SPECTRORADIOMETRIC ANALYSIS OF RADIO SIGNALS.

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ABSTRACT.

When a radiation, regularly repeated, is received by a simple circuit possessing inductance, capacity, and resistance there is induced in it an electric current whose effective value may be measured by a hot wire ammeter, provided the signal is repeated for a sufficient time to take readings of the instrument, and provided that the time between the signals is very small compared to the period of the ammeter. If, while the radiation continues, the circuit be tuned to various frequencies in a certain range, and the current observed in each case, then these current readings may be plotted against resonance frequency of the circuit. If the latter has a relatively high resolving power compared to the spectral structure of the wave, such a curve will be proportional to the spectral distribution of energy in the radiation. As a matter of fact this is not usually the case, and to derive the energy distribution of the wave requires that a certain integral equation be solved in which the above-mentioned curve is the known function. Several methods of obtaining a practicable solution of this equation are developed here and applied to experimental observations. The function of this paper is to develop a method of spectrum analysis which may be applied to the study of the interference-producing quality of a given sending station.

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I. INTRODUCTION—DERIVATION OF FUNDAMENTAL EQUATION.

The question as to what constitutes interference quality in a radio wave is one of such importance as to deserve both experimental and theoretical study. Undoubtedly a thoroughgoing spectrum analysis of the radiation when properly interpreted ought to furnish a logical basis for rating the interference-producing qualities of a sending station. It must be confessed that, in spite

of the urgency of the need of it, no satisfactory meaning for the term "coefficient of interference of a sending station" has yet been arrived at, either from a scientific or a legal aspect. The accepted notion of "decrement" seems to have been considerably overworked, and it is hoped that the somewhat larger conception of this article may prove useful.

Some time ago R. T. Cox, of this bureau, began an experimental analysis of certain simple types of radiation with the object of developing a method by which the interference-producing quality of the wave might be characterized. In interpreting the results obtained, it immediately became evident that, even with a circuit of the finest attainable resolving power, the record partakes of the nature of the receiving circuit as well as of that of the incoming wave. To disentangle the latter from the former requires the solution of a certain integral equation, and this constitutes the aim of this article.

If the radiation from a given sending radio station is received simultaneously by a set of simple circuits whose constants are properly chosen, the arrangement constitutes a spectroscope. From measurements of the effective current induced in each circuit, the spectral distribution of energy in the wave may be found. Or if the message can be repeated, a single circuit may be used and its constants varied, the effective square of the current being measured for each value of the constant which characterizes the circuit.

When the circuit possesses very high selectivity so that it may be considered to respond to a definite frequency only, then if it be tuned successively to various frequencies and the mean square of the current be measured for each, this will be proportional to the spectral distribution of intensity in the radiation. Such an ideal case is not always attainable in practice. The observed current as a function of frequency is often profoundly affected by the character of the receiving circuit. The resonance curve of every circuit possess a finite width, due to the fact that there is a dissipation of energy in the circuit. A spectral structure in the radiation, which is fine compared to the resonance curve of the circuit, may be entirely masked. The problem which then presents itself is to find how the circuit impresses its character upon the record in order to make corrections for it, and draw inferences as to the character of the radiation which may be, at first glance, entirely hidden in the observations. It is analogous to attempt-

ing to resolve a light source by means of a spectroscope of insufficient resolving power, where one makes up for lack of resolving power by a quantitative study of the energy distribution in the record, a procedure which is feasible in the range of radio frequencies.

The current in a circuit possessing resistance, inductance, and capacity, R, L, and C, respectively, oscillating freely, is of the form $Ae^{-2\pi at} \sin 2\pi x\ (t+\gamma)$
Where

$$a=\frac{R}{4\pi L} \text{ and } x=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}} \tag{1}$$

The quantity $\delta=\frac{2\pi a}{x}$ is usually called the logarithmic decrement of the circuit. If the capacity only is varied, the damping coefficient, $2\pi a$, remains constant, and the characteristic frequency, x, may be given any value, real or imaginary. It will become evident that if a radiation is examined for all real values of x, this is sufficient to determine its energy distribution.

The wave produces in the circuit an electromotive force $\mathfrak{E}(t) = E_n \sin 2\pi\ x_n\ (t+\alpha_n)$ where E_n is referred to in the following as the Fourier component of the radiation. It is proportional to the electromagnetic field in the case where the receiving circuit consists of antenna with capacitative coupling to earth, but in the case of coil aerials E_n will be proportional to the time derivative of the field and hence contains the frequency as a factor. This will maintain in the circuit the current $\mathfrak{I}\ (t) = I_n \sin 2\ \pi\ x_n\ (t+\beta_n)$ whose mean square is

$$\overline{\mathfrak{I}^2}=\frac{I_n^2}{2}=\frac{1}{2}\frac{E_n^2}{R^2+\left(2\pi x_n L-\frac{1}{2\pi x_n C}\right)^2}$$
$$=\frac{E_n^2}{8\pi^2L^2\left[4a^2+\left(x_n-\frac{X^2}{x_n}\right)^2\right]}$$

where $X=\frac{1}{2\pi}\sqrt{\frac{1}{LC}}$ = the resonance frequency of the circuit, or the frequency which would elicit the greatest response from the circuit. It is connected with the characteristic frequency, x, by the relation

$$x^2=X^2-a^2 \tag{2}$$

There is very little difference between these two frequencies for a good receiving circuit, but the exact relations which follow are more simple when expressed in terms of x. The frequency, X, is the one given by wave-meter measurements.

If for brevity we let

$$f(x) = 8RL\,(x\overline{\mathfrak{I}^2}) \tag{3}$$

then

$$f(x) = \frac{4ax}{\pi}\,\frac{E^2_n}{4a^2 + \left(x_n - \dfrac{a^2+x^2}{x_n}\right)^2}$$

for a simple harmonic wave of frequency x_n and amplitude (of emf) E_n.

A similar formula holds, however, if the electromotive force is expressed as a finite or infinite number of such components, if $\overline{\mathfrak{I}^2}$ denotes the time average of current squared over a large number of periods. Making the usual assumption that it holds when there are present all frequencies from $x' = O$ to $x' = \infty$ gives the integral equation

$$f(x) = \frac{4ax}{\pi}\int_0^\infty \frac{E^2(x')\,dx'}{4a^2 + \left(x' - \dfrac{a^2+x^2}{x'}\right)^2} \tag{4}$$

to determine $E^2(x)$ in terms of the function $f(x)$ which is given by observations.

The proof in the case of a Fourier's series

$$\mathfrak{I}(t) = \sum_{n=1} I_n \sin\frac{2\pi n}{T}(t+\beta_n)$$

is effected by squaring the series and noting that the integral

$$\frac{1}{NT}\int_0^{NT} \sin\frac{2\pi m}{T}(t+\beta_m)\sin\frac{2\pi n}{T}(t+\beta_n)\,dt$$

taken over a large number of periods, NT, is practically zero if m is not equal to n and is one-half if $m = n$. This gives

$$\overline{\mathfrak{I}^2} = \frac{1}{NT}\int_0^{NT} \mathfrak{I}^2(t)\,dt = \Sigma\frac{I^2_n}{2} = \frac{1}{2}\Sigma\,\frac{E^2_n}{R^2 + \left(2\pi x_n L - \dfrac{1}{2\pi x_n C}\right)^2}$$

and holds no matter how large the finite value of T which measures the range of convergence of the series. It is assumed to hold even when the range of convergence is infinite and the series passes into a Fourier integral.

