATED LOADS

Among the papers which have been published giving the mathematical theory of the steel tape may be mentioned those of Woodward,[2] Jäderin,[3] Guillaume,[4] Professor and Captain Henrici,[5] and Young.[6] Of these only the last treats the case of concentrated load. Two mathematical treatments, unpublished as yet, which deal with this special case have been communicated to the author. One of these is by Dr. Chester Snow, of the Bureau of Standards, the other

[1] Woodward, R. S., On the Measurement of the Base Lines at Holton, Ind., and at St. Albans, W. Va. App. 8, U. S. Coast and Geodetic Survey Report for 1892.

[2] Woodward, R. S., Mathematical Theory of Metallic Tapes, U. S. Coast and Geodetic Survey Report for 1892, Appendix 8, pp. 480 to 489.

[3] Jäderin, E., On the Measurement of Base Lines with Steel Tapes and with Steel and Brass Wires (Translated by J. H. Gore), U. S. Coast and Geodetic Survey Report for 1893, Appendix 5.

[4] Guillaume, C. A., La Mesure Rapide des Bases Géodésiques (published by Gauthier-Villars, Paris, 1917).

[5] Henrici, Prof. O., and Henrici, Capt. E. O., Theory of Tapes in Catenary. Professional Papers No. 1 (New Series), Ordnance Survey of Great Britain, 1912.

[6] Young, A. E., On the Form of a Suspended Wire or Tape, including the Effect of Stiffness, Phil. Mag. 29, p. 96; 1915.

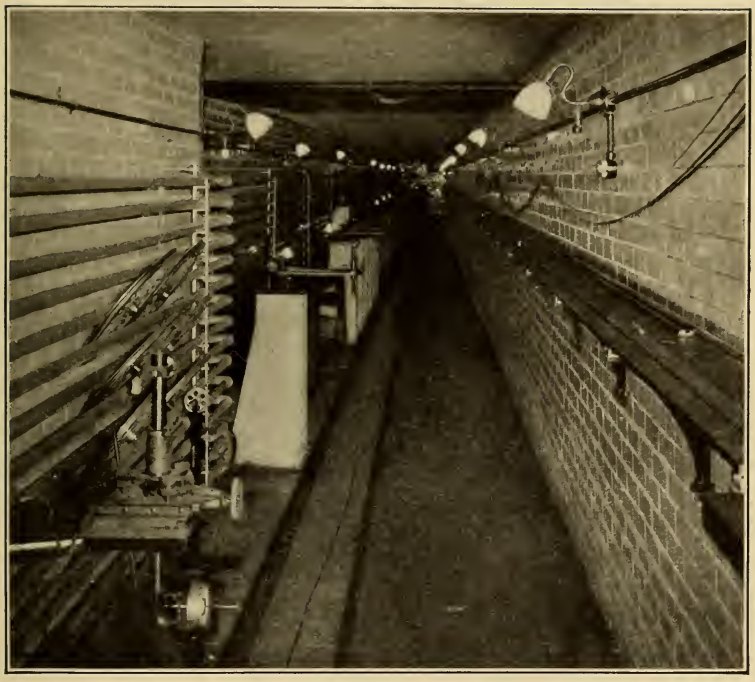

FIG. 1.—*Tunnel laboratory for length comparison of measuring tapes*

is by W. D. Lambert, of the United States Coast and Geodetic Survey, to each of whom the author desires to express his thanks for courtesies extended.

Since, both in the laboratory and in the field, base-line tapes are commonly used when suspended in loops with a thermometer placed 1 m inside the terminal graduations at each end, it is important to know the effect upon the length of the tape of these concentrated loads. This is especially necessary if it is desired to calculate the length of the tape when supported throughout its entire length. The length of a tape is defined as the straight-line distance between the terminal graduations, not the length along the tape.

In his article which appeared in the Philosophical Magazine, Young [7] considered a concentrated load to be equivalent to a point condensation of a portion of the tape sufficiently heavier than the remainder of the tape to fulfill the requirements. His formula for the shortening of a tape suspended in a single loop, when a single concentrated load is applied, is

$$\frac{l}{24T_0^2}\left[12wl\,Wn\,(1-n)+12\,W^2n\,(1-n)\right] \tag{1}$$

where

l=horizontal distance between terminal marks,
T_0=horizontal component of tension,
W=concentrated load,
n=fractional part of total length from end to point of loading,
w=linear density of tape.

