ALTERNATING-CURRENT RESISTANCE AND INDUCTANCE OF SINGLE-LAYER COILS.

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ABSTRACT.

An integral equation method is used to develop formulas for the alternating-current resistance and inductance of a system of parallel "go and return" circuits. It is shown how these formulas may be applied to single-layer coils. The formulas are applied to several coils in which the "skin effect" is quite pronounced. These coils are measured at several frequencies and the results compared with the computed values. The methods of measurements are explained and a discussion is given pertaining to the distribution of the effective resistance in a coil. The eddy current losses due to dead ends are measured and discussed.

Other formulas are applied to the coils and the results vary from 50 to several hundred per cent.

The frequencies for which the formulas are suitable depend on the diameter of the conductor and its material. For No. 2 copper wire 3,000 is the highest frequency applicable, but for No. 22 copper wire frequencies as high as 300,000 may be used.

A table of part of the functions occurring in the formulas is given.

A simple formula for the mutual inductance between coaxial circles of the same diameter is also developed.

CONTENTS.

I. INTRODUCTION.

Several formulas for the alternating-current resistance and inductance of single-layer coils have been published. However, the results obtained by applying these formulas to specific coils are so widely different that it has been thought advisable to treat the subject in an entirely different manner. Most of the past writers on the subject started with the field equations and by making various assumptions were able to obtain a solution of the problem. An integral equation method is used in this paper to develop formulas for the alternating-current resistance and inductance of single-layer coils. Owing to the paucity of experimental data, several coils were constructed and the resistance and inductance of them were measured at several frequencies.

One of the first writers on this subject was Sommerfeld.[1] His assumptions are that the field about the coil under consideration is the same as if the coil were infinitely long. Furthermore, he neglects the effect of the spacing. This leads to results which are far from correct. In his original work he obtained a formula for the resistance only. His method was used by Coffin[2] to develop a formula for the inductance. In Sommerfeld's later work he takes into consideration the size and shape of the wire with which the coil is wound and also the spacing between the wires. Even this formula does not check experimental results obtained by this bureau. Breit[3] has applied an end correction to Sommerfeld's formula which gives much better values but which are still quite different from our measured values.

Cohen[4] has developed a formula for high frequencies but it is based on the assumption that the field about the wire is the same as if the wire were square and the spacing zero.

Butterworth[5] has published a formula for the resistance of a single-layer coil, the length of which is small compared to its diameter. This formula is in reality one which he develops for a set of parallel wires all in the same plane and all carrying currents in the same direction at any instant. He also develops a formula for the resistance of a single-layer coil which is applicable to coils of any length. This formula shows that the resistance decreases as the diameter of the coil is increased. This is not borne out by experi-

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[1] Annallen der Physik, 15, p. 673, 1904; 24, p. 609, 1907.

[2] B. S. Bull., 2, No. 2, p. 275; B. S. Sci. Paper, No. 37.

[3] Unpublished. To appear in a forthcoming Scientific Paper of this bureau.

[4] B. S. Bull., 4, No. 1, p. 161; B. S. Sci. Paper, No. 76.

[5] "Eddy current losses in cylindrical conductors," Philosophical Transactions, Royal Soc. of London, Series A, 222, pp. 57–100.

mental results obtained by the author. In developing these formulas, he assumes that the magnetic field from a conductor is uniform over the cross section of any other wire however close. The assumption might be satisfactory for conductors far from the point under investigation but could not be expected to hold for those conductors quite close.

Dr. H. L. Curtis, of this bureau, has developed formulas for the alternating-current resistance and inductance of a return circuit[6] and also for a three-phase [7] current. The formulas for the return circuit have been checked by experimental results. The method is a decided departure from the usual treatment of such problems. He considers the conductors made up of infinitesimal filaments carrying currents. He assumes a current distribution in the wires of a Fourier and power series type, and knowing the mutual inductance between filaments he obtains a solution by an integral equation method.

This paper is an attempt to apply the integral equation method to coils. Since the formulas for the mutual inductance between two circular rings are rather complicated and the integration becomes very difficult, after several attempts to make use of them it was finally decided to try to find a relation between the conditions existing in a coil and those in straight parallel wires in the same plane with a similar return set of wires, the spacing between wires of the same plane corresponding to that in the coil. It is evident that the field about the wires in a coil may be approximated in such a system if the proper distance between the two floors is determined.

II. RELATION BETWEEN PARALLEL FLOOR SYSTEMS AND COILS.

Two relations between the parallel floor system and the coil will be developed from which the alternating-current resistance of a coil may be obtained by finding it for a particular parallel floor system. Both relations give the distance between the parallel floors as a function of the diameter of the coil. The results obtained by either assumption check measured values within experimental errors.

These methods of determining the proper distance between the parallel floors to represent the conditions existing in the coil

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[6] B. S. Sci. Papers, 16, pp. 93–124, 1920; B. S. Sci. Papers, No. 374.

[7] Physical Review, 18, p. 156; 1921.

are not set forth as a rigorous mathematical solution. However, the results seem reasonable and are justified by the experimental results.

1. FIRST RELATION.

Consider the self-inductance of a circular ring of circular cross section. The approximate inductance L of this ring is given by the equation:

$$L = 2\pi D' \left(\log \frac{4D'}{a} - 1.75\right)$$

where D' is the diameter of the ring and a is the radius of the wire.

The inductance of a very long return circuit is given by the equation:

$$L = 4l \left(\log \frac{D}{a} + 0.25\right)$$

where l is the length of the circuit; D is the distance between the two wires; and a as before is the radius of the conductor. Now if

$$l = \frac{\pi D'}{2},$$

by equating the two expressions for the inductance L

$$\log \frac{D}{a} = \log \frac{4D'}{a} - 2 \text{ or, } \log D = \log 4D' - 2$$

From which

$$D = 0.541\ D'$$

This means that a portion of a return circuit of length equal to one-half the circumference of the ring and spacing of 0.541 of the diameter of the ring has the same inductance as the ring itself, or that the inductance per unit length of the wire in either case is the same.

Furthermore, if additional turns are added to the ring under consideration, the conditions existing in a coil may be approximately duplicated by a system of parallel floors of wires, the spacing between the floors having the same ratio to the diameter of the coil as indicated above.

A very simple formula for the mutual inductance between two circles may be derived from the foregoing assumptions. This formula is given and discussed in the appendix.

The fact that this relation gives an approximate formula for the mutual inductance between two coaxial circles is another proof that the assumptions are justified. By making use of this mutual-inductance formula, the problem of finding the alternating-current resistance of a coil could be solved by dividing each turn into filament rings and integrating over each turn; that is, filament rings taking the place of straight filaments in Dr. Curtis's work. The integrations are more difficult to carry through by this method so that it is preferred to treat the problem for the coil by solving an equivalent parallel floor system.

2. SECOND RELATION.

Another method of determining the proper distance between the floors to represent the coil is as follows:

Let D' = the diameter of the coil

and D = the distance between the floors which is to be determined.

Consider a portion of the parallel floor system of length $\frac{\pi D'}{2}$; that is, one-half the circumference of the coil. If there are a large number of turns of wires in the coil and also in the parallel floor system, the field in the coil will be uniform and constant, and equal to the field between the parallel floors in every respect. The field outside of either system will be zero. If we choose a distance D between the parallel floors so that the rectangular section will contain the same number of lines of force as the coil, we may assume that the current distribution in the wires of the parallel floor system will be about the same as in the turns of the coil.

In order to meet these conditions D must equal $\frac{D'}{2}$. This is not the same relation obtained for the first method but when used in the formulas it does not give results far different from the other. The change in resistance and inductance of a long coil with frequency is not a strong function of the diameter. However, the diameter of all finite coils must be taken into consideration in computing these changes.

If the coil were infinitely long, the variation of the current distribution with the diameter would probably be very small. In like manner, the distance between the parallel floor systems would probably be unimportant.

When we consider coils of finite length the current distribution is no longer independent of the diameter of the coil or the distance between the parallel floors. The two relations given above, for which it is claimed the current distribution in the conductors will be approximately the same for the parallel floor system as for the coil, give about the same results. One is based on making the inductance per unit length of the conductor of the two systems equal and is derived by considering a single turn. The other relation comes from a consideration of many turns and is based on an attempt to make the total magnetic field within the coil equal to the total magnetic field within a corresponding length of the parallel floor system. The close agreement between two relations of so widely different nature, together with the results obtained when applied to specific coils, justify the assumptions.

Therefore, in order to develop formulas for the resistance and inductance of a coil, it is only necessary to develop formulas for parallel floors of wires, one floor being a return at a distance D from the other. The integral equation method is easily applied to this case.

III. DERIVATION OF FORMULAS.

Consider a group of parallel cylindrical conductors having the same diameters, equally spaced, having their axes all in the same plane and all carrying the same current; and another similar group in a plane parallel to the first and at a distance D from it carrying the return circuit. If the current is alternating, the resistance and inductance of it may be found by the integral equation method.

In Figure 1 let $\boldsymbol{p}$ with coordinates ρ, θ, be a point in the middle conductor at which the current density is to be determined, $\boldsymbol{p}_o$ another point in the same conductor, distant from $\boldsymbol{p}$ by $\boldsymbol{d}_o$ and $\boldsymbol{p}_r$, p_r, $\boldsymbol{P}_r$, and P_r points in the other conductors distance from $\boldsymbol{p}$ by $\boldsymbol{d}_r$, d_r, $\boldsymbol{D}_r$, and D_r, respectively.

Bold-faced letters ($\boldsymbol{A}$, $\boldsymbol{a}$) are used for the system to the right of the middle conductor and light-faced letters (A, a) for the system to the left. Lower-case letters are used for the upper floor and upper-case letters for the lower floor.

