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Formulas for Computing Capacitance and Inductance

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Explicit formulas are given for the computation of (1) the capacitance between conductors having a great variety of geometrical configurations, (2) the inductance, both self- and mutual, of circuits of various shapes, and (3) the electrodynamic forces acting between coils when carrying current. Formulas for skin effect and proximity effect in concentric cables and parallel wires are included. The formulas for the simpler configurations are given in terms of the elementary functions, whereas more complex shapes involve the use of Legendre polynominals, Legendre functions, and elliptic functions. One section is devoted to a discussion of the relation between the Legendre and the elliptic functions.

Introduction

This collection of formulas contains some that are commonly used in electrical work and some that have been specially developed for precision work at this Bureau. This is no attempt at completeness, for there is now available (since 1948) a revised third edition of the earlier compilation of formulas for inductance by Rosa and Grover [1].[1] This may be consulted for references to original memoirs and also for discussion of the most suitable formula for a given configuration or relative dimensions. Reference may also be made to Dr. Grover's [2] additions to these formulas in 1918 and to his book "Inductance calculations working formulas and tables."

Formulas for capacitance may be found in the second edition, 1924, of a work by J. H. Dellinger, L. E. Whittemore, and R. S. Ould [3]. This contains formulas for inductance and a few for capacitance. It is possible that the aggregate of researches on capacitance up to this time might amount to a collection of capacitance formulas as comprehensive as that of Rosa and Grover for inductance.

The formulas given here contain, in addition to elementary functions, the Legendre polynomials P_n, the Legendre functions $Q_{n-\frac{1}{2}}$ and $P_{n-\frac{1}{2}}$ and elliptic functions. It is shown in section 4 how the latter two may be found by use of tables of the two complete elliptic integrals K and E.

These, together with the incomplete integrals $F(\phi, k)$ and $E(\phi, k)$, and sn u, cn u, dn u, etc., may be readily found from the 1947 Smithsonian elliptic functions tables by G. W. and R. M. Spenceley [4]. One point of superiority of this work over that of R. L. Hippisley [5] is that it proceeds by increments of 1° in the modular angle instead of 5°.

Another purely mathematical table of elliptic functions and theta functions that has been found very useful is table 1, (1922) by H. Nagaoka and S. Sakurai [6]. The same authors in (1927) [7, table 2] produced a volume more directly applicable to the calculation

[1] Figures in brackets indicate the literature reference at the end of the paper.

of the force between coils and their self and mutual inductance. For the latter, Nagaoka [8] has also published three formulas that make use of the remarkable convergence rate of the series defining the theta functions in powers of the Jacobian parameter q.

Short tables of theta functions are given by Jahnke and Emde [9]. Also short tables of $F(\phi, k)$ and $E(\phi, k)$ were given by B. O. Peirce [10]. The recent work of W. Magnus and F. Oberhettinger [11] is very useful.

These volumes, especially the work of the Spenceleys put the computation of formulas with elliptic functions in quite a different light. Such formulas are not more difficult than those with sines, cosines, and logarithms.

In section 5 are placed a few notes on methods of deriving some of the formulas given here that are not generally available, or perhaps are unpublished. Where space permits, it has been attempted to summarize the entire electric field, on which capacitance is based, or the entire magnetic field underlying the inductance constants. Such a scheme seems desirable on a larger scale than is possible here. Each formula for capacitance requires the evaluation of the electric potential or field at every point of space. For each inductance L or M, one must find the vector potential or magnetic field everywhere. The constants C, L, or M represent a small byproduct, since they are derived from the fields by direct processes. A summary of the more important electric and magnetic fields that have been evaluated to date would probably fit present requirements better than further tabulation of capacitance and inductance.

1. Capacitance

The formulas for capacitance given in this paper are expressed in the centimeter-gram-second electrostatic system of units (unrationalized). If lengths in centimeters are substituted for the corresponding symbols in a formula, the resulting value of C will be the capacitance in cgs electrostatic units. This value should be multiplied by 10/9 (more precisely $10/c = 1.11277$) to obtain the capacitance in micromicrofarads. The formulas assume a dielectric constant of unity (in the cgs-esu system). If the space between electrodes is filled with a dielectric of permittivity ϵ_r relative to empty space, the value of capacitance as computed from the formula should be multiplied by ϵ_r.

Alternatively, when expressed in the rationalized meter-kilogram-second-ampere (Giorgi) system of units each formula for capacitance would have an additional factor of 4π (by reason of the rationalization) and also a factor of $10^7/4\pi c^2$ (by reason of the conventionally chosen permittivity of free space). The net result is that with the dimensions expressed in meters, and after multiplying by the combined factor 1.11277×10^{-10}, the resulting value of C is in farads.