A general formula was not published by Young, the only extension of the above formula being that for a number of equal concentrated loads spaced at equal distances along a tape which is suspended in a single span.

Doctor Snow conceived the idea of expressing the mathematical equation in terms of the change in angle of the tape at the support when the load is added. He limited the treatment to the case of geodetic importance in which the quantity $\frac{\rho l}{W}$ is sufficiently small so that only the 4th order of these small quantities need be retained, ρ being the weight per unit length of the tape, l half of the distance between two adjacent supports, and W the terminal tension. The supports are assumed to be on the same level and equally spaced. In the course of the derivation there is proven what by some may be considered to be sufficiently evident, namely, that the effect of any one load is of negligible importance upon the length of any span other than the one in which it is located and that the combined effect of two or more separate loads at any points on the tape is

[7] See footnote 6, p. 386.

equal to the sum of the several individual effects calculated independently.

Doctor Snow's formula is

$$L_n^1 = L_n - \frac{\rho}{2W^2} \sum_{k=1}^{k=N} w_k \left[\left(\frac{L}{2n} \right)^2 - x_k^2 \right] \tag{2}$$

where

L_n^1=length of tape supported in n loops with N concentrated loads,

L_n=length of tape supported in n loops without concentrated loads,

n=number of loops in which tape is supported,

ρ=linear density of tape,

W=terminal tension,

L=length of tape supported throughout at tension W.

(Note.—In some cases L may be taken as the nominal length of the tape.)

x_k=distance of the kth load from the lowest point of the loop in which it is placed.

Mr. Lambert derived his formula by writing the equation of two catenaries passing through a common point on the tape which is that point at which the load is applied. These equations are written in terms of the same parameter c, which is the quantity obtained by dividing the horizontal tension at either point of support by the weight per unit length of the tape measured along the catenary. For convenience, the origin was taken at the middle point of the line joining the two points of support which are assumed to be in the same level, and the equations are simplified by expanding in power series. In order to extend the formula for a single load, Lambert points out that "The curves between the points of application are all portions of catenaries having the same parameter c, but different vertices. The equations for the equilibrium of the points of application of the weights are sufficient to determine the coordinates of the vertices." His formula for a single load is

$$S - S_0 = \left(\frac{a^2 - h^2}{2c^2} \right) \left(l + \frac{l^2}{2a} \right) \tag{3}$$

where

S=length along catenary with load,

S_0=length along catenary without load,

a=one-half distance between terminal supports,

h=horizontal distance from the load to midpoint of the loop,

$$c = \frac{\text{horizontal tension}}{\text{weight per unit length of catenary}}$$

$$l = \frac{\text{concentrated load}}{\text{weight per unit length of catenary}}$$

The formula of Young and that of Lambert for a single load may be shown to be identical and also equal to that of Doctor Snow multiplied by the factor $\left(1+\frac{nw}{\rho L}\right)$. The additional term in the formula of Young and the one of Lambert is of a higher order than was considered by Snow.

Adopting the notation of equation (2) above, the formulas of Young, Snow, and Lambert in the most general form given by these authors are, respectively:

$$L_n^1 = L_n - \frac{\rho}{2W^2}\sum_{k=1}^{k=N}\left\{\left[\left(\frac{L}{2n}\right)^2 - x_k^2\right]\left[w_k + \frac{nw_k^2}{\rho L}\right]\right\} \tag{1'}$$

where the summation is restricted to cases in which the loads are uniformly distributed along the tape and $n=1$.

$$L_n^1 = L_n - \frac{\rho}{2W^2}\sum_{k=1}^{k=N}\left\{\left[\left(\frac{L}{2n}\right)^2 - x_k^2\right]\left[w_k\right]\right\} \tag{2'}$$

where no restrictions are placed on k, x_k, or n.

$$L_n^1 = L_n - \frac{\rho}{2W^2}\sum_{k=1}^{k=N}\left\{\left[\left(\frac{L}{2n}\right)^2 - x_k^2\right]\left[w_k + \frac{nw_k^2}{\rho L}\right]\right\} \tag{3'}$$

with no restrictions on k, x_k, or n.

