THE FIELD RADIATED FROM TWO HORIZONTAL COILS

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ABSTRACT

The landing of airplanes can be facilitated by emitting from the landing field a vertical beam of electromagnetic radiation of a radio frequency. A type of transmitting coil antenna has been devised for this purpose by J. A. Willoughby, of the Bureau of Standards. The behavior of this transmitting coil antenna is calculated in the following paper. Formulas are worked out for the current received in a coil aerial and in an antenna oriented in a known manner at a sufficient distance from the transmitter. It is found that for coil reception if—

I_s=sending current.
n_s=number of turns in either coil of the transmitter.
h=distance between the planes of the two coils of the transmitter.
M_s=area of either transmitting coil.
l=distance from transmitter to receiver.
θ=angle made by the length l with the vertical.
ϕ=angle made by plane of receiving coil with length l.
λ=wave length used.
R=resistance of receiving coil.
n_r=number of turns in receiving coil.
M_r=area of receiving coil.
I_r=received current.

Then

$$I_r = 4.675 \times 10^4 \frac{I_s\, n_s\, n_r\, M_s\, M_r\, h_s \sin\theta \cos\theta \cos\phi}{\lambda^4\, l\, R}$$

and for open antenna reception

$$I_r = \frac{7.44 \times 10^3\, I_s\, n_s\, M_s\, h_s\, h_r \sin\theta \cos\theta \cos\alpha}{\lambda^3\, R\, l}$$

where h_r=length of antenna.
α=angle made by antenna with normal to plane of l and vertical.
R=resistance of antenna.

It is found that if a vertical coil aerial is used for reception and if the airplane flies horizontally a maximum signal is obtained when the distance l makes an angle of 30° with the vertical. The influence of the earth is also discussed for the case when the earth may be considered as a perfect conductor. In both cases it was found that the energy was sent out by the transmitter into a region which is very similar to what would be obtained by constructing two coaxial circular cones having a common apex and a vertical axis, the cones themselves being inverted and cut off at a proper height (perhaps a mile or more, depending on the sensitivity of the receiving apparatus).

CONTENTS

I. INTRODUCTION

The usefulness of a beam of radio waves directed upward from an airplane landing field for assisting an airplane to locate the landing area in times of poor visibility is apparent. During the course of some experimental work along this line J. A. Willoughby, then laboratory assistant at the Bureau of Standards, suggested the use of a special type of antenna for this purpose. The theory of the radiation from such an antenna is given in the present paper.

Mr. Willoughby proposed the use of an antenna consisting of two horizontal coils, one placed somewhat above the other, the current in one coil flowing in the opposite direction to that of the current in the other. The current was intended to be of a fairly high radio frequency, such as is ordinarily employed in radio communication with airplanes, so that signals could be received with an ordinary radio receiving set.

In general, it was believed that the waves would radiate in the form of a beam or inverted cone above the antenna. Experiment showed this to be true, signals being received on the airplane only when it was nearly above and in the immediate vicinity of the transmitting antenna.

To determine from theoretical considerations the nature of the radiated field and the region over which signals can be heard at various heights above such an aerial, the following calculation was made. The calculation justifies the belief that the radiation would form a beam in the form of an inverted cone above the transmitting antenna. The results of the calculation are expressed in inverse powers of the distance from the transmitting antenna. The lowest power gives the radiation component of the field. The next higher power may be looked at as a first approximation to the induction component. In the final formulas it is always assumed that the dimensions of the transmitting coils and the distance between them are small in comparison with the wave length used. However, it has been made sure by additional computations that the directional properties of the radiation do not depend on this approximation within a wide range.

II. CALCULATION OF FIELD INTENSITY

The principle of the calculation is based on the solution of Lorentz of the general problem of finding the electric field intensity **E** and the magnetic field intensity **H** at a given point if the distribution of charges and currents is known. The fundamental equations of the electromagnetic field are:

$$\text{curl } \mathbf{H} = \frac{4\pi}{c}\left(\rho\mathbf{v} + \frac{K}{4\pi}\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\text{curl } \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\text{div } \mathbf{E} = \frac{4\pi\rho}{K}$$

$$\text{div } \mathbf{H} = O$$

where ρ is the volume density of charge, K the dielectric constant, μ the permeability, t the time, $\mathbf{E}$ the electric intensity, $\mathbf{H}$ the magnetic intensity, c a constant depending upon the choice of units, and $\mathbf{v}$ is the velocity of the charge.[1]

A solution of this system of equations is

$$\mathbf{H} = \text{curl } \mathbf{A}$$

$$\mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{A}}{\partial t} - \Phi$$

where

$$\mathbf{A} = \frac{1}{c}\int \frac{\{\rho\mathbf{v}\}}{r}\,d\tau$$

$$\Phi = \frac{1}{K}\int \frac{\{\rho\}}{r}\,d\tau$$

the symbol $\{\psi\}$ denoting the value of ψ at the element of volume $d\tau$ at the time $t - \frac{r\sqrt{K\mu}}{c}$, r being the distance from the element of volume $d\tau$ to the point at which $\mathbf{A}$ or Φ is computed. The integrals are to be extended throughout all space. A surface distribution of charge is only a special case of volume distribution. Thus, the above solution for volume distribution applies in all cases.

