A RELATION CONNECTING THE DERIVATIVES OF PHYSICAL QUANTITIES[1]

By Mayo D. Hersey

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I. INTRODUCTION

1. *Scope of the Paper.*—In this paper it is shown how the theory of dimensions may be used in a differential form; a procedure which appears fruitful particularly in investigating the effect of given sources of error on the performance of measuring instruments.

The examples which led to the necessity for developing this method are discussed at the end of the paper and illustrated by experimental data.

2. *Statement of the Problem.*—Given the fact that some relation of unknown form

$$Q_0 = f\,(Q_1, Q_2, \quad . \quad . \quad . \quad Q_{N-1}) \tag{1}$$

subsists between N physical quantities $Q_0, Q_1, Q_2, \quad . \quad . \quad . \quad Q_{N-1}$, no others being involved, it is required to deduce a relation of known form

$$\frac{\partial Q_0}{\partial Q_1} = F\left(\frac{\partial Q_0}{\partial Q_2}, Q_0, Q_1, Q_2, \quad . \quad . \quad . \quad Q_{N-1}\right) \tag{2}$$

[1] This work was done at the Jefferson Physical Laboratory, Harvard University, in 1916, and presented as an interesting application of Buckingham's Π theorem during a series of four lectures on Dimensional Reasoning given at the Friday evening conferences. It was first published in the Journal Wash. Acad. Sci., 6, pp. 620–629, 1916, and is reprinted here to provide the mathematical basis for a forthcoming paper on the effect of temperature on bodies of constant shape.

such that at any point whose generalized coordinates, Q_0, Q_1, Q_2, etc., are given, the value of any one of the $N-1$ partial derivatives of Q_0 can be computed from some other one. Thus, it is required to calculate one of the component slopes of the generalized surface (1) from a knowledge of another, although the equation of the surface is not available. The interest of the problem to the physicist lies in the fact that he may wish to learn the value of a derivative not readily accessible to experiment, in a case where some other derivative of the same quantity can easily be observed. It will be shown that a definite solution can always be obtained, provided certain dimensionless products of the N quantities are held constant.

3. *Other Classes of Relations Among Derivatives.*—The proposition, that relations may be found connecting the derivatives of quantities in the absence of a primitive equation, is not new. There are two other classes of such relations. One consists of mathematical identities, applicable to any set of related quantities, whether physical or not. To this class belongs the identity

$$\frac{\partial}{\partial Q_1}\frac{\partial Q_0}{\partial Q_2}=\frac{\partial}{\partial Q_2}\frac{\partial Q_0}{\partial Q_1} \tag{3}$$

as well as the triple product relation

$$\frac{\partial Q_0}{\partial Q_1}\cdot\frac{\partial Q_1}{\partial Q_2}\cdot\frac{\partial Q_2}{\partial Q_0}=-1 \tag{4}$$

The other class comprises relations requiring the explicit use of physical laws, such as the two laws of thermodynamics, or Hamilton's principle. To this class belong Maxwell's four thermodynamic relations, and the reciprocal relations of generalized dynamics.[2] The relations to be presented here are of a nature intermediate between the other two classes, in that they require a knowledge only of the dimensions of the quantities.

II. THEORETICAL DISCUSSION.

1. *Derivation of the New Relation.*—The present result depends upon and is a corollary to Buckingham's Π-theorem,[3] according to which any complete physical equation is reducible to the form

$$\text{funct.}\ (\Pi_1, \Pi_2, \ . \ . \ . \ \Pi_i)=0 \tag{5}$$

[2] J. J. Thomson, Applications of dynamics to physics and chemistry, Chap. 5.

[3] Journal Wash. Acad. Sci., 4, pp. 347–353, 1914; Phys. Rev., 4, pp. 345–376, 1914; Trans. Am. Soc. Mech. Engs., 37, pp. 263–296; 1915. Anyone who can sufficiently visualize the meaning of the Π-theorem will be able to treat each concrete problem by itself, dispensing with the formulas of the present paper save as a check.

in which the Π's are all the independent dimensionless products which can be built up by combining in any way the N physical quantities involved. Further, the total number of such products, or dimensionless arguments, will always be the same, no matter how the quantities are grouped. This number will be

$$i = N - k \tag{6}$$

if k is the number of fundamental units needed for measuring the N quantities.[4]