Since Young did not extend his formula to a tape in several loops, or to weights in any possible position, the most general formula is that indicated by Lambert. This may be combined with the classical formula which expresses the difference in length of a tape when supported throughout and when it is supported at a number of equidistant points.

Using the same notation as that of equation (2) this combined formula is

$$L_n^1 = L - \frac{L}{6}\left(\frac{\rho L}{2nW}\right)^2 - \frac{\rho}{2W^2}\sum_{k=1}^{k=N}\left\{\left[\left(\frac{L}{2n}\right)^2 - x_k^2\right]\left[w_k + \frac{nw_k^2}{\rho L}\right]\right\} \tag{4}$$

To investigate the magnitude of the term

$$\frac{\rho}{2W^2}\sum_{k=1}^{k=N}\left\{\left[\left(\frac{L}{2n}\right)^2 - x_k^2\right]\left[\frac{nw_k^2}{\rho L}\right]\right\}$$

the following table has been prepared for a 50 m tape when at a tension of 15 kg, with k equal to unity:

TABLE 1.—*Magnitude of quadratic term in the concentrated load formula*

[L=50 m, W=15,000 g; one concentrated load]

N	x	w	$\frac{1}{2W^2}\left[\left(\frac{L}{2}\right)^2-x^2\right]\frac{Nwk^2}{L}$
	m	*g*	*μ*
4	5.25	25	1
4	5.25	50	5
4	5.25	150	46
4	0.00	50	17
4	0.00	150	156
2	11.50	50	5
2	11.50	150	48
2	5.50	50	28
2	5.50	150	252
2	0.00	50	34
2	0.00	150	312
1	24.00	50	5
1	24.00	100	22
1	24.00	150	49
1	10.00	50	58
1	10.00	150	525
1	0.00	50	69
1	0.00	150	625

It is seen that for ordinary geodetic uses in computing the effect of the weight of thermometers placed on the tape near the ends upon the tape length the quadratic term is of little practical importance, and considering only the effect of the concentrated loads the three formulas reduce to (2). As the quadratic term is independent of ρ, the above table applies to uniform 50 m tapes of all linear densities.

IV. EXPERIMENTAL VERIFICATION OF THE FORMULA

The term $-\frac{L}{6}\left(\frac{\rho L}{2nW}\right)^2$, in formula (4) given above is a well-known expression for obtaining the length of a tape supported throughout when its length supported at equidistant points is known. Recognized as entirely legitimate for the ordinary engineer, its validity for the modern geodesist has occasionally been questioned, probably because of difficulties in obtaining precise standardizations of a tape when supported throughout. The following data, a comparison of the measured change in length obtained for tapes supported in three and in five loops and the calculated change using the formula, shows that the reliability of the formula is amply sufficient for all geodetic purposes.

TABLE 2.—*Effect of changing the number of spans in a 50 m tape*

Tape	Weight of tape	Length in four spans minus length in two spans		Difference
		Observed	Computed from formula	
	g/m	*μ*	*μ*	*μ*
A	21.8	2,070	2,060	+10
B	25.1	2,740	2,733	+7
C	25.5	2,826	2,822	+4
D	25.8	2,878	2,888	−10
E	25.8	2,905	2,888	+17
F	26.6	3,049	3,071	−22

That equation (2) expresses the facts to a very good approximation for the usual cases encountered in the standardization of geodetic tapes at the Bureau of Standards is proven by the following data:

TABLE 3.—*Effect of thermometers upon length of tape*

[Two thermometers weighing 28.6 g each hung 1 m from each end of tape]

Tape	Linear density of tape	Change in length in microns					
		Tape in two loops			Tape in four loops		
		Observed	Computed	Observed minus computed	Observed	Computed	Observed minus computed
	g/m						
A	25.7	−74.8	−78.5	+3.7	−44.2	−37.6	−6.6
B	25.4	−89.7	−77.7	−12.2	−46.4	−37.2	−9.2
C	25.8	−75.1	−78.8	+3.7	−43.8	−37.7	−6.1
D	25.7	−80.4	−78.5	−1.9	−39.6	−37.6	−2.0
E	25.6	−79.7	−78.2	−1.5	−44.6	−37.4	−7.2
F	25.8	−84.6	−78.8	−5.8	−42.7	−37.7	−5.0
G	25.7	−77.7	−78.5	+.8	−40.6	−37.6	−3.0
H	25.5	−76.9	−77.8	+.9	−40.2	−37.3	−2.9
I	25.6	−76.4	−78.2	+1.8	−41.6	−37.4	−4.2
J	25.7	−75.1	−78.5	+3.4	−39.4	−37.6	−1.8
K	25.8	−75.6	−78.8	+3.2	−40.2	−37.7	−2.5
L	25.2	−81.9	−76.9	−5.0	−42.8	−36.9	−5.9
Mean				−.2			−4.7