The integral in equation (4) involves x^2 as a parameter under the integral sign, so that $\frac{1}{x} f(x)$ is really a function of x^2 (and of a). Therefore, E^2 will be determined as a function of x^2, and $E^2(-x') = E^2(x')$. Since $E^2(x')$ is an even function of x' the equation (4) may be written

$$f(x) = \frac{2ax}{\pi} \int_{-\infty}^{\infty} \frac{E^2(x')dx'}{4a^2 + \left(x' - \frac{a^2 + x^2}{x'}\right)^2} = \frac{a}{2\pi} \int_{-\infty}^{\infty} \frac{x'E^2(x')dx' \frac{4x}{x'}}{4a^2 + \left(x' - \frac{a^2 + x^2}{x'}\right)^2}$$

$$= \frac{a}{2\pi} \int_{-\infty}^{\infty} \frac{x'E^2(x')dx'}{a^2 + (x' - x)^2} - \frac{a}{2\pi} \int_{-\infty}^{\infty} \frac{x'E^2(x')dx'}{a^2 + (x' + x)^2}$$

If the variable of integration be changed in the second integral, letting $x' = -\mu$, it becomes

$$+ \frac{a}{2\pi} \int_{-\infty}^{\infty} \frac{\mu E^2(-\mu)d\mu}{a^2 + (\mu - x^2)} = \frac{a}{2\pi} \int_{-\infty}^{\infty} \frac{x'E^2(x')dx'}{a^2 + (x' - x)^2}$$

Hence, if we let $u(x) = x\,E^2(x)$ the fundamental equation becomes

$$f(x) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{u(x')dx'}{a^2 + (x' - x)^2} \tag{5}$$

where it is known that $f(x)$ and $u(x)$ are odd functions of x. The equation (5) formulates the problem here undertaken, which is to find the unknown function $u(x)$ in terms of the observed or given function $f(x)$. The remainder of this paper consists in the development of several forms of solutions of this equation, together with illustrative examples and applications to experimental results.

II. FIRST FORM OF SOLUTION—FOURIER'S INTEGRAL METHOD.

It may be verified by direct integration that

$$\frac{a}{a^2+(x'-x)^2} \doteq \int_0^\infty e^{-a\alpha} \cos \alpha (x'-x) \, d\alpha \text{ if } a > 0$$

Hence, the equation (5) may be written

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(x') \, dx' \int_0^\infty e^{-a\alpha} \cos \alpha (x'-x) \, d\alpha \tag{6}$$

Comparing this with the Fourier integral identities for $f(x)$ and $u(x)$

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \, dx' \int_0^\infty \cos \alpha (x'-x) \, d\alpha \tag{7}$$

$$u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(x') \, dx' \int_0^\infty \cos \alpha (x'-x) \, d\alpha \tag{8}$$

leads to the inference that

$$u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \, dx' \int_0^\infty e^{a\alpha} \cos \alpha (x'-x) \, d\alpha \tag{9}$$

or

$$u(x) = \frac{1}{\pi} \int_0^\infty f(\mu) \, d\mu \int_0^\infty e^{a\alpha} \sin \alpha\mu \sin \alpha x d\alpha \tag{10}$$

The latter form is a consequence of the fact that $f(x)$ is an odd function of x. That the formula (10) does give the solution of (5) may be verified by making use of Laplace's integrals [1]

$$a \int_{-\infty}^{\infty} \frac{\cos \alpha t}{a^2+t^2} dt = \int_{-\infty}^{\infty} \frac{t \sin \alpha t}{a^2+t^2} dt = \pi e^{-a\alpha} \text{ if } a \alpha > 0 \tag{11}$$

Replacing x by x' in (10) and substituting in the right hand side of (5) gives

$$\frac{2a}{\pi^2} \int_0^\infty f(\mu) \, d\mu \int_0^\infty e^{a\alpha} \sin \alpha \mu \, d\alpha \int_{-\infty}^{\infty} \frac{\sin \alpha x' \, dx'}{a^2+(x'-x)^2}$$

[1] Meyer, Bestimmte integrale, p. 198.

To perform the integration first with respect to x' let $x'-x=t$. Then

$$\int_{-\infty}^{\infty} \frac{\sin \alpha x'}{a^2+(x'-x)^2} dx' = \int_{-\infty}^{\infty} \frac{\sin \alpha\ (x+t)}{a^2+t^2} dt$$

$$= \sin \alpha\ x \int_{-\infty}^{\infty} \frac{(\cos \alpha\ t}{a^2+t^2} dt = \frac{\pi\ e^{-a\alpha}}{a} \sin \alpha\ x$$

The triple integral last written then reduces to

$$\frac{2}{\pi} \int_0^{\infty} f\ (\mu)\ d\ \mu \int_0^{\infty} \sin \alpha\ \mu \sin \alpha\ x\ d\ \alpha$$

which is identically equal to $f(x)$ and therefore (10) is the solution of (5). The understanding in (10) is that the integration is to be performed first with respect to x', otherwise it is meaningless. This form of solution has the advantage that it is applicable when $f(x)$ has any finite number of finite discontinuities. When the observed function $f(x)$ is given by a curve, the formal solution (9) is unsatisfactory. It is desirable to work out a method whereby the function $u(x)$ may be derived from the curve for $f(x)$ by the use of graphical or mechanical methods. This is done in the following.

III. SECOND FORM OF SOLUTION—ADAPTATION FOR EVALUATION WITH A HARMONIC ANALYZER.

In practice we are usually concerned with a relatively small range of high frequencies, to which the observations are confined. As far as these observations are concerned, we may assume that $u(x')=0$ outside this range $x_1<x'<x_2$. It is to be noticed that in the original equation (5) the nucleus involves $(x-x')$, the difference of frequencies, and therefore we may take the origin of x anywhere, and the form of (5) and of the solution (10) will be unaltered. Let us take it at the middle of the observation range, which is supposed to extend a distance $\pm T$ from this origin. The curve $f(x)$ having been plotted according to its definition $f(x) = 8\ R\ L\ x\mathfrak{J}^2$ where x is actual frequency, we shall denote it by $F(x)$, where x now represents frequency difference from the new origin. Similarly, we shall use $v(s)$ instead of $u(x')$ for the function which determines the energy of the wave as a function of this frequency difference.

Since $v(s)$ is zero, if $s < -T$ or $s > T$, the fundamental equation to determine energy $v(s)$ when the observed curve $F(x)$ has been plotted is

$$F(x) = \frac{a}{\pi}\int_{-T}^{T} \frac{v(s)\,ds}{a^2+(s-x)^2} \tag{12}$$

where $v(x)$ and $F(x)$ are no longer known to be either even or odd functions of x. The assumption that $v(s)$ is zero outside the range $x_1 < x' < x_2$ is contradictory to the fact already noted that $E^2(x')$ is an even function of x'. In keeping with this fact there should be added to the right side of equation (12) the corresponding "image" of the term there present corresponding to

$$\frac{a}{\pi}\int_{-x_2}^{-x_1} \frac{u(x')\,dx'}{a^2+(x'-x)^2}$$

which may be put in the form

$$\frac{a}{\pi}\int_{-T}^{T} \frac{v(s)\,ds}{a^2+(s+x)^2}$$

which is utterly negligible when x is large and positive compared to the term retained in (12). The nucleus of the integral in (12) may be written $K(x\,x')$ where

$$K(x\,y) = \frac{a}{\pi}\,\frac{1}{a^2+(x-y)^2}$$

This has a sharp maximum $\frac{1}{\pi a}$ at $x=y$, reduces to half this value when $x-y=a$ (the "half width of the resonance peak") and rapidly decreases when $(x-y)$ becomes large compared to a. This shows that the fraction

$$\frac{a}{\pi}\,\frac{1}{a^2+(x+y)^2}$$

must be negligible. If the wave is undamped, its spectrum is a single line at the frequency x_1 say. The corresponding observed function is

$$\frac{a}{\pi}\,\frac{A}{a^2+(x-x_1)^2}$$

which has the maximum value $\frac{A}{\pi a}$ at the resonance frequency $x = x_1$ and reduces to half this value when $x = x_1 \pm a$. Hence a is called the half width of the resonance curve. If the spectrum consists of two equal lines separated by the distance $\triangle x$, in frequency, the observed curve would be the sum of two such terms

$$F(x) = \frac{A}{\pi} \frac{a}{a^2 + (x - x_1)^2} + \frac{A}{\pi} \frac{a}{a^2 + (x - x_2)^2}$$

and if $|x_1 - x_2| > a$ this curve would have two visible peaks, which would coalesce (practically) into one peak if $\triangle x$ or $|x_2 - x_1| \leqq a$. Therefore, a measures the *limit* of *resolution* of the receiving circuit as an analyzer and $\frac{\triangle x}{x} = \frac{a}{x} = \frac{\delta}{2\pi}$ is the resolving power of the circuit as a spectroscope. In the notation immediately following the limits will not be written, but it is understood that the integrals are all definite integrals over the range from $s = -T$ to $s = +T$.