Consider each conductor made up of infinitesimal filaments and assume that $\boldsymbol{u}'$, $\boldsymbol{u}'_o$, $\boldsymbol{u}'_r$, u_r, $\boldsymbol{U}'_r$, and U'_r is the instantaneous current density at $\boldsymbol{p}$, $\boldsymbol{p}_o$, $\boldsymbol{p}_r$, p_r, $\boldsymbol{P}_r$, P_r, respectively, and that $\boldsymbol{m}_o$, $\boldsymbol{m}_r$, m_r, $\boldsymbol{M}_r$, M_r is the mutual inductance between the filament at $\boldsymbol{p}$ and those at the other points, respectively. The general equation

$$E' = U'_x \sigma l + \iint M_{xy} \frac{dU'_y}{dt} dS$$

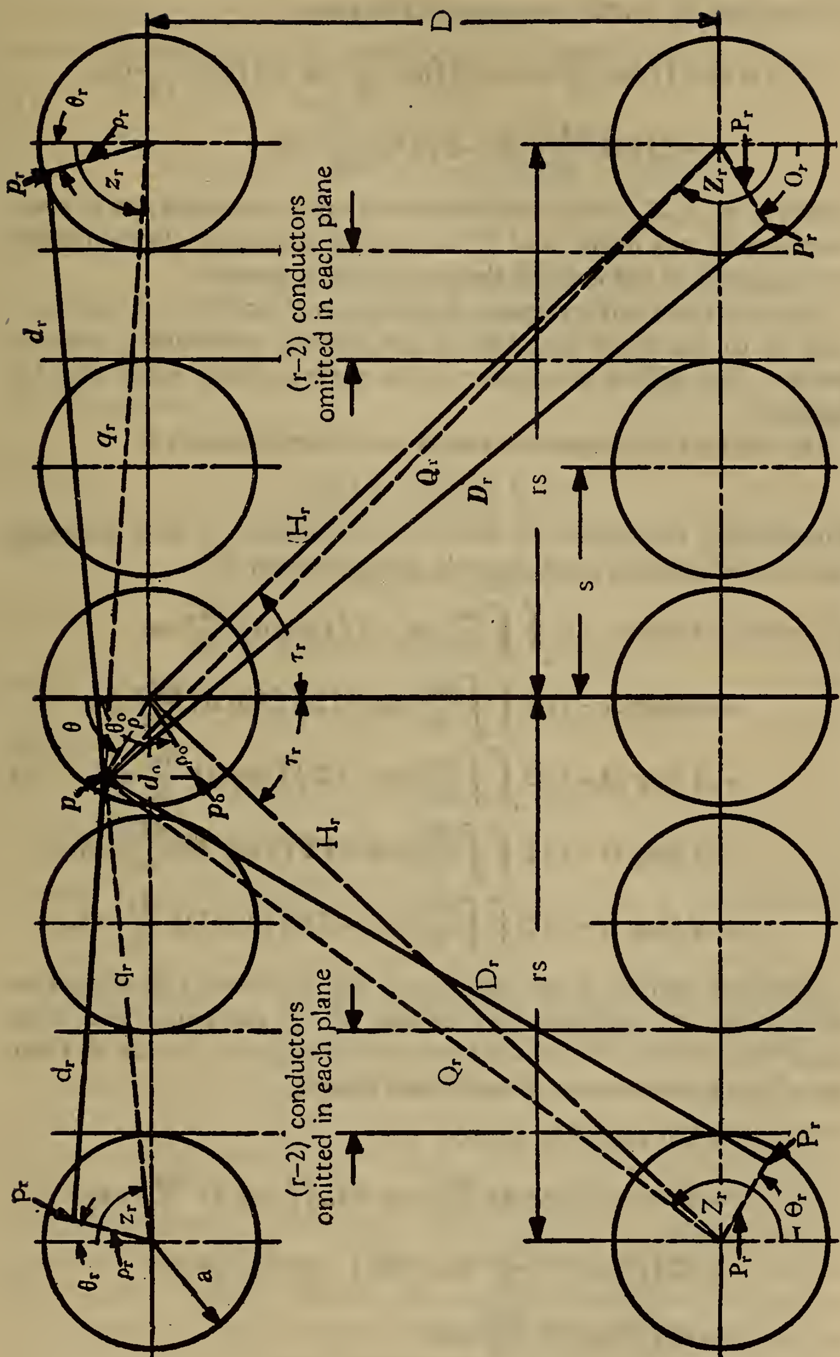

FIG. 1.—Cross section of a go and return system of parallel conductors.

as found by Dr. Curtis[8] becomes in this case:

$$E' = u'\sigma l + \int\int m_o \frac{du'_o}{dt} ds_o + \Sigma \int\int m_r \frac{du'_r}{dt} ds_r + \Sigma \int\int m_r \frac{du'_r}{dt} ds_r$$
$$- \Sigma \int\int M_r \frac{dU'_r}{dt} dS_r - \Sigma \int\int M_r \frac{dU'_r}{dt} dS_r \qquad (1)$$

Where s_o, s_r, s_r, S_r, and S_r are elements of area, σ = resistivity in electromagnetic cgs units, and E' is the instantaneous electromotive force applied at the ends of the parallel floor system.

The sums are with respect to the r's and are for all the conductors to the right and left of the middle conductors, respectively. The middle conductor of the return circuit must also be included.

The mutual inductance between two long filaments is:

$$m = 2l\ (\log 2l) - l \log d^2$$

Substituting the values of the m's in equation (1) and denoting the total number of conductors in each plane by N

$$E' = u'\sigma l + 2l\ (\log 2l - 1) \int\int \frac{du'_o}{dt} ds_o - l \int\int \log d_o^2 \frac{du'_o}{dt} ds_o$$
$$+ 2l\ (\log 2l - 1)\ \Sigma \int\int \frac{du'_r}{dt} ds_r - l\Sigma \int\int \log d_r^2 \frac{du'_r}{dt} ds_r$$
$$+ 2l\ (\log 2l - 1)\ \Sigma \int\int \frac{du'_r}{dt} ds_r - l\Sigma \int\int \log d_r^2 \frac{du'_r}{dt} ds_r \qquad (2)$$
$$- 2l\ (\log 2l - 1)\ \Sigma \int\int \frac{dU'_r}{dt} dS_r - l\Sigma \int\int \log D_r^2 \frac{dU'_r}{dt} dS_r$$
$$- 2l\ (\log 2l - 1)\ \Sigma \int\int \frac{dU'_r}{dt} dS_r - l\Sigma \int\int \log D_r^2 \frac{dU'_r}{dt} dS_r$$

Since the current is the same in all the conductors all the terms above that are multiplied by $2l\ (\log 2l - 1)$ are equal and since there will always be the same number of negative terms as there are of the positive ones, these terms vanish.

Equation (2) therefore reduces to:

$$E' = u'\sigma l - l \int\int \log d_o^2 \frac{du'_o}{dt} ds_o - l\Sigma \int\int \log d_r^2 \frac{du'_r}{dt} ds_r$$
$$- l\Sigma \int\int \log d_r^2 \frac{du'_r}{dt} ds_r + l\Sigma \int\int \log D_r^2 \frac{dU'_r}{dt} dS_r \qquad (3)$$
$$+ l\Sigma \int\int \log D_r^2 \frac{dU'_r}{dt} dS_r$$

But $E' = Ee^{j\omega t}$, $u' = ue^{j(\omega t - \phi)}$

[8] B. S. Sci. Papers, 16, p. 96, 1920; B. S. Sci. Papers, No. 374.

Referring to Figure 1 for the manner in which the angles are measured and the significance of the various symbols:

$$\begin{aligned}
d_0^2 &= \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\theta - \theta_0) \\
d_r^2 &= (\mathrm{sr} + \rho \sin\theta - \rho_r \sin\theta_1)^2 + (\rho\cos\theta - \rho_r\cos\theta_r)^2 \\
&= (q_r \sin z_r - \rho_r \sin\theta_r)^2 + (q_r \cos z_r - \rho_r \cos\theta_r)^2 \\
&= q_r^2 + \rho_r^2 - 2q_r\rho_r \cos(z_r - \theta_r)
\end{aligned}$$

Where

$$\begin{aligned}
q_r \sin z_r &= \mathrm{s}_r + \rho \sin\theta, \\
q_r \cos z_r &= \rho \cos\theta \\
q_r^2 &= \mathrm{r}^2\mathrm{s}^2 + 2\rho\mathrm{sr}\sin\theta + \rho^2
\end{aligned}$$

In like manner

$$d_r^2 = \mathrm{q}_r^2 + \rho_r^2 - 2\mathrm{q}_r\rho_r \cos(\mathrm{z}_r - \theta_r)$$

Where

$$\mathrm{q}_r^2 = \mathrm{r}^2\mathrm{s}^2 - 2\rho\mathrm{sr}\sin\theta + \rho^2$$

Also

$$D_r^2 = Q_r^2 + 2Q_r\mathrm{P}_r \cos(\Theta_r - Z_r) + \mathrm{P}_r^2$$

where

$$Q_r^2 = \rho^2 + \mathrm{H}_r^2 + 2\mathrm{PH}_r \cos(\tau_r - \theta)$$

and

$$\mathrm{D}_r^2 = \mathrm{Q}_r^2 + 2\mathrm{Q}_r\mathrm{P}_r \cos(\Theta_r - \mathrm{Z}_r) + \mathrm{P}_r^2$$

where

$$\mathrm{Q}_r^2 = \rho^2 + \mathrm{H}_r^2 + 2\rho\mathrm{H}_r \cos(\tau_r + \theta)$$

Substituting these values in equation (3)

$$\begin{aligned}
\mathrm{E} = \sigma l u e^{\neg j\phi} &- j\omega l \int_0^a \rho_0 d\rho_0 \int_0^{2\pi} \log[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\theta - \theta_0)] u_0 e^{\neg j\phi_0} d\theta_0 \\
&- j\omega l \Sigma \int_0^a \rho_r d\rho_r \int_0^{2\pi} \log[q_r^2 + \rho_r^2 - 2q_r\rho_r \cos(z_r - \theta_r)] u_r e^{\neg j\phi_r} d\theta_r \\
&- j\omega l \Sigma \int_0^a \rho_r d\rho_r \int_0^{2\pi} \log[\mathrm{q}_r^2 + \rho_r^2 - 2\mathrm{q}_r\rho_r \cos(\mathrm{z}_r - \theta_r)] u_r e^{\neg j\phi_r} d\theta_r \qquad (4) \\
&+ j\omega l \Sigma \int_0^a \mathrm{P}_r d\mathrm{P}_r \int_0^{2\pi} \log[Q_r^2 + \mathrm{P}_r^2 - 2Q_r\mathrm{P}_r \cos(\Theta - Z_r)] U_r e^{\neg j\phi_r} d\Theta_r \\
&+ j\omega l \Sigma \int_0^a \mathrm{P}_r d\mathrm{P}_r \int_0^{2\pi} \log[\mathrm{Q}_r^2 + \mathrm{P}_r^2 - 2\mathrm{Q}_r\mathrm{P}_r \cos(\Theta - \mathrm{Z}_r)] \mathrm{U}_r e^{\neg j\phi_r} d\Theta_r
\end{aligned}$$