Let Π_0 and Π designate any two of the i products in (5) which contain between them the three quantities Q_0, Q_1, and Q_2 in which we are interested. Let Q_0 appear to the first power in Π_0 and not at all in any other product, which can always be done, for Buckingham has shown that a certain standard arrangement is possible in which each product contains to the first power some one quantity of type P which occurs nowhere else.[5] We shall then have

$$\Pi_0 = Q_1^{\alpha_0} Q_2^{\beta_0} \cdot \cdot Q_k^{\kappa_0} \cdot Q_0 \tag{7}$$

and

$$\Pi = Q_1^{\alpha} Q_2^{\beta} \cdot \cdot Q_k^{\kappa} \cdot Q_{k+1} \tag{8}$$

The exponents are abstract numbers fixed by the dimensions of the N quantities; in any particular problem some of them may be zero. If we now agree to keep the remaining $i-2$ products constant, (5) becomes

$$\Pi_0 = \phi(\Pi) \tag{9}$$

in which the form of ϕ is unknown. The restriction to constant products can always be fulfilled in theory, but it may lead to difficulties in practice; it will be discussed in a later section. Differentiating (9) and then (8) gives in succession

$$\frac{\partial \Pi_0}{\partial Q_1} = \frac{d\phi}{d\Pi} \cdot \frac{\partial \Pi}{\partial Q_1} = \frac{d\phi}{d\Pi} \frac{\alpha \Pi}{Q_1} \tag{10}$$

From (7)

$$\frac{\partial \Pi_0}{\partial Q_1} = \frac{\partial Q_0}{\partial Q_1} \frac{\Pi_0}{Q_0} + \frac{\alpha_0 \Pi_0}{Q_1} \tag{11}$$

Comparing (10) and (11)

$$\Pi \frac{d\phi}{d\Pi} = \frac{Q_1 \Pi_0}{\alpha} \left(\frac{1}{Q_0} \frac{\partial Q_0}{\partial Q_1} + \frac{\alpha_0}{Q_1} \right) \tag{12}$$

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[4] The question of the number of fundamental units needed has been discussed by Riabouchinsky, Rayleigh, and Buckingham; see Nature, **96**, pp. 396–397; 1915.

[5] Trans. Am. Soc. Mech. Engs., **37**, pp. 291–292; note equation (11) and its discussion.

Similarly

$$\Pi\frac{d\phi}{d\Pi}=\frac{Q_2\Pi_0}{\beta}\left(\frac{1}{Q_0}\frac{\partial Q_0}{\partial Q_2}+\frac{\beta_0}{Q_2}\right) \tag{13}$$

Comparing (12) and (13)

$$\frac{\partial Q_0}{\partial Q_1}=\left(\frac{\alpha}{\beta}\beta_0-\alpha_0\right)\frac{Q_0}{Q_1}+\frac{\alpha}{\beta}\frac{Q_2}{Q_1}\frac{\partial Q_0}{\partial Q_2} \tag{14}$$

Hence the desired relation (2) has the linear form

$$\frac{\partial Q_0}{\partial Q_1}=a+b\frac{\partial Q_0}{\partial Q_2} \tag{15}$$

in which the coefficients

$$a=\left(\frac{\alpha}{\beta}\beta_0-\alpha_0\right)\frac{Q_0}{Q_1} \tag{16}$$

and

$$b=\frac{\alpha}{\beta}\frac{Q_2}{Q_1} \tag{17}$$

involve none of the N quantities save Q_0, Q_1, and Q_2.

Evidently (14) can be written also

$$\frac{\partial \log Q_0}{\partial \log Q_1}=\left(\frac{\alpha}{\beta}\beta_0-\alpha_0\right)+\frac{\alpha}{\beta}\frac{\partial \log Q_0}{\partial \log Q_2} \tag{18}$$

in which the coefficients are independent of the coordinates. Thus the relation connecting the logarithmic derivatives is the same all over the generalized surface.

If no two independent products can be found which do contain between them the three quantities Q_0, Q_1, and Q_2, either α or β will vanish, showing that the derivatives are independent. Thus, while it is always possible to find a relation connecting any derivative with some other, it is not always possible to find a relation connecting a given derivative with any other desired.

2. *Extension to Higher Derivatives.*—Differentiating (14) with respect to Q_1 and using the identity (3) gives

$$\frac{\partial^2 Q_0}{\partial Q_1^2}=A+B\frac{\partial Q_0}{\partial Q_2}+C\frac{\partial^2 Q_0}{\partial Q_2^2} \tag{19}$$

in which the coefficients are

$$A=\frac{Q_0}{Q_1^2}\left(\frac{\alpha}{\beta}\beta_0-\alpha_0\right)\left(\frac{\alpha}{\beta}\beta_0-\alpha_0-1\right) \tag{20}$$

$$B=\frac{Q_2}{Q_1^2}\frac{\alpha}{\beta}\left(\frac{\alpha}{\beta}-1\right)(1+2\alpha_0) \tag{21}$$

and

$$C=\left(\frac{Q_2\alpha}{Q_1\beta}\right)^2 \tag{22}$$

Thus the curvature with respect to Q_1 can be calculated from the slope and the curvature with respect to Q_2.