The general theory of integral equations shows that the solution of the integral equation

$$F(x) = \int K(xs)\, v(s)\, ds \tag{13}$$

may be found in terms of the normal functions $\phi_n(x)$, and the associated functions $\psi_n(x)$ and the characterisitc constants λ_n where

$$\int \phi_n(x)\, \phi_m(x)\, dx = 0 \text{ if } n \neq m, = 1 \text{ if } n = m \tag{14}$$

and $\phi_n(x)$ and $\psi_n(x)$ are solutions of the pair of simultaneous homogeneous integral equations.

$$\begin{aligned} \psi_n(x) - \lambda_n \int K(x\,y)\, \phi_n(y)\, dy &= 0 \\ \phi_n(x) - \lambda_n \int K(y\,x)\, \psi_n(y)\, dy &= 0 \end{aligned} \tag{15}$$

and where the constants λ_n are the roots of the characteristic determinant $D(\lambda) = 0$ which is found when $K(x\,y)$ is known by Fredholm's formula.

If the nucleus $K(x\,y)$ is a symmetric function of x and y, so that $K(x\,y) = K(y\,x)$, then $\psi_n(x)$ becomes identical with $\phi_n(x)$.

If the roots of $D(\lambda) = 0$ are infinite in number, there are an infinite number of normal functions $\phi_n(x)$ and the associated ones

$\psi_n(x)$. In this case any function $v(x)$ may be developed in a series of these normal functions, analogous to Fourier's series.

$$v(x) = \Sigma v_n \phi_n(x) \tag{16}$$

provided Σv^2_n is convergent. The constant coefficient v_n is found by multiplying this equation by $\phi_n(x)\,dx$ and integrating over the complete range. This gives

$$v_n = \int v(x)\,\phi_n(x)\,dx. \tag{17}$$

Multiplying the first of the equations (15) by $\psi_m(x)\,dx$ and integrat ing gives

$$\int \psi_n(x)\,\psi_m(x)\,dx = \lambda_n \int \phi_n(y)\,dy \int K(x\,y)\,\psi_m(x)\,dx$$
$$= \int \phi_n(y)\,\phi_m(y)\,dy \text{ by } (15)$$

which shows that the $\psi_n(x)$ themselves constitute a set of normal functions. Consequently

$$F(x) = \sum_n F_n\,\psi_n(x) \text{ where } F_n = \int F(x)\,\psi_n(x)\,dx \tag{18}$$

The work of Hilbert and Schmidt has shown that the nucleus may be expressed in terms of these functions by the formula

$$K(x\,y) = \sum_n \frac{\psi_n(x)\,\phi_n(y)}{\lambda_n} \tag{19}$$

To solve the original integral equation, assume that the unknown function, which is to be found, may be expressed by a convergent series of the form

$$v(x) = \sum_k v_k \phi_k(x) \tag{20}$$

The solution consists in finding the coefficients v_k. To do this substitute this series for $v(x)$ in the integral equation, substituting at the same time the series for $F(x)$ in terms of the associated functions $\psi_n(x)$. Equation (13) becomes

$$F(x) = \sum_n F_n \psi_n(x) = \sum_n \frac{\psi_n(x)}{\lambda_n} \sum_k v_k \int \phi_n(s)\phi_k(s)\,ds = \sum_n \frac{v_n}{\lambda_n}\psi_n(x)$$

This equation will be satisfied if $\frac{v_n}{\lambda_n} = F_n$ or

$$v_n = \lambda_n F_n = \lambda_n \int F(s)\psi_n(s)\,ds \text{ (by 18)}$$

Hence, the solution is

$$v(x) = \sum_n \lambda_n F_n \phi_n(x) = \Sigma \lambda_n \phi_n(x) \int F(s) \psi_n(s)\, ds \tag{21}$$

provided this series is convergent. The coefficients F_n must first be found and then the sum of the series taken. An exceedingly great gain in convenience would be made if it were possible to interchange the order of limits in (21) so that we could sum the integrand first and then integrate with respect to x giving

$$v\ (x) = \int F\ (s)\ ds \sum_n \lambda_n\ \phi_n\ (x)\ \psi_n\ (s) = \int F\ (s)\ N\ (x\ s)\ ds \tag{22}$$

where

$$N\ (x\ y) = \Sigma\ \lambda_n\ \phi_n\ (x)\ \psi_n\ (y)$$

for this function N could be computed once for all for a given receiving circuit and a given range of integration, and the work of applying it to various observed functions $F\ (s)$ by (22) would involve considerably less routine work than in applying (21). However, it is to be noticed that this series for $N\ (x\ y)$ can seldom converge, for λ_n increases with order, n, and if

$$\sum_n \frac{\phi_n\ (x)\ \psi_n\ (y)}{\lambda_n}$$

converges it is not to be expected that $\Sigma\ \lambda_n\ \phi_n\ (x)\ \psi_n\ (y)$ will converge in general.

In order that the preceding developments shall be valid, it is necessary that the system of normal functions be "closed," which will be the case if no function, $h\ (x)$, exists such that $\int K\ (x\ s)\ h\ (s)\ ds = 0$. The normal functions are then infinite in number.

The above form of solution is of no practical use for computing unless one can find the set of constants $\lambda_1\ \lambda_2$ .... and the corresponding normal functions.

However, in case the nucleus is a function of $x-y$ or of $x+y$, or is the sum of such functions, the normal functions are trigonometric functions. The harmonic analysis of the nucleus (either mechanically or analytically) then amounts to a determination of the characteristic constants. In the present case $K\ (x\ y)$ is an even function of $(x-y)$ and may be resolved as follows:

$$K\ (x\ y) = \frac{1}{2} K_0 + \sum_n K_n \cos \frac{n\ \pi\ (x-y)}{T}$$

or if

$$K_n = \frac{a}{\pi T} \int_{-T}^{T} \frac{\cos \frac{n \pi \mu}{T} d \mu}{a^2 + \mu^2}$$

then

$$K(x\,y) = \frac{1}{2} K_0 + \sum_n K_n \cos \frac{n \pi x}{T} \cos \frac{n \pi y}{T} + \sum_n K_n \sin \frac{n \pi x}{T} \sin \frac{n \pi y}{T} \tag{23}$$

Also $F(x)$ may be developed in a Fourier's series .....
$T < x < T$

$$\left.\begin{aligned} F(x) &= \frac{1}{2} F_0 + \sum_n F_n \cos \frac{n \pi x}{T} + \sum F_n' \sin \frac{n \pi x}{T} \\ &\text{where} \\ F_n &= \frac{1}{T} \int_{-T}^{T} F(\mu) \cos \frac{n \pi \mu}{T} d \mu \\ &\text{and} \\ F_n' &= \frac{1}{T} \int_{-T}^{T} F(\mu) \sin \frac{n \pi \mu}{T} d \mu \end{aligned}\right\} \tag{24}$$

If a value of $v(x)$ is sought in the form

$$v(x) = \frac{1}{2} v_0 + \sum_k v_k \cos \frac{k \pi x}{T} + \sum_k v_k' \sin \frac{k \pi x}{T} \tag{25}$$

if

$$-T < x < T$$

the coefficients v_0, v_n, and v_n' may be found by substituting all these series in the equation

$$F(x) = \int_{-T}^{T} K(x\,y) v(y)\, dy$$

which gives the equation

$$\frac{1}{2} F_0 + \sum_n F_n \cos \frac{n \pi x}{T} + \sum_n F' \sin \frac{n \pi x}{T}$$

$$= \frac{K_0 v_0}{2} T + T \sum_n K_n v_n \cos \frac{n \pi x}{T} + T \sum_n K_n v'_n \sin \frac{n \pi x}{T}$$

This will be satisfied if we take

$$\frac{v_0}{2}=\frac{F_0}{2K_0T} \qquad v_n=\frac{F_n}{K_nT} \qquad v'_n=\frac{F'_n}{K_nT}$$

and the solution is

$$v(x)=\frac{1}{T}\left[\frac{1}{2}\frac{F_0}{K_0}+\sum_n\frac{F_n}{K_n}\cos\frac{n\pi x}{T}+\sum_n\frac{F_n'}{K_n}\sin\frac{n\pi x}{T'}\right] \tag{26}$$

if this series is convergent.