U and ϕ are functions of ρ and θ but since they have the same values for $-\theta$ as for θ, they may be expressed in a Fourier series in terms of the cosine series only. This is equivalent to saying that the current distribution is symmetrical about the line drawn through the center of the conductor, perpendicular to it and to the planes of the conductors. This is exact for the conductor under investigation; that is, the conductor at the middle of the system, and is nearly so for all conductors near by. The distribution of the current in the middle conductor will be little affected by the manner in which the current is distributed in those conductors which are at a considerable distance from it. Therefore, in determining the effect on the middle wire, the current distribution in the other wires is assumed to be the same as in the middle wire.

$$\begin{aligned}Ue^{-j\phi} = & \, a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \cdots\cdots \\ & + \rho\,(b_0 + b_1 \cos\theta + b_2 \cos 2\theta + \cdots\cdots) \\ & + \rho^2\,(c_0 + c_1 \cos\theta + c_2 \cos 2\theta + \cdots\cdot) \\ & + \rho^3\,(d_0 + d_1 \cos\theta + d_2 \cos 2\theta + \cdots\cdot) \\ & + \cdots\cdots\cdots\cdots\cdots \text{etc.}\end{aligned}$$

Since it is known that the current distribution is finite and single valued at the origin (that is, at the center of each conductor), it can not be a function of θ at that point, so that:

$$a_1 = a_2 = a_3 = \cdots\cdots \text{etc.} = 0$$

Also it may be assumed that:

$$\left|\left(\frac{dU}{dx}\right)_{x=+0}\right| = \left|\left(\frac{dU}{dx}\right)_{x=-0}\right|;\ \left|\left(\frac{d^2U}{dx^2}\right)_{x=+0}\right| = \left|\left(\frac{d^2U}{dx^2}\right)_{x=-0}\right| \ \&c.$$

And since:

$$\left(\frac{dU}{dx}\right)_{x=+0} = \left(\frac{dU}{d\rho}\right)_{\substack{\rho=+0\\ \theta=\text{const.}}}$$

And

$$\left(\frac{dU}{dx}\right)_{x=-0} = -\left(\frac{dU}{d\rho}\right)_{\substack{\rho=+0\\ \theta=\text{const.}+\pi}}$$

As ρ approaches zero from $\theta=$ const. and from $\theta=$ const. $+\pi$ U is of opposite sign and this can be true only for $\cos\theta$, $\cos 3\theta$, $\cos 5\theta$, etc. Therefore

$$b_0 = b_2 = b_4 \text{ etc.} = 0$$

And also since:

$$\left(\frac{d^2U}{dx^2}\right)_{x=+0} = \frac{d}{d\rho}\left(\frac{dU}{d\rho}\right)_{\substack{\rho=+0\\ \theta=\text{const.}}} = \left(\frac{d^2U}{d\rho^2}\right)_{\substack{\rho=+0\\ \theta=\text{const.}}}$$

And

$$\left(\frac{d^2U}{dx^2}\right)_{x=-0} = -\frac{d}{d\rho}\left(-\frac{dU}{d\rho}\right)_{\substack{\rho=+0\\ \theta=\text{const.}+\pi}} = \left(\frac{d^2U}{d\rho^2}\right)_{\substack{\rho=+0\\ \theta=\text{const.}+\pi}}$$

As ρ is decreased from $\theta=$ const. and from $\theta=$ const. $+\pi$, the sign is the same. This can be true only for $\cos 2\theta$, $\cos 4\theta$, $\cos 6\theta$, etc.

Therefore: $c_1 = c_3 = c_5 =$ etc. $=0$.

In like manner it may be shown that:

$$d_0 = d_2 = d_4 = \text{etc.} = 0$$

$$e_1 = e_3 = e_5 = \text{etc.} = 0 \cdots\cdots\cdots \text{etc.}$$

Also the terms of higher multiple than two of the angle are negligible.

The expression for U $e^{-j\phi}$ therefore simplifies to:

$$
\begin{aligned}
Ue^{-j\phi} = {} & a_0 + c_0\rho^2 + e_0\rho^4 + g_0\rho^6 + \cdots\cdots \\
& + (b_1\rho + d_1\rho^3 + f_1\rho^5 + \cdots\cdots) \cos\theta \\
& + (c_2\rho^2 + e_2\rho^4 + g_2\rho^6 + \cdots\cdots) \cos 2\theta \\
& + \cdots\cdots\cdots \text{ etc.}
\end{aligned}
$$

Substituting in equation (4) and integrating by making use of the integrals given in the Appendix:

$$
\begin{aligned}
E = \sigma l \, [& a_0 + c_0\rho^2 + e_0\rho^4 + g_0\rho^6 + \cdots\cdots\cdots \\
& + (b_1\rho + d_1\rho^3 + f_1\rho^5 + \cdots\cdots\cdots) \cos\theta \\
& + (c_2\rho^2 + e_2\rho^4 + g_2\rho^6 + \cdots\cdots\cdots) \cos 2\theta \\
& + \cdots\cdots\cdots\cdots \text{ etc.}] \\
-j\omega l & \left[4\pi \log a \left(\frac{a_0 a^2}{2} + \frac{c_0 a^4}{4} + \frac{e_0 a^6}{6} + \cdots \right) \right] \\
+j\omega l & \left[4\pi \left(\frac{a_0 a^2}{4} + \frac{c_0 a^4}{16} + \frac{e_0 a^6}{36} + \cdots\cdots\cdots \right) \right] \\
-j\omega l & \left[4\pi \left(\frac{a_0 \rho^2}{4} + \frac{c_0 \rho^4}{16} + \frac{e_0 \rho^6}{36} + \cdots\cdots\cdots \right) \right] \\
-j\omega l & \left[4\pi \left(\frac{b_1 \rho^3}{8} + \frac{d_1 \rho^5}{24} + \frac{f_1 \rho^7}{48} + \cdots\cdots \right) \right. \\
& \left. - 2\pi\rho \left(\frac{b_1 a^2}{2} + \frac{d_1 a^4}{4} + \frac{f_1 a^6}{6} + \cdots\cdots\cdots \right) \right] \cos\theta \\
-j\omega l & \left[4\pi \left(\frac{c_2 \rho^4}{12} + \frac{e_2 \rho^6}{32} + \frac{g_2 \rho^8}{60} + \cdots\cdots \right) \right. \\
& \left. - \pi\rho^2 \left(\frac{c_2 a^2}{2} + \frac{e_2 a^4}{4} + \frac{g_2 a^6}{6} + \cdots\cdots\cdots \right) \right] \cos 2\theta \\
+ & \left[\cdots\cdots\cdots\cdots\cdots \text{ etc.} \right. \\
-j\omega l & \left[4\pi\Sigma \, (\log \boldsymbol{q}_r + \log q_r) \left(\frac{a_0 a^2}{2} + \frac{c_0 a^4}{4} + \frac{e_0 a^6}{6} + \cdots \right) \right] \\
+j\omega l & \left[2\pi\Sigma \left(\frac{\cos \boldsymbol{z}_r}{\boldsymbol{q}_r} + \frac{\cos\ z_r}{q_r} \right) \left(\frac{b_1 a^4}{4} + \frac{d_1 a^6}{6} + \frac{f_1 a^8}{8} + \cdots \right) \right] \\
+j\omega l & \left[\pi\Sigma \left(\frac{\cos 2\ \boldsymbol{z}_r}{\boldsymbol{q}_r^2} + \frac{\cos 2\ \ z_r}{q_r^2} \right) \left(\frac{c_2 a^6}{6} + \frac{e_2 a^8}{8} + \frac{g_2 a^{10}}{10} + \cdots \right) \right] \\
+ & \cdots\cdots\cdots\cdots\cdots\cdots \text{ etc.} \qquad (5) \\
+j\omega l & \left[4\pi\Sigma \, (\log \boldsymbol{Q}_r + \log Q_r) \left(\frac{a_0 a^2}{2} + \frac{c_0 a^4}{4} + \frac{e_0 a^6}{6} + \cdots \right) \right] \\
-j\omega l & \left[2\pi\Sigma \left(\frac{\cos \boldsymbol{Z}_1}{\boldsymbol{Q}_r} + \frac{\cos\ Z_r}{Q_r} \right) \left(\frac{b_1 a^4}{4} + \frac{d_1 a^6}{6} + \frac{f_1 a^8}{8} + \cdots \right) \right] \\
-j\omega l & \left[\pi\Sigma \left(\frac{\cos 2\, \boldsymbol{Z}_r}{\boldsymbol{Q}_r^2} + \frac{\cos 2\ Z_r}{Q_r^2} \right) \left(\frac{c_2 a^6}{6} + \frac{e_2 a^8}{8} + \frac{g_2 a^{10}}{10} + \cdots \right) \right]
\end{aligned}
$$

By equating to zero coefficients of like terms of ρ^n and $\rho^m \cos n\,\theta$, many of the undetermined coefficients are easily evaluated.

TABLE I.—Coefficients of Terms Involving ρ and θ from Equation (5).