3. *Integral Form of the Relation.*[6]—Integrating (14) at the point $(Q_0=q_0, Q_1=q_1, Q_2=q_2)$ over an interval so short that $\frac{\partial Q_0}{\partial Q_2}$ may be treated as constant, and denoting its value by the symbol $\frac{\partial q_0}{\partial q_2}$, gives for the primitive equation of an element of the surface

$$\frac{Q_0}{q_0}=\left(\frac{Q_1}{q_1}\right)^h \tag{23}$$

in which

$$h=\frac{\alpha}{\beta}\left(\frac{q_2}{q_0}\frac{\partial q_0}{\partial q_2}+\beta_0\right)-\alpha_0 \tag{24}$$

The use of (23) would permit a direct comparison of any new results obtained by the present method with empirical results previously published in one-term, constant-exponent formulas.

4. *Discussion of the Constant-Product Restriction.*—Let Π_c denote any one of the $i-2$ arguments which we have agreed to hold constant, and let Q stand for either Q_1 or Q_2. Then, unless Π_c can be so chosen that it does not contain Q, it must be so chosen that it will contain some additional quantity Q_c not occurring in any other product. The rule for keeping Π_c constant will then be: Vary Q_c simultaneously in such a manner as to compensate the changes due to Q.

If Q enters Π_c to the nth power and Q_c enters it to the first, the derivatives in (15) and elsewhere are subject to one or more conditions of the type $Q_c \propto Q^{-n}$. For such a derivative let us adopt from now on the notation $\left(\frac{\partial Q_0}{\partial Q}\right)_{Q_c \propto Q^{-n}}$. There are two experimentally independent methods for getting its numerical value: First, by directly observing the change in Q_0 with Q while simultaneously changing Q_c in the prescribed manner; second, by calculating it from separate observations on the change in Q_0 with Q at constant Q_c, and the change in Q_0 with

[6] If instead of an isolated value of $\frac{\partial Q_0}{\partial Q_2}$ we were furnished with the entire curve $Q_0=f_2(Q_2)$, the direct use of the Π-theorem would be preferable, and would give the whole curve $Q_0=f_1(Q_1)$. If successively furnished with additional curves, $Q_0=f_3(Q_3)$ and so on, we could gradually build up generalized cross sections of the surface (1) until, when $N-k$ independent curves had been given, we should have the whole of it. The problem of developing empirical equations synthetically has not been treated in the available papers. That problem is a general one, of which the problem of the present paper is a special case; this situation is illustrated by the fact that our final result (23) applies only to an infinitesimal piece of the curve $Q_0=f_1(Q_1)$.

Q_c at constant Q. Expanding the conditioned derivative into the form $\left(\frac{\partial Q_0}{\partial Q}\right)_{Q_c}+\left(\frac{\partial Q_0}{\partial Q_c}\right)_Q\frac{dQ_c}{dQ}$ and taking account of the fixed relation between Q_c and Q leads to the working formula

$$\left(\frac{\partial Q_0}{\partial Q}\right)_{Q_c\propto Q^{-n}}=\left(\frac{\partial Q_0}{\partial Q}\right)_{Q_c}-n\frac{Q_c}{Q}\left(\frac{\partial Q_0}{\partial Q_c}\right)_Q \tag{25}$$

for the second method. In the most general case where there are $i-2$ arguments to be kept constant, the second term on the right of (25) will be replaced by $-\frac{1}{Q}$ times the summation of $i-2$ terms of the type $nQ_c\left(\frac{\partial Q_0}{\partial Q_c}\right)_Q$.

While the procedure outlined in this section is always possible and sufficient, it is not always necessary or even desirable. For example: if the number of quantities, N, does not exceed the number of fundamental units, k, by more than 2, there will be no other arguments than Π_0 and Π; again, if the remaining $i-2$ arguments do not involve Q (i. e., Q_1 or Q_2), their constancy will not be disturbed at all by the fact that Q_1 and Q_2 do vary. Further expedients for simplifying the work will suggest themselves upon examining each particular case by itself.