The formula (26) gives a practical method of solving the problem. The coefficients K_n may be evaluated once for all for a given circuit (a given) and a given range 2 T. However, this must be done by approximate methods, or mechanically, since the integral which defines K_n can not be integrated in finite terms if T is finite. While it would be very convenient to reverse the order of limits in (26) and write

$$v(x)=\int_{-T}^{T}F(s)\ N\ (x-s)\ ds$$

where

$$N(x)=\frac{1}{2K_0T^2}+\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi x}{T}}{T^2K_n}$$

it is easy to see that such a procedure is, in general, not allowable, since this series will not converge in general. It is easy to show by means of Laplace's integrals that the limit of

$$K_nT=\pi e^{-a\frac{n\pi}{T}} \text{ when } T=\infty$$

and if we let $dx=\frac{\pi}{T}\alpha=\frac{n\pi}{T}$ the solution (26) approaches as a limit when $T=\infty$

$$v(x)=\frac{1}{\pi}\int_{\infty}^{\infty}F(s)ds\int_{0}^{\infty}e^{a\alpha}\cos\alpha(s-x)d\alpha$$

which has a meaning only if the integration is performed first with respect to s. This is identical with the solution previously obtained (9).

IV. THIRD FORM OF SOLUTION—OPERATION WITH $\cos a D_x$.

By multiplying the identity

$$\frac{1}{\pi}\int_0^\infty e^{-a\alpha} \cos \alpha (x'-x)\, d\alpha = \frac{a}{\pi} \frac{1}{a^2+(x'-x)^2}$$

by $f(x')\, dx'$ and integrating one obtains the identity

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(x')\, dx' \int_0^\infty e^{-a\alpha} \cos \alpha (x'-x)\, d\alpha - \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{f(x')\, dx'}{a^2+(x'-x)^2} = 0 \quad (27)$$

Adding this identically zero quantity to the right side of (9) gives

$$u(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')\, dx' \int_0^\infty (e^{a\alpha}+e^{-a\alpha}) \cos \alpha (x'-x)\, d\alpha - \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{f(x')\, dx'}{a^2+(x'-x)^2} \quad (28)$$

The last integral in this expression may be evaluated graphically or mechanically. It is desirable to find a method of evaluating the first integral. For this purpose it may be noticed that

$$\frac{(-1)^n a^{2n} D_x^{2n}}{\underline{|2n}} \cos \alpha x = \frac{(a\alpha)^{2n}}{\underline{|2n}} \cos \alpha x$$

and hence if we symbolize the infinite series of differential operations

$$1 - \frac{a^2 D_x^2}{\underline{|2}} + \frac{a^4 D_x^4}{\underline{|4}} \ldots \text{ or } \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} D_x^{2n}}{\underline{|2n}} \text{ by } \cos a D_x \quad (29)$$

then

$$2 \cos a D_x = (e^{iaD_x} + e^{iaD_x}),$$

and

$$2 \cos a D_x \cos \alpha (x-x') = (e^{a\alpha} + e^{-a\alpha}) \cos \alpha (x-x')$$

With this in mind, the form of the first integral in (28) suggests that it is the result of applying the above operator to the Fourier double integral for $f(x)$, equation (7), applying the infinite series of differentiations with respect to x inside the sign of integration, a process not always allowable even with differentiations of a

finite order. Nevertheless this suggests the second important form of solution.

$$u(x) = 2 \cos a D_x f(x) - \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{f(x')\, dx'}{a^2 + (x' - x)^2} \tag{30}$$

This form, if correct, may often be more convenient to evaluate than the first form of solution (9), since the operations on the right of (30) may all be performed mechanically. By applying Leibnitz's theorem for the derivative of a product, one finds that

$$\frac{D_x^{2n}}{\underline{|2n}} x \mathfrak{J}^2(x) = x \frac{D_x^{2n}}{\underline{|2n}} \mathfrak{J}^2(x) + \frac{D^{2n-1}}{\underline{|2n-1}} \mathfrak{J}^2(x)$$

so that

$$\frac{2}{x} \cos a D_x x \mathfrak{J}^2(x) = 2 \cos a D_x \mathfrak{J}^2(x) - \frac{2a}{x} \sin a D_x \mathfrak{J}^2(x)$$

Also it may be noted that for large positive values of x' and x the term

$$\frac{a}{a^2 + (x' + x)^2}$$

is utterly negligible compared to

$$\frac{a}{a^2 + (x' - x)^2}$$

and, therefore, for any actual range of frequencies encountered in practice the energy distribution will be given by

$$\frac{E^2(x)}{8RL} = 2 \cos a D_x \mathfrak{J}^2(x) - \frac{2a}{x} \sin a D_x \mathfrak{J}^2(x) - \frac{a}{\pi x} \int_0^{\infty} \frac{x' \mathfrak{J}^2(x')\, dx'}{a^2 + (x' - x)^2} \tag{31}$$

or

$$\frac{E^2(x)}{8RL} = 2\left[\mathfrak{J}^2(x) - \frac{a^2 D_x^2}{\underline{|2}} \mathfrak{J}^2(x) + \frac{a^4 D_x^4}{\underline{|4}} \mathfrak{J}^2(x) \ldots\right] - \frac{2a}{x}\left[a D_x \mathfrak{J}^2(x) - \frac{a^3 D_x^3}{\underline{|3}} \mathfrak{J}^2(x) + \ldots\ldots\right] - \frac{a}{\pi x} \int_0^{\infty} \frac{x' \mathfrak{J}^2(x')\, dx'}{a^2 + (x' - x)^2} \tag{32}$$

provided, of course, that (30) is correct. In some cases one or two terms of the series are sufficient.

Having arrived by intuitive reasoning at a form of operation which will simplify the solution of the original equation, we may now inquire more carefully as to the result of applying the operator $2 \cos b\, D_x$ to the original equation (5) where $0 < b < a$. Since

$$\frac{1}{\pi} \frac{a}{a^2 + (x' - x)^2} = \text{real part of } \frac{1}{\pi i} \frac{1}{x' - x - ia}$$

the equation (5) may be written in the form

$$f(x) = \text{real part of } \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u\,(x')\,dx'}{x' - x - ia}$$

Assume that at the point x, $2 \cos b\, D_x\, f(x)$ has a meaning; that is, the result of applying the real operator $2 \cos b\, D_x$ to $f(x)$ gives a single valued finite result, though not necessarily a continuous function of x and b.

Regarding $f(x)$ as defined by the integral in the above equation, $2 \cos b\, D_x\, f(x)$ is the value of $2 \cos b\, D_x$ applied to the integral, and this operation may be performed inside the integral sign when it leads to finite integrand and a uniformly convergent integral. Since

$$\cos b\, D_x \text{ or } \sum_{n=0}^{\infty} \frac{(-1)^n\, b^{2n}\, D_x^{2n}}{\underline{|2n}}$$

is a real operator, the result of applying it to the real part of a complex function must be equal to the real part of the result of the operation upon the complex function. It is easily seen that

$$\frac{(-1)^n b^{2n} D^{2n}{}_x}{\underline{|2n}} \left(\frac{1}{x' - x - ia} \right) = \frac{(-1)^n b^{2n}}{(x' - x - ia)^{2n+1}}$$

Hence,

$$\frac{2}{\pi i} \cos b\, D_x \frac{1}{x' - x - ia} = \frac{2}{\pi i} \sum_{n=0}^{\infty} \frac{(ib)^{2n}}{(x' - x - ia)^{2n+1}}$$

provided this series is convergent. The series will converge if b is less than the modulus of $(x' - x - ia)$; that is, $\sqrt{a^2 + (x' - x)^2}$ and the least value of this is a (when $x = x'$). Hence, if $b < a$, the

series will converge for every positive value of x and x'. It is a geometrical series and is readily seen to be the expansion of

$$\frac{2}{\pi i}\left[\frac{1}{x'-x-ia}\cdot\frac{1}{1-\left(\frac{ib}{x'-x-ia}\right)^2}\right]$$

or

$$\frac{1}{\pi i}\left[\frac{1}{x'-x-i(a-b)}+\frac{1}{x'-x-i(a+b)}\right]$$

the real part of which is

$$\frac{a-b}{\pi}\frac{1}{(a-b)^2+(x'-x)^2}+\frac{a+b}{\pi}\frac{1}{(a+b)^2+(x'-x)^2}$$

consequently

$$\begin{aligned}2\cos b\, D_x f(x) &= \frac{(a-b)}{\pi}\int_{-\infty}^{\infty}\frac{u(x')\,dx'}{(a-b)^2+(x'-x)} \\ &+\frac{a+b}{\pi}\int_{-\infty}^{\infty}\frac{u(x')\,dx'}{(a+b)^2+(x'-x)^2}\end{aligned} \tag{33}$$