Let $\frac{\pi\,\omega}{\sigma}=\eta$

Coefficients of—	Values obtained.	Coefficients of—	Values obtained.
ρ^2	$c_o=j\,\eta\,a_o$	$\rho^9\cos\theta$	$j_1=\frac{\eta^4\,b_1}{2880}$
ρ^4	$e_o=-\frac{j\,\eta^2 a_o}{4}$	$\rho^{11}\cos\theta$	$l_1=\frac{j\,\eta^5\,b_1}{86400}$
ρ^6	$g_o=-\frac{j\,\eta^3\,a_o}{36}$	$\rho^{13}\cos\theta$	$n_1=-\frac{\eta^6\,b_1}{3628800}$
ρ^8	$i_o=\frac{\eta^4\,a_o}{576}$	$\rho^4\cos 2\theta$	$e_2=\frac{j\,\eta\,c_2}{3}$
ρ^{10}	$k_o=\frac{j\,\eta^5\,a_o}{14400}$	$\rho^6\cos 2\theta$	$g_2=-\frac{\eta^2\,c_2}{24}$
ρ^{12}	$m_o=-\frac{\eta^6\,a_o}{518400}$	$\rho^8\cos 2\theta$	$i_2=-\frac{j\,\eta^3\,c_2}{360}$
$\rho^3\cos\theta$	$d_1=\frac{j\,\eta\,b_1}{2}$	$\rho^{10}\cos 2\theta$	$k_2=\frac{\eta^4\,c_2}{8640}$
$\rho^5\cos\theta$	$f_1=-\frac{\eta^2\,b_1}{12}$	$\rho^{12}\cos 2\theta$	$m_2=\frac{j\,\eta^5\,c_2}{302400}$
$\rho^7\cos\theta$	$h_1=-\frac{j\,\eta^3\,b_1}{144}$	$\rho^{14}\cos 2\theta$	$o_2=-\frac{\eta^6\,c_2}{14515200}$

Substituting the values of these coefficients which occur in equation (5) and letting $\eta\,a^2=\lambda$, the series occurring in equation (5) reduce as follows:

$$\frac{a_oa^2}{4}+\frac{c_oa^4}{16}+\frac{e_oa^6}{36}+\cdots=\frac{a_oa^2}{4}\left(1+\frac{j\lambda}{4}-\frac{\lambda^2}{36}-\frac{j\lambda^3}{576}+\cdots\right)\equiv\frac{a_oa^2}{4}P_1$$

$$\frac{b_1a^2}{2}+\frac{d_1a^4}{4}+\frac{f_1a^6}{6}+\cdots=\frac{b_1a^2}{2}\left(1+\frac{j\lambda}{4}-\frac{\lambda^2}{36}-\frac{j\lambda^3}{576}+\cdots\right)\equiv\frac{b_1a^2}{4}P_1$$

$$\frac{c_2a^2}{2}+\frac{e_2a^4}{4}+\frac{g_2a^6}{6}+\cdots=\frac{c_2a^2}{2}\left(1+\frac{j\lambda}{6}-\frac{\lambda^2}{72}-\frac{j\lambda^3}{1440}+\cdots\right)\equiv\frac{c_2a^2}{2}P_2$$

$$\frac{a_oa^2}{2}+\frac{c_oa^4}{4}+\frac{e_oa^6}{6}+\cdots=\frac{a_oa^2}{2}\left(1+\frac{j\lambda}{2}-\frac{\lambda^2}{12}-\frac{j\lambda^3}{144}+\cdots\right)\equiv\frac{a_oa^2}{2}Q_1$$

$$\frac{b_1a^4}{4}+\frac{d_1a^6}{6}+\frac{f_1a^8}{8}+\cdots=\frac{b_1a^4}{4}\left(1+\frac{j\lambda}{3}-\frac{\lambda^2}{24}-\frac{j\lambda^3}{360}+\cdots\right)\equiv\frac{b_1a^4}{4}Q_2$$

$$\frac{c_2a^6}{6}+\frac{e_2a^8}{8}+\frac{g_2a^{10}}{10}+\cdots=\frac{c_2a^6}{6}\left(1+\frac{j\lambda}{4}-\frac{\lambda^2}{40}-\frac{j\lambda^3}{720}+\cdots\right)\equiv\frac{c_1a^6}{6}Q_3$$

Where

$$Q_n = \sum_{k=0}^{\infty} \frac{\lfloor n \, j^k \lambda^k}{\lfloor k \lfloor k+n}$$

And

$$P_n = \sum_{k=0}^{\infty} \frac{\lfloor n \, j^k \lambda^k}{\lfloor k+1 \lfloor k+n}$$

Substituting these values in equation (5), putting in the values of

$$(\log \boldsymbol{q}_r + \log q_r), \left(\frac{\cos n\boldsymbol{z}_r}{\boldsymbol{q}_r^n} + \frac{\cos nz_r}{q_r^n}\right),$$

$$(\log \boldsymbol{Q}_r + \log Q_r), \left(\frac{\cos n\boldsymbol{Z}_r}{\boldsymbol{Q}_r^n} + \frac{\cos nZ_r}{Q_r^n}\right).$$

given in the appendix, dividing by σl and noting that $\frac{\pi\omega a^2}{\sigma} = \lambda$ equation (5) becomes

$$\begin{aligned}
\frac{E}{\sigma l} &= \boldsymbol{a}_0 + \boldsymbol{b}_1\rho\cos\theta + \boldsymbol{c}_2\rho^2\cos 2\theta + \cdots\cdots \\
&- 2j\lambda\boldsymbol{a}_0Q_1\log a + j\lambda\boldsymbol{a}_0P_1 + j\lambda\boldsymbol{b}_1P_1\rho\cos\theta + \frac{j\lambda\boldsymbol{c}_2P_2}{2}\rho^2\cos 2\theta \\
&- \Sigma 4j\lambda\boldsymbol{a}_0Q_1\log rs - \sum\frac{2j\lambda\boldsymbol{a}_0Q_1}{(rs)^2}\rho^2\cos 2\theta + \cdots\cdots\cdots \\
&+ \sum\frac{j\lambda\boldsymbol{b}_1a^2Q_2}{(rs)^2}\rho\cos\theta - \sum\frac{j\lambda\boldsymbol{c}_2a^4Q_3}{3(rs)^2} \\
&+ \sum\frac{j\lambda\boldsymbol{c}_2a^4Q_3}{(rs)^4}\rho^2\cos 2\theta + \cdots\cdots\cdots\cdots \qquad (6) \\
&+ \Sigma 4j\lambda\boldsymbol{a}_0Q_1\log H_r + \sum\frac{4j\lambda\boldsymbol{a}_0Q_1D}{H_r^2}\rho\cos\theta \\
&- \sum\frac{4j\lambda\boldsymbol{a}_0Q_1D^2}{H_r^4}\rho^2\cos 2\theta + \sum\frac{2j\lambda\boldsymbol{a}_0Q_1}{H_r^2}\rho^2\cos 2\theta \\
&+ \sum\frac{j\lambda\boldsymbol{b}_1a^2Q_2D}{H_r^2} + \sum\frac{j\lambda\boldsymbol{b}_1a^2Q_2}{H_r^2}\rho\cos\theta \\
&- \sum\frac{2j\lambda\boldsymbol{b}_1a^2Q_2D^2}{H_r^4}\rho\cos\theta - \sum\frac{2j\lambda\boldsymbol{c}_2a^4Q_3D^2}{3H_r^4} + \sum\frac{j\lambda\boldsymbol{c}_2a^4Q_3}{3H_r^2} \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots \text{etc.}
\end{aligned}$$

Equating coefficients of like powers of $\rho^n \cos n\theta$ to zero, and, noting that $1+j\lambda P_1=Q_o$ and $1+\frac{j\lambda P_2}{2}=Q_1$

$$\frac{E}{\sigma l}=\boldsymbol{a}_oQ_o-2j\lambda\boldsymbol{a}_oQ_1\log a-\Sigma\,4j\lambda\boldsymbol{a}_oQ_1\log rs+\Sigma\,4j\lambda\boldsymbol{a}_oQ_1\log H_r$$
$$+\sum\frac{j\lambda\boldsymbol{b}_1a^2Q_2D}{H_r^2}-\frac{j\lambda\boldsymbol{c}_2a^4Q_3}{3}\sum\left[\frac{1}{r^2s^2}-\frac{1}{H_r^2}+\frac{2D^2}{H_r^4}\right]\tag{7}$$

$$\boldsymbol{b}_1=\frac{-\sum\frac{4j\lambda\boldsymbol{a}_oQ_1D}{H_r^2}}{Q_o+j\lambda a^2Q_2\sum\left[\frac{1}{r^2s^2}+\frac{1}{H_r^2}-\frac{2D^2}{H_r^4}\right]}\tag{8}$$

$$\boldsymbol{c}_2=\frac{2j\lambda\boldsymbol{a}_oQ_1\sum\left[\frac{1}{r^2s^2}-\frac{1}{H_r^2}+\frac{2D^2}{H_r^4}\right]}{Q_1+\sum\frac{j\lambda a^4Q_3}{r^4s^4}}\tag{9}$$

Substituting the values of $\boldsymbol{b}_1$ and $\boldsymbol{c}_2$ in equation (7):

$$\frac{E}{\sigma l}=\boldsymbol{a}_oQ_o-2j\lambda\boldsymbol{a}_oQ_1\log a-\Sigma\,4j\lambda\boldsymbol{a}_oQ_1\log rs+\Sigma\,4j\lambda\boldsymbol{a}_oQ_1\log H_r$$
$$+\frac{4\lambda^2\boldsymbol{a}_oa^2Q_1Q_2\left[\sum\frac{D}{H_r^2}\right]^2}{Q_o+j\lambda a^2Q_2\sum\left[\frac{1}{r^2s^2}+\frac{1}{H_r^2}-\frac{2D^2}{H_r^4}\right]}\tag{10}$$
$$+\frac{\frac{2}{3}\lambda^2\boldsymbol{a}_oa^4Q_1Q_3\sum\frac{1}{r^2s^2}\sum\left[\frac{1}{r^2s^2}-\frac{2}{H_r^2}+\frac{4D^2}{H_r^4}\right]}{Q_1+\sum\frac{j\lambda a^4Q_3}{r^4s^4}}$$

But E is equal to the impedance times the total current through the conductor, so that

$$E=(R+j\omega L)\,\pi a^2\boldsymbol{a}_oQ_1=\frac{(R+j\omega L)\,\boldsymbol{a}_oQ_1\sigma l}{R_o}$$

where R and L are, respectively, the alternating-current resistance and inductance of the system of parallel wires, and R_o is the direct-current resistance which is equal to $\frac{\sigma l}{\pi a^2}$.

Substituting the above values of E in equation (10), dividing by a_0Q_1 and noting in Figure 1 that $H_r^2 = D^2 + r^2s^2$

$$
\begin{aligned}
\frac{R+j\omega L}{R_0} = {} & \frac{Q_0}{Q_1} - 2j\lambda \log a - 4j\lambda\Sigma \log rs + 2j\lambda\Sigma \log (D^2+r^2s^2) \\
& + \frac{4\lambda^2a^2\left[\sum \frac{D}{D^2+r^2s^2}\right]^2}{\frac{Q_0}{Q_2} + j\lambda a^2 \sum\left[\frac{1}{r^2s^2} + \frac{1}{D^2+r^2s^2} - \frac{2D^2}{(D^2+r^2s^2)^2}\right]} \\
& + \frac{\frac{2\lambda^2a^4}{3}\sum\frac{1}{r^2s^2}\sum\left[\frac{1}{r^2s^2} - \frac{2}{D^2+r^2s^2} + \frac{4D^2}{(D^2+r^2s^2)^2}\right]}{\frac{Q_1}{Q_3} + j\lambda a^4\sum\frac{1}{r^4s^4}}
\end{aligned}
\tag{11}
$$

Since $\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$ and $\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}$, if the number of conductors N is large, these values may be substituted in the above equation.