[1] In the Gaussian system in this notation $K=1$ and $\mu=1$ for air, and c is numerically equal to the velocity of light (approximately 3×10^{10}). For the electromagnetic system $\mu=1$, $K=\frac{1}{(\text{velocity of light})^2}$ for air, and $c=1$. On the electrostatic system $c=1$, and for air $K=1$, $\mu=\frac{1}{(\text{velocity of light})^2}$. Results in the international system of electric and magnetic units can not be obtained from this set of equations.

Out of the infinite variety of shapes which the coils of the transmitting aerial which is under consideration can have there are two most easily amenable to analysis. These are: (1) Circular coils, (2) rectangular coils. The first has the advantage of symmetry; the second has the advantage of easy practical construction and of the possibility of obtaining particular solutions by elementary means, which also hold to the first order of magnitudes for the general case.

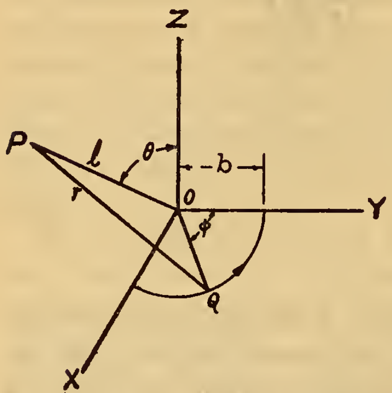

FIG. 1.—*Reference system used in discussing radiation from single circular coil*

1. CIRCULAR COILS

Consider a single circular coil of radius b in the plane of XY, with its center at the origin of coordinates. (See Fig. 1.) Let the dielectric constant of the space around the coil be K.

Let P be any point in the XZ plane,

Q any point on the circle.

Also denote

$\angle QOY$ by ϕ

$\angle POZ$ by θ

OP by l

PQ by r

If the current in the coil is in the sense indicated by the arrow, and if its amount is $I \cos \omega t$ at all points in the wire, then the components of the vector potential **A** at P along OX, OY, OZ are

$$\mathrm{A_x} = -\frac{I}{c}\int_0^{2\pi} \frac{b \cos \omega\left(t - \frac{r}{V}\right) \cos \phi \, d\phi}{r} = 0$$

$$\mathrm{A_y} = \frac{I}{c}\int_0^{2\pi} \frac{b \cos \omega\left(t - \frac{r}{V}\right) \sin \phi \, d\phi}{r}$$

$$\mathrm{A_z} = 0$$

because the only currents which exist are horizontal, and therefore $\mathrm{A_z} = 0$; $\mathrm{A_x}$ is also zero, as can be seen from considerations of symmetry. It is sufficient, therefore, to consider only the expression for A_y.

The integral in this expression can be written as a power series in $\frac{1}{l}$ for large values of l. From the triangle PQO we have

$$r=\sqrt{l^2+b^2-2bl\sin\theta\sin\phi}$$

$$r=l\left[1-\frac{b}{l}\sin\theta\sin\phi+\frac{b^2}{2l^2}(1-\sin^2\theta\sin^2\phi)\cdots\cdots\right]$$

and

$$\frac{1}{r}=\frac{1}{l}\left[1+\frac{b}{l}\sin\theta\sin\phi-\frac{b^2}{2l^2}(1-3\sin^2\theta\sin^2\phi)\cdots\right]$$

Consequently, $\cos\omega\left(t-\frac{r}{V}\right)=$

$$\cos\left[\omega\left(t-\frac{l}{V}+\frac{\omega b}{V}\sin\theta\sin\phi-\frac{\omega b^2}{2Vl}(1-\sin^2\theta\sin^2\phi)\right)\cdots\right]$$

Neglecting the terms $\frac{b}{2l}(1-\sin^2\theta\sin^2\phi)$ and $\frac{b}{2l^3}(1-3\sin^2\theta\sin^2\phi)$

$$A_y=\frac{I\cos\omega\left(t-\frac{l}{V}\right)}{c}\int_0^{2\pi}b\cos\left(\frac{\omega b\sin\theta\sin\phi}{V}\right)\sin\phi\left[\frac{1}{l}+\frac{b}{l^2}\sin\theta\sin\phi\right]d\phi$$

$$-\frac{I\sin\omega\left(t-\frac{l}{V}\right)}{c}\int_0^{2\pi}b\sin\left(\frac{\omega b\sin\theta\sin\phi}{V}\right)\sin\phi\left[\frac{1}{l}+\frac{b}{l^2}\sin\theta\sin\phi\right]d\phi$$

But

$$\int_0^{2\pi}\cos\left(\frac{\omega b\sin\theta\sin\phi}{V}\right)\sin\phi\,d\phi=0$$

$$\int_0^{2\pi}\cos\left(\frac{\omega b\sin\theta\sin\phi}{V}\right)\sin^2\phi\,d\phi$$

$$=\frac{1}{2}\int_0\cos\left(\frac{\omega b\sin\theta\sin\phi}{V}\right)(1-\cos 2\phi)\,d\phi$$

$$=\pi\left\{J_0\left(\frac{\omega b\sin\theta}{V}\right)-J_2\left(\frac{\omega b\sin\theta}{V}\right)\right\}$$

$$=2\pi J_1'\left(\frac{\omega b\sin\theta}{V}\right)$$

where J_n denotes the Bessel function of the first kind of order n and J_n' is the first derivative of that function as to its argument.