Now it is possible to prove by direct integration (and it will be proven later as an easy special case) that

$$\frac{ab}{\pi^2}\int_{-\infty}^{\infty}\frac{ds}{[a^2+(x'-s)^2][b^2+(s-x)^2]}=\frac{a+b}{\pi}\frac{1}{(a+b)^2+(x'-x)^2} \tag{34}$$

Hence, if we change x to s in (5) and multiply the equation by

$$\frac{b}{\pi}\frac{ds}{b^2+(s-x)^2}$$

and integrate, we get

$$\begin{aligned}\frac{b}{\pi}\int_{-\infty}^{\infty}\frac{f(s)\,ds}{b^3(+s-x)^2} &= \frac{ab}{\pi^2}\int_{-\infty}^{\infty}\frac{ds}{b^2+(s-x)^2}\int_{-\infty}^{\infty}\frac{u(x')\,dx'}{a^2+(x'-s)^2} \\ &= \frac{ab}{\pi^2}\int_{-\infty}^{\infty}u(x')\,dx'\int_{-\infty}^{\infty}\frac{ds}{[b^2+(s-x)^2][a^2+(x'-s)^2]} \\ &= \frac{a+b}{\pi}\int_{-\infty}^{\infty}\frac{u(x')\,dx'}{(a+b)^2+(x'-x)^2}\ \text{(by 34)}\end{aligned}$$

and this is the last integral on the right side of (33). Its value is therefore

$$\frac{b}{\pi}\int_{-\infty}^{\infty}\frac{f(s)\,ds}{b^2+(s-x)^2}$$

or changing the name of the variable of integration

$$\frac{b}{\pi}\int_{-\infty}^{\infty}\frac{f(x')dx'}{b^2+(x'-x)^2}$$

Therefore, the result of operating upon (5) by 2 cos bD_x where $0<b<a$ gives by means of (33)

$$\frac{(a-b)}{\pi}\int_{-\infty}^{\infty}\frac{u(x')dx'}{(a-b)^2+(x'-x)^2}=2\cos bD_xf(x)-\frac{b}{\pi}\int_{-\infty}^{\infty}\frac{f(x')dx'}{b^2+(x'-x)^2} \quad (35)$$

By letting $x'-x=(a-b)\ t$ in the left side of this equation, it becomes

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u[x+(a-b)t]}{1+t^2}dt$$

and by taking b sufficiently close in value to a, this may be made to approach as closely as we please to $\frac{u(x+0)+u(x-0)}{2}$ or $u(x)$ if u is continous at x.

The integral on the right side of (35) is easily proven to be a continous function of b and x even if $f(x)$ has finite discontinuities at certain points.

The value of $\lim\limits_{b\to a} 2\cos bD_xf(x)$ is assumed to exist, in general, but it may not exist at certain points. It need not, in general, be continuous either as a function of x or b.

We may, therefore, conclude from (35) that: Whenever 2 cos aD_x exists and is a continuous function of x, then the solution $u(x)$ is a continuous function of x, and is given by equation (30). At particular values of x, where the function $f(x)$ or its x-derivatives of any order have a finite discontinuity, 2 cos $aD_xf(x)$ has a finite discontinuity, and this is also the discontinuity of the function $u(x)$. For example, suppose that in the neighborhood of the point x_1, the function $f(x)$ is given by the equation

$$f(x)=\phi(x) \text{ when } x<x_1 \text{ and } f(x)=\phi(x)+A+B(x-x_1)^4$$

when $x > x_1$ where $\phi(x)$ is continuous with all its derivatives at x_1. Then $f(x)$ jumps by an amount A as we pass through the point x_1 and its fourth derivative jumps an amount $\lfloor 4 B$. Hence, $u(x)$ jumps by an amount $2(A - a^2B)$. If B equals zero, the jump in $u(x)$ is twice that in $f(x)$, and, in general, the discontinuity in $u(x)$ is greater than that in $f(x)$, although in this case if $B = a^2A$ there would be no discontinuity in $u(x)$. The function $f(x)$ would, however, be continuous if A =o, together with its first three derivatives, and yet $u(x)$ would have a discontinuity of $-2a^2B$. The ordinates of the curve $f(x)$, its slope and its curvature would all be continuous at x_1 and there would be nothing in its appearance to suggest the discontinuity in the spectrum.

Example illustrating this method: Suppose for example that the observed curve is given by

$$f(x) = \frac{A}{\pi}\left[\tan^{-1}\frac{x_2 - x}{b} - \tan^{-1}\frac{x_1 - x}{b}\right]$$

where

$$x_2 > x_1 > 0 \text{ and } b > a$$

Now

$$\tan^{-1}\left(\frac{x_2 - x}{b}\right) = \text{imaginary part of } \log\left[b + i(x_2 - x)\right]$$

Hence if $a < b$

$$2 \cos a D_x \tan^{-1}\left(\frac{x_2 - x}{b}\right) = (e^{iaD_x} + e^{-iaD_x}) \tan^{-1}\left(\frac{x_2 - x}{b}\right)$$

$$= \text{imaginary part of } (e^{iaD_x} + e^{-iaD_x}) \log\left[b + i(x_2 - x)\right]$$
$$= \text{imaginary part of } \log\left[b + i(x_2 - x - ia)\right] + \log\left[b + i(x_2 - x + ia)\right]$$
$$= \text{imaginary part of } \log\left[b + a + i(x_2 - x)\right] + \log\left[b - a + i(x_2 - x)\right]$$
$$= \tan^{-1}\left(\frac{x_2 - x}{b + a}\right) + \tan^{-1}\left(\frac{x_2 - x}{b - a}\right)$$

Hence

$$2 \cos a D_x f(x) = \frac{A}{\pi}\left[\tan^{-1}\left(\frac{x_2 - x}{b + a}\right) - \tan^{-1}\left(\frac{x_1 - x}{b + a}\right)\right]$$
$$+ \frac{A}{\pi}\left[\tan^{-1}\left(\frac{x_2 - x}{b - a}\right) - \tan^{-1}\left(\frac{x_1 - x}{b - a}\right)\right]$$

This gives the first term on the right side of (30). The second term on the right side of (30) is the integral.

$$\frac{Aa}{\pi^2}\int_{-\infty}^{\infty}\frac{dx'}{a^2+(x'-x)^2}\left[\tan^{-1}\frac{x_2-x'}{b}-\tan^{-1}\frac{x_1-x'}{b}\right].$$

It is next necessary to evaluate these integrals. To do this note that the following is identically true:

$$\frac{1}{\pi}\left[\tan^{-1}\left(\frac{x_2-x'}{b}\right)-\tan^{-1}\left(\frac{x_1-x'}{b}\right)\right]=\frac{b}{\pi}\int_{x_1}^{x_2}\frac{ds}{b^2+(s-x')^2}$$

Multiplying this identity by $\frac{Aa}{\pi}\frac{dx'}{a^2+(x'-x)}$ and integrating from $x'=-\infty$ to $x'=+\infty$ gives the term desired as the double integral

$$\frac{A\,ab}{\pi^2}\int_{-\infty}^{\infty}\frac{dx'}{a^2+(x'-x)^2}\int_{x_1}^{x_2}\frac{ds}{b^2+(s-x')^2}$$

$$=\frac{A\,ab}{\pi^2}\int_{x_1}^{x_2}ds\int_{-\infty}^{\infty}\frac{dx'}{[b^2+(x'-s)^2][a^2+(x'-x)^2]}$$

$$=\frac{A\,(a+b)}{\pi}\int_{x_1}^{x_2}\frac{ds}{(a+b)^2+(s-x)^2}\quad\text{(by 34)}$$

$$=\frac{A}{\pi}\left[\tan^{-1}\left(\frac{x_2-x}{a+b}\right)-\tan^{-1}\left(\frac{x_1-x}{a+b}\right)\right]$$

Hence, the formal solution (30) gives the spectral distribution of energy $u\,(x)$ by

$$u\,(x)=2\cos a\,D_x f(x)-\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{f\,(x')\,dx'}{a^2+(x'-x)^2}$$

$$=\frac{A}{\pi}\left[\tan^{-1}\left(\frac{x_2-x}{b-a}\right)-\tan^{-1}\left(\frac{x_1-x}{b-a}\right)\right]$$

when the observed function is

$$f\,(x)=\frac{A}{\pi}\left[\tan^{-1}\left(\frac{x_2-x}{b}\right)-\tan^{-1}\left(\frac{x_1-x}{b}\right)\right]\text{ where } b>a$$

These two curves are of the same type and are somewhat similar to the curve showing the magnetic field along the axis of a solenoid. The observed curve is broader than the energy curve. The latter becomes infinitely sharp at the edges x_1 and x_2 when b approaches a. In this case, the energy distribution is zero outside the range $x_1 < x < x_2$ and has the constant value A within that range. The observed curve, however, approaches a limiting curve which is continuous. The parameter $b-a$ which occurs in the energy curve is vaguely analogous to what is called the decrement of the wave. This must be increased by the damping coefficient, a, of the receiving circuit in order to obtain the observed curve. This enlargement of what is usually called decrement is dealt with in more general fashion later. In many simple types of waves it represents a quantity in the wave, which is augmented by a when received by the circuit.