In order to include the effect of the middle conductor of the return circuit, the sums must include the condition where $H_r = D$ or where r is zero. However, the corresponding r^{th} wires of the two sides have been added so that the coefficient multiplying the terms where $H_r = D$ must be divided by 2. Therefore

$$\sum \frac{1}{D^2+r^2s^2} = \frac{.5}{D^2} + \sum_{r=1}^{r=\frac{N-1}{2}} \frac{1}{D^2+r^2s^2}$$

$$\sum \frac{1}{(D^2+r^2s^2)^2} = \frac{.5}{D^4} + \sum_{r=1}^{r=\frac{N-1}{2}} \frac{1}{(D^2+r^2s^2)^2}$$

$$\Sigma \log (D^2+r^2s^2) = \log D + \sum_{r=1}^{r=\frac{N-1}{2}} \log (D^2+r^2s^2)$$

A very close approximation to the value of these sums may be obtained by integration.

Let $rs = x$

Then

$$\sum_{r=1}^{r=\frac{N-1}{2}} \frac{1}{D^2+r^2s^2} = \frac{1}{s}\int_{x=s/2}^{x=\frac{Ns}{2}} \frac{dx}{D^2+x^2}$$

$$= \frac{1}{s}\left[\frac{1}{D}\tan^{-1}\frac{x}{D}\right]_{x=\frac{s}{2}}^{x=\frac{Ns}{2}} = \frac{1}{Ds}\left[\tan^{-1}\frac{Ns}{2D} - \tan^{-1}\frac{s}{2D}\right]$$

Therefore

$$\sum \frac{1}{D^2+r^2s^2} = \frac{1}{D}\left[\frac{.5}{D} + \frac{1}{s}\left(\tan^{-1}\frac{Ns}{2D} - \tan^{-1}\frac{s}{2D}\right)\right]$$

In like manner it may be shown that

$$2D^2\sum\left(\frac{1}{D^2+r^2s^2}\right)^2 - \sum\frac{1}{D^2+r^2s^2} = \frac{2N}{N^2s^2+4D^2} - \frac{2}{s^2+4D^2} + \frac{.5}{D^2}$$

that

$$\Sigma \log (D^2+r^2s^2) = (N-1) \log s + S$$

and that

$$\Sigma \log rs = \log s \left(\tfrac{N-1}{2}\right) + \log\left(\frac{N-1}{2}\right)! = \frac{N-1}{2}\log s + \log\left(\frac{N-1}{2}\right)!$$

Where

$$S = \frac{N}{2}\log\left(\frac{N^2}{4}+\frac{D^2}{s^2}\right) - N + \frac{2D}{s}\tan^{-1}\frac{Ns}{2D}$$
$$-\frac{1}{2}\log\left(\frac{1}{4}+\frac{D^2}{s^2}\right) + 1 - \frac{2D}{s}\tan^{-1}\frac{s}{2D} + \log D$$

Substituting the values of these sums in equation (11)

$$\frac{R+j\omega L}{R_0} = \frac{Q_0}{Q_1} + 2j\lambda \log a - 4j\lambda \log\left(\frac{N-1}{2}\right)! + 2j\lambda S$$
$$+\frac{4\lambda^2a^2\left[\frac{.5}{D}+\frac{1}{s}\left(\tan^{-1}\frac{Ns}{2D}-\tan^{-1}\frac{s}{2D}\right)\right]^2}{\frac{Q_0}{Q_2}+j\lambda a^2\left[\frac{\pi^2}{6s^2}-\frac{2N}{N^2s^2+4D^2}+\frac{2}{s^2+4D^2}-\frac{.5}{D^2}\right]}$$
$$+\frac{\frac{\pi^2\lambda^2a^4}{9s^2}\left[\frac{\pi^2}{6s^2}+\frac{4N}{N^2s^2+4D^2}-\frac{4}{s^2+4D^2}+\frac{1}{D^2}\right]}{\frac{Q_1}{Q_3}+\frac{j\lambda\pi^4a^4}{90\,s^4}} \qquad (12)$$

The series $Q_0, Q_1, Q_2 \cdots\cdots\cdots Q_n$ are related to each other in the following manner:

$$Q_n = Q_{n+1} + \frac{j\lambda Q_{n+2}}{(n+1)\,(n+2)}$$

$$Q_n = \frac{\underline{|n}}{j}\,\frac{dQ_{n-1}}{d\lambda}$$

Representing the real and imaginary parts of any ratio $\frac{Q_s}{Q_n}$ as follows:

$$\frac{Q_s}{Q_n} = U_{s/n} + j\lambda V_{s/n}$$

Then

$$\frac{Q_0}{Q_1} = U_{0/1} + j\lambda V_{0/1}$$

$$\frac{Q_0}{Q_2} = U_{0/2} + j\lambda V_{0/2}$$

$$\frac{Q_1}{Q_3} = U_{1/3} + j\lambda V_{1/3}$$

It has been shown by Dr. Chester Snow[9] that

$$\frac{Q_{n-1}}{Q_n} = \frac{1}{n}\,\frac{\sum_0^\infty \frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+n-1}\,\underline{|2k+n}}}{\sum_0^\infty \frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+n}\,\underline{|2k+n}}} + \frac{j\lambda}{n}\,\frac{\sum_0^\infty \frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+n}\,\underline{|2k+n+1}}}{\sum_0^\infty \frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+n}\,\underline{|2k+n}}}$$

By means of this formula and the relations given above it is possible to express the real and imaginary parts of any ratio $\frac{Q_s}{Q_n}$ in the form of one series divided by another. The real part of the above formula is denoted by $\frac{U_{n-1}}{n}$ and the imaginary part by $\frac{\lambda V_{n-1}}{n}$

$$\text{Let } G = 4\,\lambda^2 a^2 \left[\frac{.5}{D} + \frac{1}{s}\left(\tan^{-1}\frac{Ns}{2D} - \tan^{-1}\frac{s}{2D}\right)\right]^2$$

$$K = \frac{\pi^2\lambda^2 a^4}{9s^2}\left[\frac{\pi^2}{6s^2} + \frac{4N}{N^2s^2+4D^2} - \frac{4}{s^2+4D^2} + \frac{1}{D^2}\right]$$

$$E = V_{0/2} + \frac{a^2\pi^2}{6s^2} - a^2\left[\frac{2N}{N^2s^2+4D^2} - \frac{2}{s^2+4D^2} + \frac{.5}{D^2}\right]$$

$$F = V_{1/3} + \frac{a^4\pi^4}{90s^4}$$

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9 Unpublished.

Substituting in equation (12) and taking the real parts

$$\frac{R}{R_0}=U_{0/1}+\frac{G}{U_{0/2}+\frac{\lambda^2E^2}{U_{0/2}}}+\frac{K}{U_{1/3}+\frac{\lambda^2F^2}{U_{1/3}}} \tag{13}$$

Taking the imaginary parts and noting that $\frac{\omega}{R_0}=\frac{\lambda}{l}$

$$L=lV_{0/1}-2l\log a-4l\log\left(\frac{N-1}{2}\right)!+2lS$$
$$-\frac{lG}{\frac{U_{0/2}^2}{E}+\lambda^2E}-\frac{lK}{\frac{U_{1/3}^2}{F}+\lambda^2F} \tag{14}$$

Where L is the self-inductance of the conductor under investigation plus the mutual inductance between it and all the other conductors, the frequency being $\frac{\omega}{2\pi}$.

A more useful formula will be obtained by assuming that the ratio of the inductance expressed by equation (14) to the inductance of the same system for zero frequency is equal to the corresponding ratio of the inductances of a coil, the length of which is great with respect to its diameter.

For zero frequency formula (14) reduces to:

$$L_0=\frac{l}{2}-2l\log a-4l\log\left(\frac{N-1}{2}\right)!+2lS$$

The change in the inductance due to the alternating current is:

$$\Delta L=L_0-L=\frac{l}{2}-lV_{0/1}+\frac{lG}{\frac{U_{0/2}^2}{E}+\lambda^2E}+\frac{lK}{\frac{U_{1/3}^2}{F}+\lambda^2F}$$

The ratio of the alternating-current inductance to the direct-current inductance is

$$\frac{L}{L_0}=\frac{L_0-\Delta L}{L_0}=1-\frac{\Delta L}{L_0}=1-\frac{\frac{1}{2}-V_{0/1}+\frac{G}{\frac{U_{0/2}^2}{E}+\lambda^2E}+\frac{K}{\frac{U_{1/3}^2}{F}+\lambda^2F}}{\frac{1}{2}+2S-2\log a-4\log\left(\frac{N-1}{2}\right)!} \tag{15}$$

Formulas (13) and (15) represent, respectively, the ratios of the alternating-current resistance and inductance to the direct-current resistance and inductance of a system of parallel go and return conductors. These formulas, as explained in the beginning, will be approximately correct for long solenoids if D represents the radius of the coil.

If the number of conductors of a system of parallel go and return circuits is small with respect to their distance apart, the effect of the return circuit on the distribution of the current in the upper conductors will be negligible.

For a system of parallel conductors without a return these formulas become

$$\frac{R}{R_0}=U_{0/1}+\frac{\pi^4 \lambda^2 a^4}{54\, s^4\left[U^2_{1/3}+\frac{\lambda^2 F^2}{U_{1/3}}\right]} \tag{16}$$

$$\frac{L}{L_0}=1-\frac{\frac{1}{2}-V_{0/1}+\frac{\pi^4 \lambda^2 a^2}{54\, s^4\left[\frac{U^2_{1/3}}{F}+\lambda^2 F\right]}}{\frac{1}{2}-2\log a+2\,(N-1)\log s-4\log\left(\frac{N-1}{2}\right)!} \tag{17}$$

These formulas will be approximately correct for single-layer coils whose lengths are short compared to their diameters.

Formula (13) gives the ratio of the alternating-current resistance to the direct-current resistance for a single-layer coil, the length of which is great with respect to its diameter.