Also

$$\int_0^{2\pi} \sin\left(\frac{\omega b \sin\theta \sin\phi}{V}\right) \sin\phi \, d\phi = 2\pi J_1\left(\frac{\omega b \sin\theta}{V}\right)$$

while

$$\int_0^{2\pi} \sin\left(\frac{\omega b \sin\theta \sin\phi}{V}\right) \sin^2\phi \, d\phi = 0$$

Therefore

$$A_y = \frac{2\pi I\, b^2 \sin\theta}{l^2 c} J_1'\left(\frac{\omega b \sin\theta}{V}\right) \cos\omega\left(t - \frac{l}{V}\right) - \frac{2\pi I\, b}{lc} J_1\left(\frac{\omega b \sin\theta}{V}\right) \sin\omega\left(t - \frac{l}{V}\right)$$

Since Φ is constant

$$E_y = -\frac{\mu}{c}\frac{\partial A_y}{\partial t} = \frac{2\pi\mu\, I\, b^2\omega \sin\theta}{l^2c^2} J_1'\left(\frac{\omega b \sin\theta}{V}\right) \sin\omega\left(t - \frac{l}{V}\right) + \frac{2\pi\mu I b\omega}{lc^2} J_1\left(\frac{\omega b \sin\theta}{V}\right) \cos\omega\left(t - \frac{l}{V}\right) \quad (1)$$

As l increases the second term becomes much more important than the first. For sufficiently large values of l it is the only term of importance. For this reason it can be called the radiation component of **E**. It must be remembered here that although attention was confined to one special point in the OXZ plane the treatment does not lose any generality on that account, because the OXZ plane can always be made to pass through any required point. The whole expression gives the electric intensity associated with a single circular current correctly to terms involving $\frac{1}{l^2}$. For points very close to the transmitting antenna it breaks down, because the terms involving $\frac{1}{l^3}, \frac{1}{l^4}$, etc., should be taken into account. The expression can be used to calculate the effect of two circular currents by superposing the effects due to each individually.

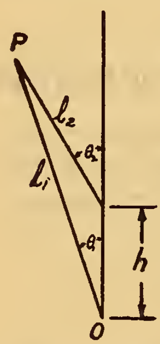

FIG. 2.—*Angles and distances used in the case of two circular coils*

If the current in the two coils is $I \cos \omega t$ at any instant, and if the quantities l, θ have values l_1, θ_1 for the first coil and l_2, θ_2 for the second coil, then the resultant value of E_y is

$$E_y = \frac{2\pi\mu\omega b^2}{c^2} I \left\{ \frac{\sin \omega \left(t - \frac{l_1}{V}\right)}{l_1^2} J_1' \left(\frac{\omega b \sin \theta_1}{V}\right) \sin \theta_1 - \frac{\sin \omega \left(t - \frac{l_2}{V}\right)}{l_2^2} J_1' \left(\frac{\omega b \sin \theta_2}{V}\right) \sin \theta_2 \right\} + \frac{2\pi\mu\omega b\, I}{c^2} \left\{ \frac{\cos \omega \left(t - \frac{l_1}{V}\right)}{l_1} J_1 \left(\frac{\omega b \sin \theta_1}{V}\right) - \frac{\cos \omega \left(t - \frac{l_2}{V}\right)}{l_2} J_1 \left(\frac{\omega b \sin \theta_2}{V}\right) \right\}$$

This is a general expression, which is correct if terms involving $\frac{1}{l^3}$ can be neglected. For special cases it can be simplified. For example, if the centers of the two circles are both on OZ, if their planes are parallel, and if the distance between their planes is h (see Fig. 2), then

$$l_1 = l_2 + h \cos \theta \quad \cdot \quad \cdot \quad \cdot$$

and $\sin \theta_1 = \frac{l_2}{l_1} \sin \theta_2 = (1 - \frac{h}{l} \cos \theta) \sin \theta$

θ being approximately the mean of θ_1 and θ_2. Therefore, neglecting powers of h higher than the first,

$$\frac{\sin \theta_1}{l_1^2} \sin \omega\left(t - \frac{l_1}{V}\right) J_1'\left(\frac{\omega b \sin \theta_1}{V}\right) - \frac{\sin \theta_2}{l_2^2} \sin \omega\left(t - \frac{l_2}{V}\right) J_1'\left(\frac{\omega b \sin \theta_2}{V}\right)$$

$$= h \cos \theta \frac{\partial}{\partial l} \left\{ \frac{\sin \theta}{l^2} J_1'\left(\frac{\omega b \sin \theta}{V}\right) \sin \omega\left(t - \frac{l}{V}\right) \right\} - \frac{h \sin \theta \cos \theta}{l} \frac{\partial}{\partial (\sin \theta)} \left\{ \frac{\sin \theta}{l^2} J_1'\left(\frac{\omega b \sin \theta}{V}\right) \sin \omega\left(t - \frac{l}{V}\right) \right\}$$

and
$$\frac{\cos \omega\left(t - \frac{l_1}{V}\right)}{l_1} J_1\left(\frac{\omega b \sin \theta}{V}\right) - \frac{\cos \omega\left(t - \frac{l_2}{V}\right)}{l_2} J_1\left(\frac{\omega b \sin \theta_2}{V}\right)$$

$$= \left[h \cos \theta \frac{\partial}{\partial l} - \frac{h \sin \theta \cos \theta}{l} \frac{\partial}{\partial (\sin \theta)} \right] \left\{ \frac{\cos \omega\left(t - \frac{l}{V}\right)}{l} J_1\left(\frac{\omega b \sin \theta}{V}\right) \right\}$$