This third form of solution (30) like the two previous forms has the advantage that it may be applied to cases where the function $f(x)$ is represented by different analytical expressions, in different parts of its range, which may or may not fit on to each other smoothly. It involves graphical differentiation and integration. To apply the second form of solution requires harmonic analysis which may be affected mechanically. However, it is sometimes the case that less labor is involved by representing the observed function by an empirical equation so as to gain the advantages which analytical tools often possess over mechanical ones. In order to find what types of elementary solutions might be most appropriate for empirical representation of the observed function, a few simple types of elementary solutions will be considered.

V. FOURTH FORM OF SOLUTION—EMPIRICAL FORMULA FOR THE OBSERVED FUNCTION.

A very simple expression for the energy distribution results when the observed function $f(x)$ is of the simple type

$$f(x) = \sum_{n} \frac{A\ a_n}{a^2_n + (x_n - x)^2}$$

where $a_n > a$ and x_n is any constant. The solution $u(x)$ is given by simply replacing a_n by the positive quantity $a_n - a$ in this equation, a form which is generalized later. This result follows as a special case of the solution (9). If we place

$$f(x') = A\ a_n \frac{1}{a^2_n + (x_n - x')^2}$$

in (9) it gives

$$u(x)=\frac{A\ a_n}{\pi}\int_{-\infty}^{\infty}\frac{dx'}{a^2{}_n+(x'-x_n)^2}\int_0^{\infty}e^{a\alpha}\cos\alpha\ (x'-x)d\alpha$$

where the integration must be performed first with respect to x' in order that this may have a meaning. To do this let $x'=x_n+t$. Then

$$u(x)=\frac{A\ a_n}{\pi}\int_0^{\infty}e^{a\alpha}\ d\ \alpha\int_{-\infty}^{\infty}\frac{\cos\alpha\ (x_n-x+t)\ dt}{a^2{}_n+t^2}$$

$$=\frac{A\ a_n}{\pi}\int_0^{\infty}e^{a\alpha}\ d\ \alpha\cos\alpha\ (x_n-x)\int_{-\infty}^{\infty}\frac{\cos\alpha\ t}{a^2{}_n+t^2}dt$$

$$-\frac{A\ a_n}{\pi}\int_0^{\infty}e^{a\alpha}\ d\ \alpha\sin\alpha\ (x_n-x)\int_{-\infty}^{\infty}\frac{\sin\alpha\ t}{a^2{}_n+t^2}dt$$

But $$\int_{-\infty}^{\infty}\frac{\sin\alpha\ t}{a^2+t^2}dt=0 \text{ and } \int_{-\infty}^{\infty}\frac{\cos\alpha\ t}{a^2{}_n+t^2}dt=\frac{\pi}{a_n}e^{-a_n\alpha}$$

(Laplace's integral). Hence

$$u(x)=A\int_0^{\infty}e^{a\alpha}\ d\ \alpha\ e^{-a_n\alpha}\cos\alpha\ (x_n-x)$$

$$=A\int_0^{\infty}e^{-(a_n-a)\alpha}\cos\alpha\ (x_n-x)\ d\ \alpha$$

$$=\frac{A\ (a_n-a)}{(a_n-a)^2+(x_n-x)^2}\ \ldots\ldots\ \text{if } a_n>a$$

$$=\text{meaningless if } a_n<a$$

It is easy to see that if a pair of functions $f(x)$ and $u(x)$ are known, such as

$$f(x)=\frac{A_n\ a_n}{a^2+(x_n-x)^2}$$

and (36)

$$u(x)=\frac{A_n\ (a_n-a)}{(a_n-a)^2+(x_n-x)^2}$$

which satisfy the original equation (5), then other pairs of functions which will satisfy (5) may be obtained by differentiating this pair with respect to a_n, x_n, or x; or by integrating with respect to these quantities. Also, by making any linear combinations of these simple solutions, other simple solutions may be found.

The simple type of solution here given is, perhaps, the one of most practical value for empirical representation of the observed function.

It might be noticed that this type of solution is not strictly compatible with the fact previously noted that $f(x)$ and $u(x)$ are both odd functions of x, whereas the pair just exhibited are neither even nor odd functions. By changing the sign of x_n and subtracting, however, we get an odd function

$$f(x) = A_n\left[\frac{a_n}{a^2_n + (x_n - x)^2} - \frac{a_n}{a^2_n + (x_n + x)^2}\right]$$

$$= 4x_n A_n \frac{a_n x}{[x^2 + x_n + a^2_n - 2x_n x]\,[x^2 + x^2_n + a^2_n + 2x_n x]}$$

$$= 4x_n A_n \frac{a_n x}{(x^2 + x_n + a^2_n)^2 - 4x^2 x_n^2}$$

which is an odd function of x, and, therefore, corresponds to a physical solution. However, as we have seen, the extra term here added, namely,

$$\frac{1}{a^2 + (x_n + x)^2}$$

is utterly negligible compared to the term

$$\frac{1}{a^2 + (x_n - x)^2}$$

when we are concerned only with a narrow range of high positive values of the frequency, and, therefore, the simpler type of solution is the more practicable.

By differentiation and integration of the fundamental types (36) many other simple types are obtainable, such as

$$\left.\begin{aligned}
\text{I } f(x) &= \frac{A\, a_n}{(x - x_n)^2 + a^2_n} \\
\text{II } f(x) &= A\left[\tan^{-1}\left(\frac{x_2 - x}{a_n}\right) + \tan^{-1}\left(\frac{x - x_1}{a_n}\right)\right] \\
\text{III } f(x) &= \frac{A\,(x - x_n)}{(x - x_n)^2 + a^2_n} \\
\text{IV } f(x) &= A \log\left[\frac{(x - x_1)^2 + a^2_n}{(x - x_2)^2 + a^2_n}\right]
\end{aligned}\right\} \qquad (37)$$

where

$$a_n > a$$

or more generally

$$f(x) = \int_{x_1}^{x_2} \frac{(x-x')\,A(x') + a_n B(x')}{(x-x')^2 + a^2{}_n}\,dx'$$

where A and B are any functions of x'. Other forms involving any power of the denominator may also be obtained. The corresponding solution $u(x)$ for all of the above types (or of sums and linear combinations of them) is found by substituting $a_n - a$ for a_n in these expressions for $f(x)$.

This rule is a generalization of the principle which forms the basis of all decrement measurements. It corresponds to the fact that peaks in the curve $f(x)$ correspond to peaks in the $u(x)$ curve, which are narrower by $2a$, the breadth of the resonance peak of the receiving circuit.