III. SOME ILLUSTRATIVE EXAMPLES

For reference in solving problems it is convenient to rewrite (5) in the form

$$Q_1^{\alpha_0}Q_2^{\beta_0}\cdots Q_k^{\kappa_0}\cdot Q_0=\text{funct. }(Q_1^{\alpha}Q_2^{\beta}\cdots Q_k^{\kappa}\cdot Q_{k+1},\text{ and other }\Pi\text{'s}) \tag{26}$$

The values of α, β, etc., can now be read off directly by identifying them with the corresponding numerical exponents in the equation, of type (26), afforded by the particular example in hand.

1. *Variation of Journal Friction with Size of Bearing.*—In the case of a journal bearing, under certain restrictions, we may expect a relation of type (1) to connect the coefficient of friction f, with the viscosity of the lubricant μ, the revolutions per unit time n, the bearing pressure p, the journal diameter D, and the volume of oil V forced through the bearing in unit time. Let it be required to calculate the effect of altering the size of the machine from a test in which nothing is varied but the rate of pumping

oil through the bearing. It may be shown by the Π-theorem that

$$f = \text{funct.}\left(\frac{D^3 n}{V}, \frac{\mu n}{p}, \text{shape}\right) \tag{27}$$

Let now f, D, and V serve, respectively, as Q_0, Q_1, and Q_2. Comparing (27) with (26), $\alpha_0 = 0$, $\beta_0 = 0$, $\alpha = 3$, $\beta = -1$; hence, by (16) and (17), $a = 0$ and $b = -3\frac{V}{D}$, or

$$\frac{\partial f}{\partial D} = -3\frac{V}{D}\frac{\partial f}{\partial V} \tag{28}$$

Also, by (20) and (22), $A = 0$, $B = 12\frac{V}{D^2}$, and $C = 9\left(\frac{V}{D}\right)^2$; therefore

$$\frac{\partial^2 f}{\partial D^2} = 12\frac{V}{D^2}\frac{\partial f}{\partial V} + 9\left(\frac{V}{D}\right)^2\frac{\partial^2 f}{\partial V^2} \tag{29}$$

Equations (28) and (29) enable us to predict the bearing losses of any slightly larger or smaller machine in the same geometrically similar series. This requirement of geometrical similarity is an instance of the constant-product restriction. The products in this case are the length ratios fixing the shape. The first of the two results in this example, namely, equation (28), follows readily enough from equation (27) without calling in the aid of the present theorem at all. For in differentiating f first with respect to D and then with respect to V the *same* unknown occurs both times and can be eliminated, leaving the desired relation between $\frac{\partial f}{\partial D}$ and $\frac{\partial f}{\partial V}$. The contemplation of this example, which can be solved by inspection because the left-hand member of equation (27) has only one quantity in it, reveals instructively the meaning of the theorem and shows that it is simple when understood.

2. *Effect of Gravity on a Rolling-Ball Viscosimeter.*—Let it be required to find the effect of gravity on a rolling-ball viscosimeter in terms of the effect produced by changing the size of the instrument. Let D, l, and θ denote, respectively, the diameter and length of the tube and its angle of inclination to the horizontal, d and ρ_0 the diameter and density of the ball, ρ and μ the density and viscosity of the liquid, and t the roll-time[7] in a locality[8] of

[7] That is, the time required for the ball to roll down. This instrument, proposed by Flowers (Proc. Am. Soc. Test. Mat., **14**, pp. 565; 1914), is further discussed by the writer in Journal Wash. Acad. Sci., **6**, pp. 527; 1916.

[8] Having set up such a viscosimeter in Cambridge, the question arose whether there would be any sensible change upon taking it to Washington, where gravity is 0.3 per cent less. The conclusion is that the roll-time in a very viscous liquid will be 0.3 per cent greater in Washington; and that the effect of gravity diminishes when the fluidity of the liquid increases, falling to 0.2 per cent for water.

gravity g. Assuming that a complete relation does subsist among these quantities, the π-theorem shows that any equation describing that relation, whether obtained theoretically or experimentally, must be reducible to the form

$$\frac{\mu}{\rho D^2} t = \text{funct.}\left(\frac{\rho_0}{\rho}, \frac{g\rho^2 D^3}{\mu^2}, \text{shape}\right) \tag{30}$$

the shape, in turn, being fixed by the arguments $\frac{d}{D}$, $\frac{l}{D}$, and θ. Taking t, g, and D, respectively, for Q_0, Q_1, and Q_2 gives $\alpha_0 = 0$, $\beta_0 = -2$, $\alpha = 1$, and $\beta = 3$; so that by (18)

$$\frac{g\partial t}{t\partial g} = -\frac{2}{3} + \frac{1}{3}\frac{D}{t}\frac{\partial t}{\partial D} \tag{31}$$