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Performing the differentiations and substituting

$$E_y = \frac{2\pi\mu I b\omega^2 h \cos\theta}{Vc^2} J_1\left(\frac{\omega b \sin\theta}{V}\right) \frac{\sin\omega\left(t-\frac{l}{V}\right)}{l}$$

$$-\frac{2\pi\mu I b\omega h \cos\theta}{l^2 c^2}\left\{J_1\left(\frac{\omega b\sin\theta}{V}\right)+2\frac{b\omega\sin\theta}{V}J_1'\left(\frac{\omega b\sin\theta}{V}\right)\right\}\cos\omega\left(t-\frac{l}{V}\right)$$

If, in addition, $\frac{\omega b}{V}$ is small compared to unity, $J_1\left(\frac{\omega b \sin\theta}{V}\right)$ can be set equal to $\frac{\omega b\sin\theta}{2V}$, while $J_1'\left(\frac{\omega b\sin\theta}{V}\right)$ can be replaced by $\frac{1}{2}$.

Therefore, if h and b are both sufficiently small

$$E_y = \frac{\pi\mu I b^2\omega^3 h \sin\theta\cos\theta}{V^2 c^2}\;\frac{\sin\omega\left(t-\frac{l}{V}\right)}{l}$$

$$-\frac{3\pi\mu I b^2\omega^2 h\sin\theta\cos\theta}{Vc^2}\;\frac{\cos\omega\left(t-\frac{l}{V}\right)}{l^2}$$

$$E_y = \frac{M\mu I h \sin\theta\cos\theta}{c^2}\left[\frac{\omega^3\sin\omega\left(t-\frac{l}{V}\right)}{lV^2}-\frac{3\omega^2\cos\omega\left(t-\frac{l}{V}\right)}{l^2 V}\right] \cdots (2)$$

where $M = \pi b^2 =$ the area of either loop.

The expression for E_y given in equation (2) enables one to compute the electric field intensity at any point whose distance from the coils is large compared with their radii and their distance apart. This is obtained in terms of the current, its frequency, the distance from the coils, the velocity of light, and the angle between the line joining the point to the coils and the vertical line through the coils.

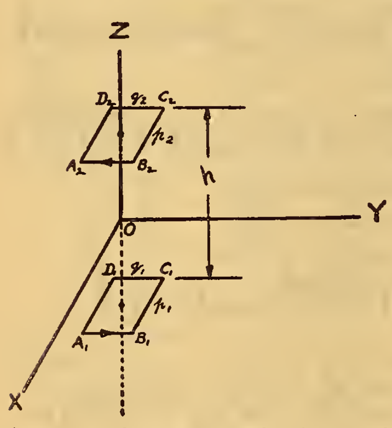

FIG. 3.—*Reference system for the case of two rectangular coils*

2. RECTANGULAR COILS

Let the sides of the two coils be parallel to each other, the lengths of the sides of one being p_1, q_1, and the lengths of the corresponding sides of the other being p_2, q_2. Let the centers of the two coils be on a line perpendicular to the planes of the coils, the planes themselves being parallel and a distance h apart. (See Fig. 3.)

Let the origin of coordinates be at a point midway between the planes of the two coils, the axis of X being parallel to the sides p_1, p_2, the axis of Y parallel to q_1, q_2, and the axis of Z being drawn in such a sense as to make the system right-handed. Let the currents in the two coils in the directions indicated by the arrowheads on Fig. 2 be $I \cos \omega t$. Consider the effect at a point (X, O, Z) in the OXZ plane. As before, $\mathsf{A}_x = \mathsf{A}_z = 0$. In computing A_y it is to be noted that the distance from (X, O, Z) to A_2B_2 is

$$\sqrt{\left(X-\frac{p_2}{2}\right)^2+\left(Z-\frac{h}{2}\right)^2}=l-\frac{p_2X}{2l}-\frac{hZ}{2l}$$

where l is the distance of (X, O, Z) from the origin and the expression for the distance does not take into account terms of higher order than the first in p_2 and h. Similarly, the distances from the same point to C_2D_2 is

$$l+\frac{p_2X}{2l}-\frac{hZ}{2l},$$

to A_1B_1 is

$$l-\frac{p_1X}{2l}+\frac{hZ}{2l},$$

and to C_1D_1 is

$$l+\frac{p_1X}{2l}+\frac{hZ}{2l}.$$

Thus, since the contributions of the sides $A_1 D_1$, $B_1 C_1$, $A_2 D_2$, $B_2 C_2$ to the vector potential at (X, O, Z) are equal and opposite, and since the distance of any point in $A_1 B_1$ from (X, O, Z) is to the first power of q the same as the distance from $A_1 B_1$ to (X, O, Z)