The physical meaning of these types may be illustrated by considering the radiation produced by another simple circuit at a distance, oscillating freely and sending out a radiation which is given by $\mathfrak{E}(t) = Ce^{-2\pi bt} \cos 2\pi x't$. The development of this function of the time in a Fourier integral for all values of the time from $-\infty$ to $+\infty$ shows that the amplitude of each harmonic component of frequency x is given by

$$E^2(x) = \frac{C^2}{2\pi^4} \frac{x^2 + x'^2 + b^2}{[(x-x')^2 + b^2][(x+x')^2 + b^2]}$$

$$= B\left[\frac{b}{(x-x')^2 + b^2} + \frac{b}{(x+x')^2 + b^2}\right] \text{exactly}$$

$$= \frac{Bb}{[(x-x')^2 + b^2]} \text{ very approximately}$$

where

$$B = \frac{C^2}{4\pi^4 b}$$

Since $u(x) \equiv x\,E^2(x)$ this gives

$$u(x) = \frac{Bbx}{(x-x')^2 + b^2} + \frac{Bbx}{(x+x')^2 + b^2}$$

$$= \frac{Bb\,(x-x') + x'\,Bb}{(x-x')^2 + b^2} + \frac{Bb\,(x+x') - x'\,Bb}{(x+x')^2 + b^2}$$

$$= \frac{A\,(x-x') + x'\,Bb}{(x-x')^2 + b^2} + \frac{A\,(x+x') - x'\,Bb}{(x+x')^2 + b^2}$$

where $A = Bb$

According to the rule above given, the observed function, when the radiation for the simple circuit is received by one whose damping constant is a, would then be given by

$$f(x) = 8\,R\,L\,x\,\mathfrak{J}^2(x) = \frac{A(x-x') + x'\,B(b+a)}{(x-x')^2 + (b+a)^2} + \frac{A(x+x') - x'\,B(b+a)}{(x+x')^2 + (b+a)^2}$$

exactly

$$= \frac{A(x-x') + x'\,B(b+a)}{(x-x')^2 + b + a)^2} = Bx' \frac{(b+a) + A\left(\frac{x'-x}{x'}\right)}{(x-x')^2 + (b+a)^2}$$

approximately

$$= x'B\,\frac{b+a}{(x-x')^2 + (b+a)^2} \text{ if the range } \frac{(x'-x)}{x'} \text{ is small.}$$

This is a combination of the types (37), I and III. Therefore the oscillations of a distant simple circuit which produces an effective current in the receiving circuit of the form

$$\mathfrak{J}^2(x) = \frac{x'B}{8\,R\,L\,x}\,\frac{a_n + \left(\frac{x-x'}{x'}\right)A}{(x-x')^2 + a^2_n}$$

or approximately

$$8\,R\,L\,\mathfrak{J}^2\,(x) = \frac{B\,a_n}{[(x-x')^2 + a^2_n]}$$

when the range $\frac{(x-x')}{x}$ is small, must have a spectral energy distribution

$$8\,R\,L\,E^2\,(x) = \frac{B\,(a_n - a)}{(x-x')^2 + (a_n - a)^2}$$

The width of the peak of this curve is $2(a_n - a)$; that is, it is narrower than the peak of the observed current by $2a$, the width of the resonance peak of the receiving circuit. The receiving circuit adds its own decrement to that of the radiation.

The symmetrical simple solution of the type (37), I, is therefore seen to correspond to one of the most important physical types of radiation to be met with, namely, damped oscillations. It is readily seen to contain the case of undamped oscillations as a limiting case, for if a_n is but very slightly greater than a, the solution

$$\frac{a_n - a}{(x - x_n)^2 + (a_n - a)^2}$$

is very small for all values of x, except $x = x_n$, which corresponds to a single spectrum line of frequency x_n; that is, an undamped wave. The unsymmetrical form (37), III, which changes sign at $x = x_n$ is also seen to possess a similar property. It approaches zero in general, but becomes $+$ or $-\infty$ when x approaches x_n from opposite sides, when a_n approaches a as a limit. Although simple solutions which change sign do not correspond to any physical case when considered alone, yet a number of them together may readily be compounded into a function which is everywhere positive inside a given range, just as in the case of sines and cosines. This type of solution is therefore useful in representing the observed function. The form (37), IV, also changes sign half way between x_1 and x_2. The form (37), II, is very useful in some cases. It is easily seen that if a_n approaches a as a limit, the solution $u(x)$ is zero everywhere outside the range between x_1 and x_2, and is constant inside this range.

VI. GENERAL RELATION BY MEANS OF COMPLEX INTEGRATION.

All of these special cases are included in a very general relation which may be derived from (30), and which gives an elegant solution in case the function $f(x)$ has been represented by an analytical expression. If the poles of $f(z)$ all lie at a distance from the real axis (in the complex z-plane), which is greater than a, then 2 cos aD_x $f(x)$ or $(e^{iaD_x} + e^{-iaD_x})$ $f(x)$ represents $f(x+ia) + f(x-ia)$ since the expression e^{iaD_x} $f(x)$ is just the symbolic form of Taylor's theorem, and in this case the expansion is valid about the point x, which is any point on the real axis. Equation (32) is therefore

$$u(x) = f(x+ia) + f(x-ia) - \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{f(x')\,dx'}{a^2 + (x'-x)^2}$$

If $f_1(z)$ represents that part of $f(z)$ whose poles are in the upper half plane and $f_2(z)$ that part whose poles are below the real axis, and if $f(z)$ is finite or zero at infinity, then by deforming the path of integration indefinitely upward in the one case and downward in the other case, and evaluating the integrals around the poles of the nucleus by Cauchy's theorem of residues, one easily arrives at a fourth form of solution

$$u(x) = f_1(x+ia) + f_2(x-ia) \tag{38}$$

This means that if a new function is formed from $f(x)$ by placing all its poles nearer to the real axis by an amount a, the new function is $u(x)$. By similarly treating the original equation (5) one finds

$$f(x) = u_1(x - ia) - u_2(x + ia)$$

which means that if a new function is formed from $u(x)$ by moving all its poles a distance a farther from the real axis, this new function is $f(x)$.

VII. APPLICATION OF FOURTH METHOD TO EXPERIMENTAL CURVES.

In the analysis of a given radiation one is usually only concerned with relative values of the induced current. When the frequency range studied is not large compared to the mean frequency, one may obtain correct relative values by identifying $f(x)$ with $\mathfrak{J}^2(x)$ and $u(x)$ with $E^2(x)$, for multiplying $\mathfrak{J}^2(x)$ by x distorts the curve in a manner which is practically all compensated for when we divide $u(x)$ by x to obtain $E^2(x)$.

When the capacity of the circuit is varied, a wave meter may be used to find the corresponding resonance frequency X, which is practically the same as the characteristic frequency $x = \sqrt{X^2 - a^2}$. This method then gives the true frequency which depends not only upon the capacity of the condenser, but upon the inherent distributed capacity of the coil.

The constant $a = \frac{R}{4\pi L}$ $\left(= \frac{x}{2\pi}\delta\right.$ where δ is the decrement of the circuit) must be determined at a value of the frequency near which it is to be used. Since the resistance and inductance of the circuit will change with frequency, due to eddy currents, the value of a will not be strictly constant, but may be considered so for relatively small ranges of frequency. If an unmodulated (simple harmonic) wave of frequency x' be studied with the circuit, the current curve should be of the form

$$\mathfrak{J}^2(x) = \frac{A}{(x - x')^2 + a^2}$$

so that

$$(x - x')^2 + a^2 = \frac{A}{\mathfrak{J}^2(x)}$$

If $(x - x')^2$ be plotted against $\frac{1}{\mathfrak{J}^2(x)}$, and the result is a straight

line, the intercept of this line gives the value of a^2 to be used in the neighborhood of the frequency x'.

Figure 1 illustrates the use of an empirical formula of the simple type:

$$f(x) = f_1(x) + f_2(x) + f_3(x)$$

where

$$f_1(x) = A_1\left[\tan^{-1}\left(\frac{x_1 - x}{a_1}\right) + \tan^{-1}\left(\frac{x - x'_1}{a_1}\right)\right]$$