The complete formula, including the quadratic term, is even more closely checked. The computed value for the change in length of each tape when the tape is in two loops is then a greater negative value by 3.5 μ and that for the tape in four loops a greater negative value by 3.4 μ. The mean variations are then +3.3 and −1.4 μ and in place of −0.2 and −4.7 μ, respectively. Five of the quantities in the last column become positive.

A table giving these changes in length may be prepared and will be found to save time if many computations are to be made. The following table based on the complete quadratic formula is being

used at the Bureau of Standards. As the quadratic term is independent of the weight per meter of the tape it needs to be computed but once for each of the columns of this table and added to the linear expression. This is a simple operation and enables the bureau to use the most precise formula in its fundamental work. In using this formula care must be taken to choose a consistent set of units. If the weight per meter of the tape is used and expressed in grams, then all lengths must be expressed in meters, and all weights and tensions in grams, and the final computed result will be in meters.

TABLE 4.—*Change in length of 50 m invar tapes under a tension of 15 kg due to two thermometers, weighing 45.0 g placed 1 m from the ends*

Weight per meter of tape in grams	Change in length in microns		Weight per meter of tape in grams	Change in length in microns	
	Three supports	Five supports		Three supports	Five supports
24.0	124	64	25.6	132	67
24.1	124	64	25.7	132	67
24.2	125	64	25.8	132	68
24.3	125	64	25.9	133	68
24.4	126	64	26.0	133	68
24.5	126	65	26.1	134	68
24.6	127	65	26.2	134	69
24.7	127	65	26.3	135	69
24.8	128	65	26.4	135	69
24.9	128	66	26.5	136	69
25.0	129	66	26.6	136	70
25.1	129	66	26.7	137	70
25.2	130	66	26.8	137	70
25.3	130	66	26.9	138	70
25.4	130	67	27.0	138	70
25.5	131	67			

NOTE.—The effect of placing the thermometer on the tape is to make its apparent length shorter.

Experiments have also shown that the formula is of more general validity if the spacing of supports, the tension, and the linear density of the tape are such that the sag of the tape is not greatly in excess of a decimeter.

It will be noted that the sensitive condition by which to test whether the term in the formula involving the square of the concentrated mass is essential in order to express physical facts would be that case in which a weight is placed at the middle point of a tape suspended at the two ends only.

Measurements made by Young of this general character were by no means conclusive on this point, as his published data had residuals as high as 0.090 link (18.034 mm), the experiments having been made out of doors, apparently under unfavorable conditions.

Our data leave but little to doubt on this question, the measurements being a set of measurements using a 50 m base-line tape having a linear density of 25.7 g per meter stretched under a tension of 15 kg.

TABLE 5.—*Effect of concentrated load at mid-point of a catenary*

Concentrated load in grams	Observed shortening	Computed shortening			
		Linear formula	Variation	Quadratic formula	Variation
	μ	μ	μ	μ	μ
10	358	357	1	360	2
20	713	714	1	725	12
30	1,085	1,071	14	1,096	11
40	1,470	1,428	42	1,472	2
50	1,837	1,785	52	1,854	17
60	2,248	2,142	106	2,242	6
70	2,638	2,499	139	2,635	3
80	3,036	2,856	180	3,034	2
90	3,436	3,212	224	3,437	1
100	3,850	3,569	281	3,847	3

V. CONCLUSION

The general formula, equation (4), expressing the effect of sag and of concentrated loads upon the distance between the terminal marks on a tape is found to be rigidly applicable even for the most precise work. For ordinary cases arising in geodetic practice, such as the application to the case of thermometers placed near the terminal graduations, the quadratic term may be omitted and the formula thus materially simplified for routine numerical computations. Care must be taken, however, to see that the quadratic term actually is negligible in any given case.

WASHINGTON, April 15, 1926.