$$\begin{aligned}\mathsf{A}_y = &\frac{q_1 I \cos \omega\left(t-\frac{l}{V}+\frac{p_1X}{2lV}-\frac{hZ}{2lV}\right)}{c\left(l-\frac{p_1X}{2l}+\frac{hZ}{2l}\right)}\\ &-\frac{q_2 I \cos \omega\left(t-\frac{l}{V}+\frac{p_2X}{2lV}+\frac{hZ}{2lV}\right)}{c\left(l+\frac{p_2X}{2l}-\frac{hZ}{2l}\right)}\\ &+\frac{q_2 I \cos \omega\left(t-\frac{l}{V}-\frac{p_2X}{2lV}+\frac{hZ}{2lV}\right)}{c\left(l+\frac{p_2X}{2l}-\frac{hZ}{2l}\right)}\\ &-\frac{q_1 I \cos \omega\left(t-\frac{l}{V}-\frac{p_1X}{2lV}-\frac{hZ}{2lV}\right)}{c\left(l+\frac{p_1X}{2l}+\frac{hZ}{2l}\right)}\end{aligned} \qquad (3)$$

If we are interested in the radiation component only, it is justifiable to write the denominators of all the foregoing fractions as l, because

$$\frac{1}{l+\delta}=\frac{1}{l}-\frac{\delta}{l^2}+\text{, etc., where } \delta<l.$$

Thus, $$A_y=\frac{-2q_1I}{cl}\sin\omega\left(t-\frac{l}{V}-\frac{hZ}{2lV}\right)\sin\frac{\omega p_1X}{2lV}$$
$$+\frac{2q_2I}{cl}\sin\omega\left(t-\frac{l}{V}+\frac{hZ}{2lV}\right)\sin\frac{\omega p_2X}{2lV}$$

Now let $\frac{\omega p_1}{V}$ and $\frac{\omega p_2}{V}$ be small compared to unity. Then, since $\frac{X}{l}<1$ we can write as an approximation

$$A_y=\frac{\omega}{clV}\frac{X}{l}\left[p_2\,q_2\sin\omega\left(t-\frac{l}{V}+\frac{hZ}{2lV}\right)-p_1\,q_1\sin\omega\left(t-\frac{l}{V}-\frac{hZ}{2lV}\right)\right]$$

If now $p_1\,q_1=p_2\,q_2=M$

$$A_y=2\frac{\omega M}{clV}\frac{X}{l}\cos\omega\left(t-\frac{l}{V}\right)\sin\left(\frac{hZ\omega}{2lV}\right)$$

And, again, if $\frac{\omega h}{V}$ is small compared to unity then since $\frac{Z}{l}<1$

$$A_y=I\frac{\omega^2Mh}{cV^2l}\frac{X}{l}\frac{Z}{l}\cos\omega\left(t-\frac{l}{V}\right)$$

Therefore

$$E_y=-\frac{\mu}{c}\frac{\partial A_y}{\partial t}=I\frac{\omega^3\mu Mh}{c^2V^2l}\frac{X}{l}\frac{Z}{l}\sin\omega\left(t-\frac{l}{V}\right)$$

$$E_y=I\frac{\omega^3\mu Mh}{c^2V^2l}\sin\theta\cos\theta\sin\omega\left(t-\frac{l}{V}\right) \tag{4}$$

in the notation of the first case. To the first power of $\frac{1}{l}$, this is precisely the same as formula (2) of the first case. This shows that for sufficiently low frequencies the shape of the coils is immaterial as long as their area is kept the same. Moreover, if in the expression (4) we carry out the calculation to the second power of $\frac{1}{l^2}$ we get in addition to the terms so far obtained in A_y

$$\left(\frac{p_1X}{2l^3}-\frac{hZ}{2l^3}\right)\frac{q_1}{c}I\cos\omega\left(t-\frac{l}{V}+\frac{p_1X}{2lV}-\frac{hZ}{2lV}\right)$$

$$-\left(\frac{p_2X}{2l^3}+\frac{hZ}{2l^3}\right)\frac{q_2}{c}I\cos\omega\left(t-\frac{l}{V}+\frac{p_2X}{2lV}+\frac{hZ}{2lV}\right)$$

$$+\left(\frac{hZ}{2l^3}-\frac{p_2X}{2l^3}\right)\frac{q_2}{c}I\cos\omega\left(t-\frac{l}{V}-\frac{p_2X}{2lV}+\frac{hZ}{2lV}\right)$$

$$+\left(\frac{p_1X}{2l^3}+\frac{hZ}{2l^3}\right)\frac{q_1}{c}I\cos\omega\left(t-\frac{l}{V}-\frac{p_1X}{2lV}-\frac{hZ}{2lV}\right)$$

which after a few reductions gives on replacing the sines of small quantities by the quantities themselves and the cosines by 1

$$\frac{3Mh\omega}{l^2cV}\frac{Z}{l}\frac{X}{l}I\sin\omega\left(t-\frac{l}{V}\right)$$

which contributes to E_y

$$\frac{-3Mh\omega^2\mu}{l^2c^2V}I\cos\theta\sin\theta\cos\omega\left(t-\frac{l}{V}\right)$$

Thus, it is seen that this also agrees with formula (2) of the first case. Therefore, the value of E_y is the same for both circular and rectangular coils within terms in the second power of $\frac{1}{l}$. It may be well to recall again the conditions under which these first terms may be taken as approximating the actual value of E_y. These are:

1. $\frac{\omega h}{V}$ is small compared to unity. This amounts to saying that the wave length is large compared to the distance between the coils.