$$f_2(x) = \frac{A_2}{a_2}\,\frac{1}{1 + \left(\frac{x - x_2}{a_2}\right)^2}$$

$$f_3(x) = \frac{A_3}{a_3}\,\frac{1}{1 + \left(\frac{x - x_3}{a_3}\right)^2}$$

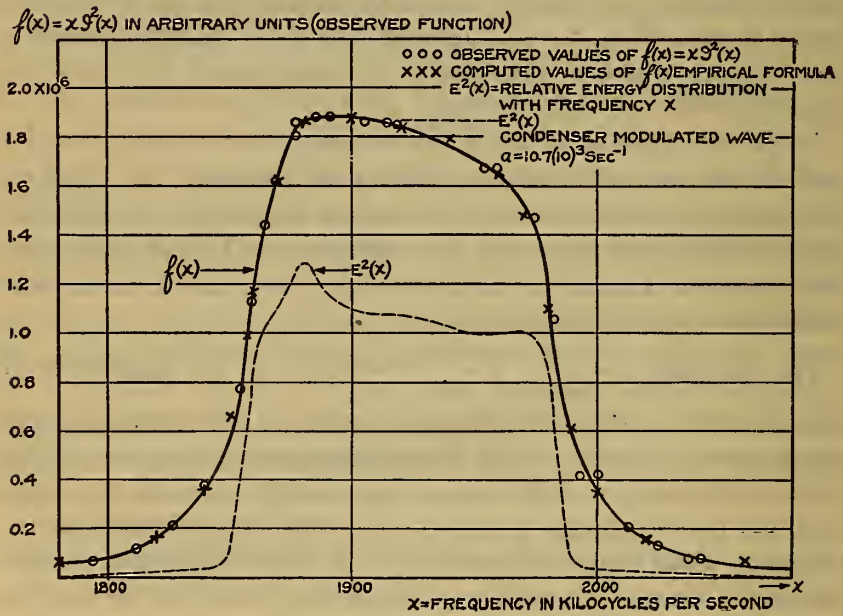

FIG. 1.—*Energy distribvtion in a condenser modulated wave.*

The observed values are represented by circles where, as in the previous notation, $f(x)$ is $x\,\overline{\mathfrak{I}^2}(x)$, and $\overline{\mathfrak{I}^2}$ is the effective current produced in the circuit, as indicated by a hot-wire ammeter. The units of current are entirely arbitrary, since only relative values are desired. The radiation in this case was a "condenser modulated," one caused by the rotation of the plates of an air condenser in a sending circuit in which there is a sustained simple harmonic wave. The experiments were performed by R. T. Cox and Miss F. Kenyon, of the Bureau of Standards radio laboratory.

The numerical value of the constants which were used to fit the observed curve were:

$A_1 = 0.672(10)^6$, $x'_1 = 1857.5$ kilocycles per sec. $a_1 = 12.5\ 10^3$ sec.$^{-1}$
$x_1 = 1982.5$ kilocycles per sec.
$A_2 = 2.56(10)^9$, $x_2 = 1882.0$ kilocycles per sec. $a_2 = 16\ 10^3$ sec.$^{-1}$
$A_3 = -1.10(10)^9$, $x_3 = 1948.0$ kilocycles per sec. $a_3 = 22\ 10^3$ sec.$^{-1}$

The circuit constant $a = 10.7\ 10^3$ sec.$^{-1}$ was determined, as explained above, from the curve obtained on the unmodulated wave. The inductance L was 56.8 microhenrys.

The empirical formula in this case fits the observed curve with no more error than those of experiment.

The energy distribution was computed from $u(x)$ where $u(x)$ was obtained by decreasing each a_n by a in the empirical formula for $f(x)$. It is represented by the dotted curve, and shows that the spectrum of the radiation consisted of a wide band of nearly uniform intensity.

The representation of a function which has well-defined peaks by an empirical equation of the type

$$\Sigma \frac{A_n}{a_n} \frac{1}{1 + \left(\frac{x - x_n}{a_n}\right)^2}$$

is generally fairly easy provided that each a_n is greater than a. If the peaks are no wider than $2a$, this would indicate the presence of a line spectrum. This was the conclusion drawn from an attempt to represent the curves of Figure 2 by terms of the above type. The two curves are drawn through the observed values for two different values of the resistance of the receiving circuit, the radiation being the same in both cases. It was produced by interrupting a simple harmonic wave with a chopper whose frequency was 612 cycles per second. The frequency of the wave which was modulated was 53 kilocycles per second. One would expect that, if the amplitude of a pure harmonic vibration were varied harmonically with a frequency of 612 cycles per second, the frequency of the original note would be absent from the spectrum, and there would appear frequencies of this value plus and minus multiples of the 612, so that the spectrum would consist of lines separated by 1,224 cycles. The separation of the lines is about 1,200 cycles, but the amplitudes are not symmetrical, showing that the action of the chopper was not so simple as this.

The crosses in Figure 2 represent the computed currents on the assumption that the spectrum consisted of three monochromatic

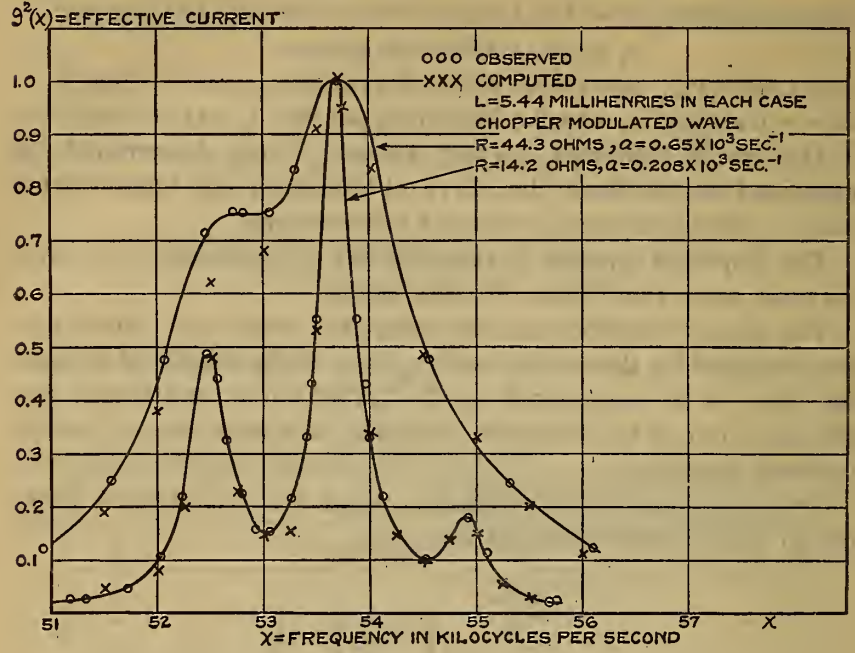

FIG. 2.—*Current in two different circuits produced by the same wave.*

lines at 52.5, 53.7, and 54.9 kilocycles per second, respectively. The equation used was

$$\mathfrak{I}^2(x) = \sum_{n=1}^{n=3} \frac{A_n}{a} \frac{1}{1 + \left(\frac{x - x_n}{a}\right)^2}$$

Where

$$\left.\begin{array}{l} A_1 = 94 \\ A_2 = 202 \\ A_3 = 30.6 \end{array}\right\} \begin{array}{l} a = 0.206\ 10^3 \text{ in lower curve} \\ a = 0.65\ 10^3 \text{ in upper curve} \end{array}$$

The agreement of computed values with observed values in the case of the circuit which had the higher resolving power justifies the inference as to a line spectrum. The deviations of computed values from observed values in the case of the upper curve might be accounted for by a slight underestimate of the resistance of this circuit. The resistances in both cases were determined from decrement measurements. All three curves were obtained by the same observers in a preliminary study, the object of which was the development of a method by which the interference-producing quality of a wave might be characterized.

VIII. CONCLUSION.

Four different methods have been shown for finding the energy spectrum of a radiation in terms of the mean current it produces in a receiving circuit, this current being observed for all real values of the characteristic frequency of the circuit. The problem consists essentially in finding practicable methods of solving the integral equation for $u(x)$

$$f(x) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{u(x')\, dx'}{a^2 + (x' - x)^2}$$

when $f(x)$ is given as an odd function of x.

It is obvious that other applications of resonance theory may involve solving this equation, which may be quite different from the purely electrical problem with which it is here associated.

The first method gives the solution of this equation in terms of a double integral analogous to Fourier's (equation 10). This form is only of value for computing when $f(x)$ is given analytically, but is useful as a basis for deriving other forms.

The second method is based upon the fact that when the nucleus of an integral equation is an even function of the sum or difference of the two variables its corresponding normal functions are trigonometric functions. Harmonic analysis of the nucleus (mechanically or graphically) amounts to a determination of the characteristic constants. A similar analysis of the observed function has been shown to be sufficient to lead to a series solution (equation 26).

The third form gives the solution in terms of differentiations and integrations of the given function $f(x)$. Like the first two forms of solution, it is applicable to cases where the functions $f(x)$ or $u(x)$ possess finite discontinuities, and has the additional advantage that the operations may be performed graphically (equation 30).

The fourth method consists in representing the observed function by empirical equations of certain simple types (37), the solution corresponding being easily obtained by subtracting the damping coefficient of the circuit from certain analogous parameters in the simple types of solutions used. This method is applicable to the limiting case of line spectra. It is applied to three different experimental curves.

WASHINGTON, July 3, 1923.