Formula (15) gives the corresponding ratio for the inductance of the coil.

Formula (16) gives the ratio of the alternating-current resistance to the direct-current resistance of a single-layer coil, the length of which is small with respect to its diameter.

Formula (17) gives the corresponding ratio for the inductance of the coil.

The following notation is used in the above formulas:

R = Alternating-current resistance.
R_0 = Direct-current resistance.
L = Alternating-current inductance.
L_0 = Direct-current inductance.
N = Total number of conductors in a plane or the number of turns on the coil.
a = Radius of conductor.
D = Distance between floors or the radius of the coil.
s = Spacing between centers of adjacent conductors.
$\lambda = \frac{\pi\,\omega\, a^2}{\sigma}$
ω = 2π times the frequency.
σ = Resistivity in electromagnetic cgs units.

$$U_{0/1}=\frac{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k}\,\underline{|2k+1}}}{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+1}\,\underline{|2k+1}}}=\frac{1+\frac{\lambda^2}{6}+\frac{\lambda^4}{480}+\cdots}{1+\frac{\lambda^2}{12}+\frac{\lambda^4}{1440}+\cdots}$$

$$V_{0/1}=\frac{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+1}\,\underline{|2k+2}}}{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+1}\,\underline{|2k+1}}}=\frac{1}{2}\,\frac{1+\frac{\lambda^2}{24}+\frac{\lambda^4}{4320}+\cdots}{1+\frac{\lambda^2}{12}+\frac{\lambda^4}{1440}+\cdots}$$

$$U_{0/2}=\frac{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+1}\,\underline{|2k+2}}}{\displaystyle 2\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+2}\,\underline{|2k+2}}}=\frac{1+\frac{\lambda^2}{24}+\frac{\lambda^4}{4320}+\cdots}{1+\frac{\lambda^2}{36}+\frac{\lambda^4}{8640}+\cdots}$$

$$V_{0/2}=\frac{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+1}\,\underline{|2k+3}}}{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+2}\,\underline{|2k+2}}}=\frac{2}{3}\,\frac{1+\frac{\lambda^2}{40}+\frac{\lambda^4}{10080}+\cdots}{1+\frac{\lambda^2}{36}+\frac{\lambda^4}{8640}+\cdots}$$

$$U_{1/3}=\frac{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+2}\,\underline{|2k+3}}}{\displaystyle 3\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+3}\,\underline{|2k+3}}}=\frac{1+\frac{\lambda^2}{60}+\frac{\lambda^4}{20160}+\cdots}{1+\frac{\lambda^2}{80}+\frac{\lambda^4}{33600}+\cdots}$$

$$V_{1/3}=\frac{\displaystyle\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+2}\,\underline{|2k+4}}}{\displaystyle 3\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{\underline{|k}\,\underline{|k+3}\,\underline{|2k+3}}}=\frac{1}{4}\,\frac{1+\frac{\lambda^2}{90}+\frac{\lambda^4}{40320}+\cdots}{1+\frac{\lambda^2}{80}+\frac{\lambda^4}{33600}+\cdots}$$

Miss C. M. Sparks has computed these U and V functions for values of λ^2 from 0 to 10. A table of these functions is given in the appendix.

$$G=4\,\lambda^2\,a^2\left[\frac{1}{s}\left(\tan^{-1}\frac{Ns}{2D}-\tan^{-1}\frac{s}{2D}\right)+\frac{.5}{D}\right]^2$$

$$K=\frac{\pi^2\lambda^2 a^4}{9\,s^2}\left[\frac{\pi^2}{6\,s^2}+\frac{4\,N}{N^2s^2+4\,D^2}-\frac{4}{s^2+4\,D^2}+\frac{1}{D^2}\right]$$

$$E=V_{0/2}+\frac{a^2\pi^2}{6\,s^2}-\frac{2\,a^2 N}{N^2s^2+4\,D^2}+\frac{2\,a^2}{s^2+4\,D^2}-\frac{.5\,a^2}{D^2}$$

$$F=V_{1/3}+\frac{\pi^4 a^4}{90\,s^4}$$

$$S=\log D+\frac{N}{2}\log\left(\frac{N^2}{4}+\frac{D^2}{s^2}\right)-\frac{1}{2}\log\left(\frac{1}{4}+\frac{D^2}{s^2}\right)$$
$$-N+1+\frac{2D}{s}\left(\tan^{-1}\frac{Ns}{2D}-\tan^{-1}\frac{s}{2D}\right)$$

$\log\left(\frac{N-1}{2}\right)!$ is approximately equal to

$$\frac{N}{2}\log\frac{N}{2}-\frac{N-3}{2}-\frac{3}{2}\log\frac{3}{2}=\frac{N}{2}\log\frac{N}{2}-\frac{N-3}{2}-0.60819$$

As an example in applying these formulas, the ratio R/R_0 is computed for 3,000 cycles for the smallest coil used.

$N=160=$ number of turns.

$a=0.259$ cm $=$ radius of wire.

$D=4.12$ cm $=$ distance between floors $=$ radius of coil.

$s=0.1$ cm $=$ spacing between centers of wire.

$\rho=1.72\times 10^3$ absolute units $=$ resistivity of copper.

$\omega=18{,}850=2\pi$ times the frequency.

$$\lambda=\frac{\pi\omega a^2}{\sigma}=2.29$$

From Table 6

$U_{0/1}=1.33$

$U_{0/2}=1.07$

$U_{1/3}=1.02$

Also by solving the equations on page 92

$G=8.56$

$K=0.326$

$E=0.960$

$F=0.285$

Substituting these values in equation (13)

$R/R_0=3.08$

To obtain the ratio L/L_0 for the same coil and at the same frequency, the following additional expressions must be obtained:

From Table 6

$V_{0/1}=0.420$

Solving the above equations.

$S=565.5$

$\log\left(\frac{N-1}{2}\right)!=273.8$

Substituting in equation (15)

$L/L_0=0.961.$

IV. EXPERIMENTAL MEASUREMENTS.

1. CONSTRUCTION OF COILS.

In order to compare values obtained by these formulas with experimental results, a few coils were constructed and their alternating-current resistance and inductances were measured at several frequencies.

Large wire was used in the construction of these coils in order to exaggerate the "proximity effects" on the resistance and inductance. The wire was wound on well-seasoned wooden cylinders. The cylinders were turned on a lathe and a groove cut for the wire by means of a milling tool. All the coils were wound with 160 turns of No. 4 (diameter 0.519 cm) bare copper wire, the pitch being 0.6 cm. Coils having the following mean diameters were constructed: 30.39, 22.62, 15.77, and 8.24 cm, respectively. The coil having a diameter of 15.77 cm was built in two sections, one section having 110 turns and the other having 50 turns. The sections were so constructed that they could be fitted together to form a coil of 160 turns and at the same time not have a discontinuity in the pitch.

2. METHOD OF MEASUREMENTS.

Figure 2 is a diagram of the bridge which was used in measuring the alternating-current resistance of the coils. R_1 and R_2 are equal noninductive resistances and were used as ratio arms of the bridge. L is the coil to be measured. L_2 is a variable inductor having a range equal to the coil L. R_4 is a variable resistance. R_3 is a variable resistance made up of a set of substitute links having a range of 0.1 of an ohm in steps of 0.01 ohm, and a slide [10] resistance having a range of 0.01 of an ohm. Both of these instruments were specially designed for work of this nature. Either can be read to 0.0001 of an ohm. L_1 is a variable inductor having a range of 0.1 mh. It is constructed of Litzendraht wire in the form of pancakes. The inductance is varied by sliding one coil over the other. When the two coils are separated one-half of an inch or more, the proximity effect is so small that the effective resistance of the coil is not changed by an appreciable amount while varying the inductance. The following procedure was followed in making the measurements of the alternating-current resistance of a coil:

With L_1 and R_3 set low and the potential terminal from the telephone at the end of the coil, L; L_2 and R_4 were varied until a

[10] C. N. Hickman, "A variable resistor of low value," Jour. of the Optical Soc. of Amer., October, 1922.

balance was obtained. The potential terminal was then moved along the coil by from 1 to 10 turns. This placed a part of the coil in the other arm, and if a balance was now obtained by varying R_3 and L_1, the change in R_3 would be twice as great as the effective resistance of that part of the coil which was added to the other arm. By a repetition of this process the effective resistance of the coil was measured turn by turn, or by any number of turns. The number of turns transferred from one arm to the other was limited by the range of the inductor L_1.

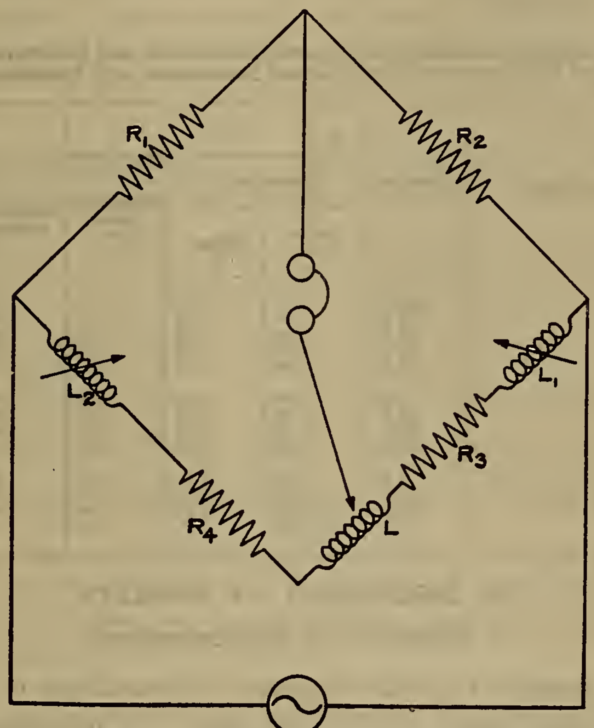

FIG. 2.—*Diagram of bridge used in measuring the alternating-current resistance of coils.*

The inductance was measured by means of Maxwell's bridge, the coil being placed in one arm of a Wheatstone bridge, the opposite arm being a resistance shunted by a capacitance.