2. $\frac{\omega q}{V}, \frac{\omega p_1}{V}, \frac{\omega p_2}{V}$ are small compared to unity. This means that the wave length is large compared to the dimensions of each coil.

3. The distance of the airplane from the coils is large compared to the dimensions of the coils and their distance apart.

The same general method will also apply to the calculation of the field radiated by any number of coils.

III. APPLICATIONS

1. REGION OF AUDIBLE SIGNAL

If an airplane is flying over the two horizontal coils, the strength of the signal received on the airplane depends on the type of receiving apparatus and upon the orientation of the aerial. A type commonly used consists of a vertical coil aerial fixed rigidly to the frame of the machine.

For simplicity assume that the airplane is so far from the transmitter that $\frac{\omega}{V}$ is large compared to $\frac{1}{l}$.

Then

$$\mathrm{E}_y = \frac{M\mu Ih \sin\theta \cos\theta}{c^2} \cdot \frac{\omega^3 \sin\omega\left(t-\frac{l}{V}\right)}{l\,V^2}$$

Let there be a receiving coil placed at P with its plane parallel to OY and let the angle which its plane makes with the plane POY be ϕ. Then the emf induced in the coil is

$$n_r M_r \frac{M\mu Ih \sin\theta \cos\theta}{c^2} \cdot \frac{\omega^4 \cos\omega\left(t-\frac{l}{V}\right)}{lV^3} \cos\phi$$

where M_r is the area of the coil, n_r is the number of turns.

The units in which this result is given depend on the choice of system of units, as explained at the beginning of this paper. The result is given in volts by

$$4.675 \times 10^4 \frac{n_r M_r M I h \sin\theta \cos\theta \cos\phi}{l\,\lambda^4} \cos\omega\left(t-\frac{l}{V}\right)$$

in which I is in amperes, $\lambda = \frac{2\pi V}{\omega}$, and the lengths are in any unit, provided the same unit is used for all including λ and the dimensions entering into M_r and M.

If I, ω, and the constants of the two circuits are unchanged, this expression is proportional to $\frac{\sin\theta \cos\theta \cos\phi}{l}$

If $\phi = 0$, i. e., if the receiving coil is kept vertical, this number is $\psi = \frac{\sin\theta \cos^2\theta}{l}$

Consider now the variations in ψ as the position of P is varied. The variation most likely to occur in practice is a displacement of P parallel to the planes of the two coils; i. e., along OX. Mathematically this amounts to making Z a constant and since $r = \frac{Z}{\cos\theta}$

$$\psi = \frac{1}{Z} \sin\theta \cos^3\theta$$

This attains a maximum when $\cos^4\theta - 3\cos^2\theta \sin^2\theta = 0$, i. e., when $\left(\text{neglecting the minimum corresponding to } \theta = \frac{\pi}{2}\right)$

$$\sin\theta = \frac{1}{2}, \quad \text{or } \theta = 30°.$$

Thus, if the airplane moves horizontally with the receiving coil vertical the signal is loudest when the airplane subtends at the transmitter an angle of 30° with the vertical.

In terms of the coordinates X and Z

$$\psi = \frac{XZ^2}{[Z^2+X^2]^2} = \frac{\frac{X}{Z}}{Z\left[1+\frac{X^2}{Z^2}\right]^2} = \frac{1}{Z}\frac{\xi}{(1+\xi^2)^2} \text{ where } \xi = \frac{X}{Z}$$

Fig. 4 shows the graph of the function $\frac{\xi}{(1+\xi^2)^2}$ plotted against $\xi = \frac{X}{Z}\cdot$ It represents the variation of the electromotive force with the distance traversed by the airplane on a scale which would correspond to a unit distance of the airplane above the ground.

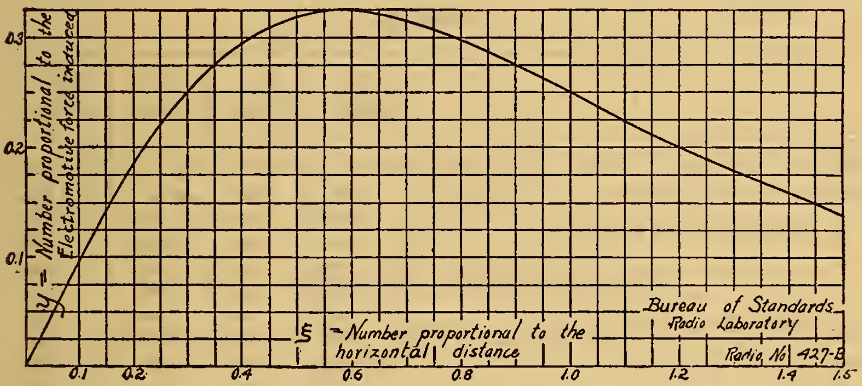

Fig. 4.—*Relation between electromotive force induced in a vertical coil and the distance traversed by the airplane in horizontal flight*