An interesting check on (31) is afforded by differentiating the empirical equation for such an instrument.[9] The equation has been presented in the form $y = a + bx$, in which x denotes $\tau\sqrt{Dg\left(\frac{\rho_0}{\rho} - 1\right)}$ and y denotes $\nu \Big/ \sqrt{D^3 g\left(\frac{\rho_0}{\rho} - 1\right)}$, τ being the roll-time per unit length $\frac{t}{l}$, ν the kinematic viscosity $\frac{\mu}{\rho}$, and a and b particular numerical values fixed by a particular choice of $\frac{d}{D}$ and θ. Recast in the form (30) it becomes

$$\frac{\mu}{\rho D^2} t = \frac{1}{b} \cdot \frac{1}{\frac{\rho_0}{\rho} - 1}\left(1 - a\sqrt{\frac{\rho_0}{\rho} - 1}\sqrt{\frac{g\rho^2 D^3}{\mu^2}}\right)\frac{\mu^2}{g\rho^2 D^3} \cdot \frac{l}{D} \tag{32}$$

or

$$t = \frac{A}{gD}\left(1 + B\sqrt{gD^3}\right) \tag{33}$$

in which A and B (both intrinsically positive) do not involve g at all, nor D except in a shape factor. The values of $\frac{g}{t}\frac{\partial t}{\partial g}$ and $\frac{D}{t}\frac{\partial t}{\partial D}$ found by differentiating (33) do satisfy (31).

3. *Effect of High Pressure on Accuracy of a Rolling-Ball Viscosimeter.*—Without knowing the empirical equation let it be required to predict the change in roll-time due to any small change in liquid density, such as would occur upon using the tube under pressure, by reference to an observation on the effect of changing the ball density. Since an expression for $\frac{\partial t}{\partial \rho}$ in terms of $\frac{\partial t}{\partial \rho_0}$ is

[9] Journal Wash. Acad, Sci., 6, pp. 528, eq. (6); 1916.

sought, t, ρ, and ρ_0 are selected for Q_0, Q_1, and Q_2, respectively. If (30) were to be used as it stands, there would be a restriction on the derivative $\frac{\partial t}{\partial \rho}$, which is hardly to be desired. An equivalent result in a more convenient form can evidently be obtained by confining ρ to a smaller number of arguments. This is done by replacing (30) by one of the alternative forms provided by the Π-theorem, such as

$$\sqrt{\frac{g}{D}}\, t = \text{funct.}\left(\frac{\rho_0}{\rho}, \frac{g\rho_0^2 D^3}{\mu^2}, \text{shape}\right) \tag{34}$$

Comparing this with (26), $\alpha_0 = 0$, $\beta_0 = 0$, $\alpha = 1$, $\beta = -1$; hence by (14)

$$\frac{\partial t}{\partial \rho} = -\frac{\rho_0}{\rho}\left(\frac{\partial t}{\partial \rho_0}\right)_{\mu \propto \rho_0} \tag{35}$$

or by (25)

$$\frac{\partial t}{\partial \rho} = -\frac{1}{\rho}\left(\rho_0 \frac{\partial t}{\partial \rho_0} + \mu \frac{\partial t}{\partial \mu}\right) \tag{36}$$

In the last transformation μ took the part of Q_c and ρ_0 of Q, while n had the value -1.

The following observations afford an experimental illustration of (36). They were made with a tube 59 cm long and 1 cm in diameter, containing a one-fourth inch (0.635 cm) ball, ordinarily of steel ($\rho_0 = 7.7$ g/cm³). The tube was filled with lard oil ($\mu = 0.74$ cgs units, $\rho = 0.92$ g/cm³). The slope $\frac{\partial t}{\partial \mu}$ was found to be 31 cgs units. Substituting now a brass ball ($\rho_0 = 8.6$ g/cm³) for the steel one, the roll-time dropped from 27.9 to 24.7 seconds, making $\frac{\partial t}{\partial \rho_0}$ equal to -3.6 cgs units. From these data, in conjunction with (36), the value $\frac{\partial t}{\partial \rho} = 5.2$ cgs units would be predicted. From (32), the actual value is found to be 5.7 cgs units. Since $\frac{\partial t}{\partial \rho}$ is itself a correction term, the agreement is sufficient.

WASHINGTON, September 23, 1918.