V. COMPARISON OF EXPERIMENTAL AND CALCULATED RESULTS.

Table 2 gives a comparison of the measured and calculated results. All of these coils had 160 turns of No. 4 bare copper wire wound with a pitch of 0.6 cm. R/R_0 is the ratio of the alternating-current resistance to the direct-current resistance. L/L_o is the ratio of the alternating-current inductance to the direct-current inductance. In the first column of the measured values, R/R_o is the ratio of the effective alternating-current resistance of the middle wire of the coil to its direct-current resistance. In the second column, R/R_o is the ratio of the average of alternating-current resistances of all the turns to the direct-current resistance of a single turn. The calculated values are for the middle turn and for floors distant from each other by an amount equal to the radius of the coil.

TABLE 2.—Ratios of the Alternating-Current Resistances and Inductances of le-Sing Layer Coils to their Direct-Current Resistances and Inductances.

Frequency (cycles).	Diameter of coil.	R/R_o.			L/L_o.	
		Measured.		Calculated, center wire.	Measured, average.	Calculated, center wire.
		Center wire.	Average.			
	cm					
1,000	8.24	1.69	1.69	1.69	0.987	0.985
	15.8	1.63	1.59	1.63	.993	.993
	22.6	1.57	1.55	1.58	.996	.995
	30.4	1.56	1.54	1.53	....	....
2,000	8.24	2.50	2.50	2.50	.966	.969
	15.8	2.40	2.33	2.38	.985	.986
	22.6	2.30	2.28	2.28	.992	.990
	30.4	2.50	2.40	2.17	....	....
3,000	8.24	3.05	3.04	3.08	.964	.961
	15.8	2.94	2.88	2.93	.981	.982
	22.6	3.00	2.84	2.81	.989	.988
	30.4	3.05	2.75	2.68	....	....

VI. DISCUSSION OF RESULTS.

1. ACCURACY OF MEASUREMENTS.

It was possible to make resistance measurements of the small coil with an accuracy of 1 or 2 per cent. The inductance of the larger coils was so high that resistance measurements could not be made with a higher degree of accuracy than 4 or 5 per cent. The largest coil was the first one constructed and the methods of measurement were not so good as were used later for the other coils. In the meantime this coil became useless due to shrinkage of the wooden cylinder. No inductance measurements were made on this coil.

The copper wire was not very uniform in diameter and this dimension may be in error by 1 per cent or more. The other dimensions of the coil are probably correct to better than 1 per cent.

2. DISTRIBUTION OF EFFECTIVE RESISTANCE.

As has been explained, the resistance values of the coils were obtained by measuring the coils turn by turn. It was possible by this means to obtain the distribution of the effective resistance along the coil. Curve *A* of Figure 3 shows the effective alternating-current resistance of the largest coil turn by turn. The frequency was 3,000 cycles per second. This case was selected because the end effects were the most pronounced.

It is interesting to note that the effective resistance of the end turn is negative. Experimentally this means when this turn was transferred from one arm of the bridge to the other that the

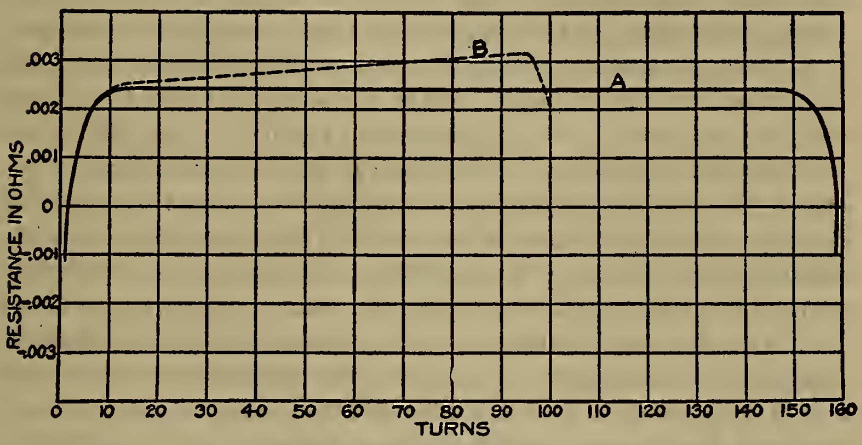

FIG. 3.—*Distribution of effective alternating-current resistance. Diameter of coil, 30.4 cm.; frequency, 3,000 cycles. No. 4 bare copper wire. Spacing 6 mm.*

A, Curve for effective alternating-current resistance when potential is applied across entire coil; *B*, curve for effective alternating-current resistance when potential is applied across only 100 turns, the other 60 turns being a "dead end."

resistance of that arm had to be decreased rather than increased as might have been expected. Physically this means that the induced electromotive force in this turn, which is in phase with the current, is greater than the counter electromotive force due to the ohmic resistance of the turn. It has been suggested that "apparent resistance" might be a better expression than "effective resistance." However, when the measured values of the turns are added, we have the real alternating-current resistance of the coil and not its apparent resistance so that if the expression "effective resistance" is properly interpreted there should be no confusion caused by its use.

Butterworth[11] in his paper on "Eddy current losses of cylindrical conductors," has shown mathematically that this apparent negative resistance at the ends of coils may exist and calls it the effective resistance of that portion of the coil. He shows that the energy loss is greatest at this portion of a coil but indicates that the energy is supplied by the other part of the coil and thus increases its effective resistance.

3. ASSUMPTIONS IN THE DEVELOPMENT OF THE FORMULAS.

It will be remembered that we have assumed that the effect of the curvature of the wire is negligible. In view of some special cases which have been worked out where the curvature was taken into consideration, it is believed that this assumption is justified.

It was also assumed that the current was distributed in the same manner in each wire. This is, of course, not true for a finite coil, but as shown by the experimental results the end effects are not large for a long coil. The losses in the end turns tend to increase the effective resistance of the middle turns but the effective resistance of the end turns is less, so that these two effects tend to neutralize each other. It should be remembered that while the current has been assumed to have the same distribution in all of the turns, this distribution is not the same as that for an infinitely long coil. For example, in computing the resistance of the largest coil, if a distribution had been assumed that exists for an infinitely long coil, the values would have been almost 50 per cent higher than those obtained. For an infinitely long coil the value of G would become

$$G=4\lambda^2a^2\left[\frac{.5}{D}+\frac{1}{s}\left(90°-\tan^{-1}\frac{s}{2D}\right)\right]^2$$

in place of

$$G=4\lambda^2a^2\left[\frac{.5}{D}+\frac{1}{s}\left(\tan^{-1}\frac{Ns}{2D}-\tan^{-1}\frac{s}{2D}\right)\right]^2$$

For the large coil $\tan^{-1}\dfrac{Ns}{2D}$ is about 71°. Thus for an infinitely long coil the value of G would be increased by about 50 per cent.

The only assumption that has been made in obtaining the inductance formula is that the change of inductance in per cent with alternating current is equal to the per cent change in the parallel floor systems.

[11] Philosophical Transactions, Royal Soc. of London, Series A, 222, pp. 57 to 100; 1921.

4. EDDY CURRENT LOSSES IN "DEAD-END" PORTION OF COILS.

Curve *B* in Figure 3 shows the effective distributed resistance in the large coil when the potential is applied across only 100 turns. The other 60 turns act as a "dead end" and their eddy current losses cause an increase in the effective resistance of the 100 turns.

The coil which was constructed in two sections was measured in the same manner and it was found that it made no difference whether the dead-end portion was electrically connected or not. However, if it were removed, the effective resistance of the remaining part would drop to a value such as would be expected for a coil of its dimensions. This coil being much smaller, the end effects were very much smaller. Even at 3,000 cycles the effective resistance of the end turns did not become negative, although it was less than the direct-current resistance.

5. IMPORTANCE OF THE VARIOUS TERMS OF THE FORMULAS.

The first terms of the resistance formula represents the "skin effect" or the ratios of the alternating-current to direct-current resistance if the wire were stretched out in a straight line. The second term represents the effect due to the field of the coil. The third term represents the "proximity effect" of the wires. This term is never very large and most of the effect is produced by the first four or five adjacent wires.

Table 3 is given in order to show the values of the separate terms for the coils which have been investigated.

TABLE 3.—Importance of the Various Terms Occurring in the Formulas.

Frequency (cycles).	Term.	R/R_o. Diameter of coils in cm. 8.2	15.8	22.6	30.4	$\Delta L/L$. Diameter of coils in cm. 8.2	15.8	22.6
1,000	1	1.05	1.05	1.05	1.05	0.0003	0.0002	0.0001
	2	.61	.55	.50	.45	.0153	.0070	.0045
	3	.03	.03	.03	.03	.0002	.0001	.0001
	Sum	1.69	1.63	1.58	1.53	.0158	.0073	.0047
2,000	1	1.17	1.17	1.17	1.17	.0011	.0005	.0004
	2	1.21	1.09	.99	.88	.0293	.0134	.0086
	3	.12	.12	.12	.12	.0009	.0004	.0003
	Sum	2.50	2.38	2.28	2.17	.0313	.0143	.0093
3,000	1	1.33	1.33	1.33	1.33	.0020	.0010	.0007
	2	1.52	1.37	1.25	1.12	.0355	.0162	.0105
	3	.23	.23	.23	.23	.0016	.0008	.0007
	Sum	3.08	2.93	2.81	2.68	.0391	.0180	.0119

6. CONVERGENCE OF THE SERIES.

The convergence of the U and V series is a function of the diameter of the wire, the frequency and the resistivity of the conductor.

For copper wire No. 2 A. W. G. (diameter 0.654 cm) these series are sufficiently convergent for frequencies as high as 3,000 cycles per second. For No. 12 wire (diameter 0.205 cm) the frequency may be as high as 30,000 and for No. 22 wire (diameter 0.0644 cm), 300,000, etc.

7. APPLICATION OF OTHER FORMULAS TO THE EXPERIMENTAL COILS.

The ratio of the alternating-current resistance to the direct-current resistance was computed for the experimental coils using the formulas of Sommerfeld and Butterworth. Table 4 gives these values, together with the measured values and those obtained by using the author's formula. According to Butterworth's formula the alternating-current resistance of a coil decreases very rapidly as the diameter is increased, all the other constants of coil remaining unchanged. This is not borne out by the measured values nor by the author's formula.

TABLE 4.—Table of Values Obtained With Other Formulas.