This curve can be used to find the surface within which the signal is audible and outside of which it is not audible. In fact, for points on the surface, $\frac{1}{Z}\frac{\xi}{(1+\xi^2)^2}$ must have a definite value K. Let a given value be assigned to $\frac{\xi}{(1+\xi^2)^2}$ and the two corresponding values of ξ be located on the curve. Let these be ξ_1 and ξ_2. Since $Z = \frac{1}{K}\frac{\xi}{(1+\xi^2)^2}$ this gives at once the values of X corresponding to ξ_1 and ξ_2, which are $\frac{\xi_1}{K}\frac{\xi}{(1+\xi^2)^2}$ and $\frac{\xi_2}{K}\frac{\xi}{(1+\xi^2)^2}$. If it is desired merely to determine the shape of the resulting surface, we can set $K=1$ and plot $\frac{\xi_1\xi}{(1+\xi^2)^2}$, and $\frac{\xi_2\xi}{(1+\xi^2)^2}$ against $\frac{\xi}{(1+\xi^2)^2}$

where ξ_1 and ξ_2 are values of ξ corresponding to $\frac{\xi}{(1+\xi^2)^2}$. Looked on in a different light this means that the curve can be traced by the parametric equations

$$Z=\frac{\xi}{(1+\xi^2)^2}, \quad X=\frac{\xi^2}{(1+\xi^2)^2}.$$

The properties of the curve can thus be easily traced from these expressions. Thus, if $\xi=0$, both Z and X are zero and, moreover, X is infinitesimal compared to Z. Consequently, here the curve is tangent to the axis of Z. Again, when $\xi=\infty$, $Z=X=0$ but now Z is infinitesimal compared to X, and hence the curve touches the axis of X. The curve is shown as curve *a* in Fig. 5.

Table 1 gives values of X and Z corresponding to a series of values of ξ.

TABLE 1

ξ	$Z=\frac{\xi}{(1+\xi^2)^2}$	$X=\frac{\xi^2}{(1+\xi^2)^2}$	$Z_1=\frac{\xi}{(1+\xi^2)^{2.5}}$	$X_1=\frac{\xi^2}{(1+\xi^2)^{2.5}}$
0	0	0	0	0
0.1	0.098	0.098	0.098	0.0098
.2	.1849	.03699	.1812	.3629
.3	.253	.0756	.2447	.0724
.4	.2972	.1189	.2761	.1105
.5	.3202	.1601	.2891	.1446
.6	.3242	.1944	.2779	.1671
.7	.3150	.2203	.2581	.1807
.8	.2973	.2379	.2321	.1859
.9	.2748	.2471	.2043	.1839
1.0	.2500	.2500	.1769	.1769
1.1	.2252	.2476	.1516	.1666
1.2	.2015	.2419	.1290	.1548
1.3	.1796	.2337	.1095	.1425
1.5	.1421	.2131	.0789	.1183
2.0	.0800	.1599	.0358	.0716
3.0	.03	.09	.00950	.02849
4.0	.0137	.0554	.003323	.01344

If the earth were a perfect conductor, its effect could be replaced by the effect of the images of the two coils in the surface of the earth. The whole transmitting system can then be replaced by four coils, the two on the inside carrying currents in the same direction and the two on the outside carrying currents in a direction opposite to that of the inside coils.

In this case the emf induced can be shown to be proportional to $\frac{\sin\theta\cos^4\theta}{Z}$. There is, therefore, a maximum signal for horizontal flight with a vertical receiving coil when

$$\cos^5\theta - 4\sin^2\theta\cos^3\theta = 0$$

or when

$$\sin\theta = \frac{1}{\sqrt{5}},\ \theta = 26°\ 34'$$

In order to trace the volume within which the signal is just audible, we must trace the curve

$$\frac{XZ^3}{r^5} = K, \text{ which becomes on letting } X = \xi Z$$

$$X = \frac{1}{K}\frac{\xi^2}{(1+\xi^2)^{2.5}}$$

$$Z = \frac{1}{K}\frac{\xi}{(1+\xi^2)^{2.5}}.$$

The surface is obtained by revolving this curve about OZ and is shown as curve b in Fig. 5. The general properties of the curve are the same as those for the case when the earth is negligible.

2. CURRENT IN RECEIVING COIL

In international electric units the current received in the coil is, within certain limitations, depending on the properties of the coil,

$$I_r = \frac{4.67 \times 10^4\ I_s n_s n_r M_s M_r h_s \sin\theta\cos\theta\cos\phi}{\lambda^4 l R} \tag{5}$$

where the symbol s refers to the sending system, R is the resistance of the receiving coil circuit, and where the same unit of length is used for λ, l, M_s, M_r, h_s. If instead of the closed aerial an open antenna is used for receiving

$$I_r = \frac{7.44 \times 10^3\ I_s n_s M_s h_s h_r \sin\theta\cos\theta\cos\alpha}{\lambda^3 R l} \tag{6}$$

where R is the resistance of the antenna circuit and α is the angle made by the antenna with the local direction of **E**, the other quantities being the same as for equation (5).

The derivation of this paper neglects the reflection of the waves from the surface of the earth and the effect of eddy currents in the earth in the neighborhood of the transmitting set.

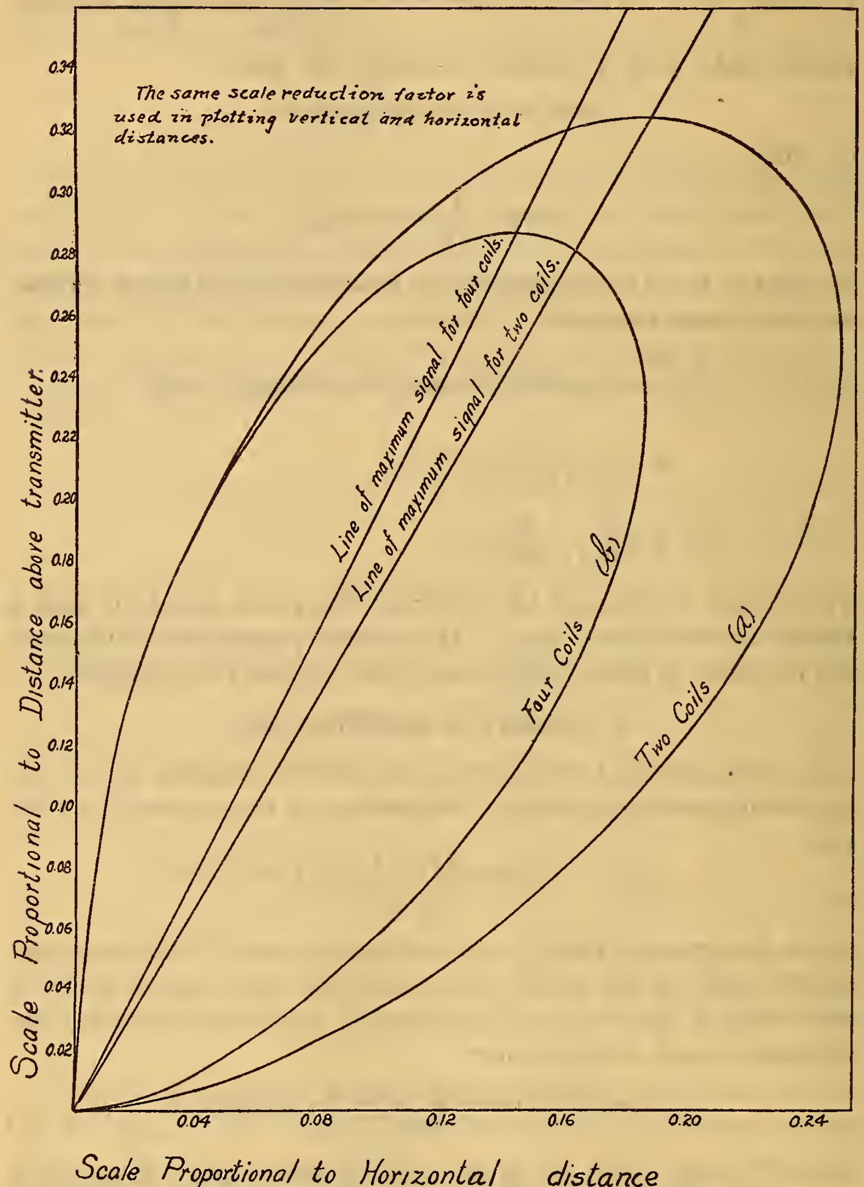

FIG. 5.—*Section of surface within which signal is audible for the case of reception with plane of coil vertical*

IV. SUMMARY

The nature of the field radiated by two horizontal coils has been calculated. The currents received in such a field by means of a coil aerial and by means of an open antenna have been computed.

The case of a receiving station installed on an airplane has been considered in detail. The portions of space where the signal is heard have been ascertained. The angle with the vertical at which the signal is a maximum has been calculated. It was found to be 30° for the case of a vertical receiving coil.

The more complicated case of four transmitting coils has been studied. Two of these coils were taken to be the image of the other two in a horizontal plane. In this case the maximum is shifted still closer to the vertical, namely, to 26° 34′. This calculation can be applied to the case of two coils situated above a perfectly conducting plane. If the earth were a perfect conductor, it would apply to the case of two coils above the surface of the earth. It is improbable that in ordinary cases the conductivity of the earth is sufficiently high to justify such an approximation. Since, however, both in the case of an earth having a perfect conductivity and in the case of an earth having zero conductivity the maximum signal strength is found approximately at 30° with the vertical, this can also be expected for ordinary conductivities.

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V. LIST OF SYMBOLS

$\mathbf{H}$ = magnetic intensity.
c = a constant depending upon the choice of units.
$\mathbf{v}$ = velocity of charge.
K = dielectric constant.
π = 3.1416.
t = time.
$d\tau$ = element of volume.
$\mathbf{E}$ = electric intensity.
μ = permeability.
ρ = volume density of charge.
$\mathbf{A}$ = vector potential at P.
$\phi = \angle QOY$ (p. 592) (Fig. 1).

A = absolute value of vector potential at P.
r = distance from element of volume $d\tau$ to the point at which **A** or ϕ is computed.
b = radius of circular coil aerial.
V = velocity of light.
R = resistance of receiving circuit.
l = distance from transmitting to receiving apparatus.
A_x = component of vector potential in direction OX.
A_y = component of vector potential in direction OY.
A_z = component of vector potential in direction OZ.
$\omega = 2\pi \times$ frequency.
(r) is a subscript referring to the receiving system.
(s) is a subscript referring to the sending system.
M_r = area of receiving coil.
M_s = area of sending coil.
n_s = number of turns in transmitting coil.
n_r = number of turns in receiving coil.
h_r = height of receiving antenna.
$h = h_s$ = distance between transmitting coils.
λ = wave length.

WASHINGTON, July 14, 1921.