Frequency (cycles).	Diameter.	R/R_0.			
		Computed values.			Measured values.
		Sommerfeld.	Butterworth.	Hickman.	
	cm				
1,000	8.24	2.12	3.75	1.69	1.69
	15.8	2.12	2.45	1.63	1.59
	22.6	2.12	2.00	1.58	1.55
	30.4	2.12	1.92	1.53	1.54
2,000	8.24	5.45	7.49	2.50	2.50
	15.8	5.45	4.47	2.38	3.33
	22.6	5.45	3.42	2.28	2.28
	30.4	5.45	3.22	2.17	2.40
3,000	8.24	11.01	10.47	3.08	3.04
	15.08	11.01	6.09	2.93	2.88
	22.06	11.01	4.57	2.81	2.84
	30.4	11.01	4.28	2.68	2.75

This error is probably due to his assumption that the field from adjacent conductors is uniform over the cross section of the conductor under investigation.

On the other hand, Sommerfeld's results are independent of the diameter of the coil. This, together with the fact that his values are too large, is due in part to considering the coil infinitely long.

A summary of the contents of this paper is given at the beginning.

The author wishes to express his appreciation to Dr. Curtis and Dr. Snow for their assistance in the mathematical part of this paper, and to Miss Sparks for her assistance in making experimental measurements and computations.

VII. APPENDIX.

1. A SIMPLE FORMULA FOR THE MUTUAL INDUCTANCE BETWEEN COAXIAL CIRCLES OF THE SAME DIAMETER.

It has been shown that if the conductors of a return circuit are spaced so that the distance between them is 0.5413 times the diameter of a circular ring made of the same size conductor, the self-inductance per unit length of one of the conductors is equal to the self-inductance per unit length of the circular ring.

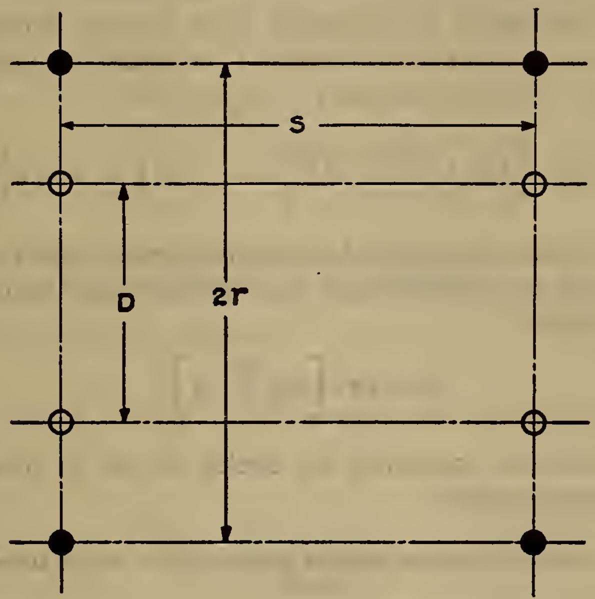

FIG. 4.—*Cross section of coaxial circles and return circuits.*

If two identical return circuits, whose planes are parallel, and distant from each other by s, are placed in a manner to correspond to two equal coaxial circles (see Fig. 4), the mutual inductance per unit length between the systems will be approximately equal to the mutual inductance per unit length of the coaxial circles, provided that the spacings between the wires of the return circuits are 0.5413 times the diameters of the coaxial circles.

But the mutual inductance per unit length between two very long return circuits placed as designated above is

$$M = 2\left[\log\frac{2}{s} - 1\right] - 2\left[\log\frac{2}{\sqrt{s^2+D^2}} - 1\right] = 2\left[\log\frac{\sqrt{s^2+D^2}}{s}\right]$$

Where s is the distance between the systems and D is the distance between the conductors of the return circuit.

Therefore, the mutual inductance between two coaxial circles having the same diameters is

$$M = 4\pi r\left[\log\frac{\sqrt{s^2+D^2}}{s}\right]$$

where r is the radius of the circles, s the distance between them and D is 0.5413 times the diameter of the circles, or 1.0826 times the radius r. This formula may then be written

$$M = 4\pi r\left[\log\frac{\sqrt{s^2+1.172\,r^2}}{s}\right] = 2\pi r\log\left(1+1.172\frac{r^2}{s^2}\right) \qquad (18)$$

Table 5 gives a comparison between the correct values and those obtained with this formula, and also those obtained by the simple Maxwell formula

$$M = 4\pi r\left[\log\frac{8r}{s} - 2\right]$$

The values are computed for circles 50 cm in diameter at various distances apart.

TABLE 5.—Mutual Induction Between Coaxial Circles Whose Diameters are 50 cm.

Spacing s.	Correct values.	Computed by Maxwell's formula.	Computed by formula (18).
1	1,036.67	1,036.0	1,036.4
4	606.07	600.7	604.1
16	216.96	165.0	212.3
20	167.09	95.0	163.5
25	122.5	25.0	121.9
30	93.2	−32.0	93.5
40	53.7	−122.0	59.3
50	35.5	−192.0	40.4

2. EVALUATION OF INTEGRALS.

The integrals occurring in this paper are of the same type as those appearing in Scientific Paper No. 374 and their evaluations are given in the appendix of that paper. They are repeated here for the convenience of those who wish to check the results of this paper.

The integral

$$I_1=\int_0^a \rho_0 d\rho_0 \int_0^{2\pi} \rho_0^m \{\log [\rho^2+\rho_0^2-2\rho\rho_0 \cos (\theta-\theta_0)]\} \cos n\theta_0 d\theta_0$$

becomes for $n>o$ and $n-m\neq 2$

$$I_1=2\pi \cos n\theta \left[\frac{2\rho^{m+2}}{(m+2)^2-n^2}-\frac{\rho^n a^{m-n+2}}{(m-n+2)\,n}\right]$$

for $n>o$ and $n-m=2$

$$I_1=\frac{2\pi\rho^n \cos n\theta}{n}\left[\log \rho-\log a-\frac{1}{2n}\right]$$

for $n=o$

$$I_1=\frac{4\pi a^{m+2} \log a}{m+2}-\frac{4\pi\,(a^{m+2}-\rho^{m+2})}{(m+2)^2}$$

For the second type of integral

$$I_2=\int_0^a \rho_r d\rho_r \int_0^{2\pi} \rho_r^m \{\log [q_r^2+\rho_r^2-2q_r\rho_r \cos (z_r-\theta_0)]\} \cos n\theta_r d\theta_r$$

When $n>o$

$$I_2=-\int_0^a \rho_r^{m+1}\left[\frac{2\pi}{n}\left(\frac{\rho_r}{q_r}\right)^n \cos nz_r\right] d\rho_r$$

When $n=o$

$$I_2=\int_0^a \rho_r^{m+1}\,(4\pi \log q_r)\, d\rho_r$$

The other integrals are of the same type.

3. EXPANSION OF LOG q_r, $\frac{\cos nz_r}{q_r}$, ETC.

The expansion of these terms may be made in a similar manner to the expansion of the similar expressions occurring in Scientific Paper No. 374.

It is found that

$$\log q_r + \log q_r = 2 \log rs + 2\left[\frac{\rho^2 \cos 2\theta}{2r^2s^2} - \frac{\rho^4 \cos 4\theta}{4r^4s^4} + \cdots\right]$$

$$\log Q_r + \log Q_r = 2 \log H_r + \frac{2D}{H_r^2}\rho \cos\theta - \frac{2D^2}{H_r^4}\rho^2 \cos 2\theta$$

$$+\frac{1}{H_r^2}\rho^2 \cos 2\theta + \cdots\cdots$$

$$\frac{\cos z_r}{q_r} + \frac{\cos z_r}{q_r} = \frac{2}{rs}\left[\frac{\rho\cos\theta}{rs} - \frac{\rho^3 \cos 3\theta}{r^3s^3} + \cdots\right]$$

$$\frac{\cos 2z_r}{q_r^2} + \frac{\cos 2z_r}{q_r^2} = -\frac{2}{r^2s^2}\left[1 - \frac{3\rho^2\cos 2\theta}{r^2s^2} + \cdots\right]$$

$$\frac{\cos 2Z_r}{Q_r} + \frac{\cos Z_r}{Q_r} = -\frac{2D}{H_r^2} - \frac{2}{H_r^2}\rho\cos\theta + \frac{4D^2\rho\cos\theta}{H_r^4} + \cdots\cdots\cdots$$

$$\frac{\cos 2Z_r}{Q_r^2} + \frac{\cos 2Z_r}{Q_r^2} = \frac{4D^2}{H_r^4} - \frac{2}{H_r^2} + \cdots\cdots$$

4. THE U AND V FUNCTIONS.

The following table gives the values of the U and V functions that occur in this paper. They are computed for various values of λ^2, being almost linear functions of this quantity.

TABLE 6.—Values of U and V Functions.

λ^2	$U_{0/1}$	$V_{0/1}$	$U_{0/2}$	$V_{0/2}$	$U_{1/2}$	$V_{1/2}$
0	1.00000	0.50000	1.00000	0.66667	1.00000	0.25000
1	1.07815	.48056	1.01362	.66484	1.00411	.24965
2	1.14730	.46357	1.02674	.66312	1.00820	.24931
3	1.20903	.44857	1.03938	.66144	1.01221	.24898
4	1.26464	.43524	1.05159	.65985	1.01616	.24866
5	1.31509	.42329	1.06338	.65830	1 02008	.24834
6	1.36117	.41250	1.07476	.65682	1.02391	.24802
7	1.40350	.40274	1.08579	.65540	1.02768	.24771
8	1.44262	.39382	1.09647	.65401	1.03141	.24741
9	1.47891	.38566	1.10681	.65268	1.03509	.24710
10	1.51275	.37816	1.11685	.65138	1.03872	.24681

WASHINGTON, September 26, 1922.

Correction Sheet for

Bureau of Standards Scientific Paper No. 472

Page 81, line 12. Bold face $\boldsymbol{d}$ should be light face d.

Page 81, line 26. Bold face $\boldsymbol{u}$ on line 3 of equation (4) should be light face u.

Page 85. Bold face $\boldsymbol{n}$ in first two equations should be light face n.

Page 85. In line ending " $=\lambda$" bold face $\boldsymbol{a}$ preceding " $=\lambda$" should be light face a.

Page 88. In line beginning "Σ log rs = " the first parenthetical expression should be an exponent of the s preceding it; viz:

$$\log s^{\frac{N-1}{2}}$$

Page 89. All bold face $\boldsymbol{n}$'s should be light face n's.

Page 89. In the line beginning "the above formula," the mathematical expression should be

$$U_{\frac{n-1}{n}}$$

In the next line, the expression should be

$$\lambda V_{\frac{n-1}{n}}$$

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