The formulas for inductance and electromagnetic force given in this paper are expressed in the centimeter-gram-second electromagnetic system of units (unrationalized). In using the formulas, lengths should be expressed in centimeters, currents in abamperes (i.e., units of 10 amperes), and the permeability of space should be taken as unity. If this is done, the inductances as computed are in units of 10^{-9} henry, forces are in dynes, and torques in dyne-centimeters.

Alternatively in the rationalized meter-kilogram-second-ampere system the formulas should be multiplied by $1/4\pi$ (by reason of rationalization) and by $4\pi \cdot 10^{-7}$ (by reason of the conventionally chosen permeability of free space). Then if dimensions are expressed in meters, and currents in amperes, the inductances as computed will be in henries, the forces in newtons, and the torques in newton-meters.

The first six figures illustrate two cases. In the axially symmetric cases the figures represent plane sections through the axis of symmetry. In the cylindrical case they are plane sections perpendicular to the endless generators. In this case the formulas give the capacitance C/l per unit length perpendicular to the plane of the figure.

1.1. Parallel Plates With Guard Planes

The separation c between the parallel plates should be small compared to the radius a_1 of the disk. Also the radius A of the plates should be large compared to a_2, so the field is practically uniform at some place between the edge of the disk and the outer edge of the plates ($A>5a_2$).

a. Coplanar Guard and Electrode [13]

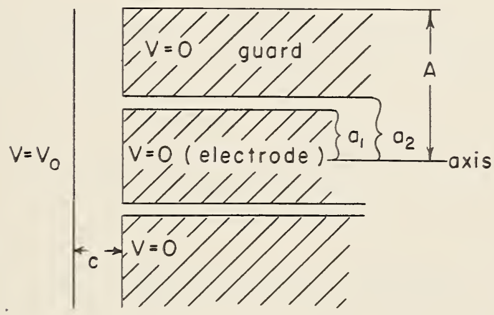


FIGURE 1

$\bar{a}=\frac{1}{2}(a_1+a_2)$, and $(a_2-a_1)/c$ is small.

Axial sym: $$C=\frac{\bar{a}^2}{4c}-\frac{\pi\bar{a}}{2}\left(\frac{a_2-a_1}{2\pi c}\right)^2\coth\frac{\pi\bar{a}}{c}. \tag{1.1}$$

Cyclindrical: $$C/l=\frac{\bar{a}}{2\pi c}-\frac{1}{2}\left(\frac{a_2-a_1}{2\pi c}\right)^2\coth\frac{\pi\bar{a}}{c}. \tag{1.2}$$

This capacitance is between the plane at potential V_0 and the electrode, including its plane face and its sides.

b. Electrode at Bottom of Hole in Guard [13]

(Capacitance Between the Plane at V_0 and the Face of Electrode at Bottom of Hole)

As in the preceding case, a/A must be small $\left(a/A \leqslant \frac{1}{5} \right)$. Also the hole is not very shallow, and the clearance between the electrode and its guard is ignored.

$$1 < \beta = \frac{d}{a} \leqslant \infty, \text{ and } 0 \leqslant \gamma = \frac{c}{a} \leqslant \frac{1}{5}.$$

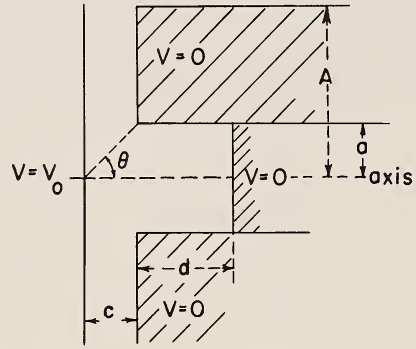


FIGURE 2

Axial sym:

$$C = \frac{a}{\gamma} \sum_{s=1}^{\infty} \frac{\sinh \alpha_s \gamma}{\alpha_s^2 \sinh \alpha_s (\beta + \gamma)}, \tag{1.3}$$

where $\alpha_1 = 2.4048$, $\alpha_2 = 5.5201$, $\alpha_3 = 8.6537$, $\alpha_4 = 11.7915$, and $J_0(\alpha_s) = 0$. The first three terms are sufficient, with the conditions given above, for an accuracy of 1 in 200. (J_0 is Bessel's function).

Cylinder:

$$C/l = \frac{4q}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n-1} \sin (2n-1)\phi}{(2n-1)[1-(q^2)^{2n-1}]}, \tag{1.4}$$

where

$$q = e^{-\left(\gamma\theta + \frac{\pi\beta}{2}\right)} \left[1 + 8 \cos^4 \theta e^{-4\left(\gamma\theta + \frac{\pi\beta}{2}\right)} \right]$$

$$\phi = \theta - 8 \sin \theta \cos \theta e^{-4\left(\gamma\theta + \frac{\pi\beta}{2}\right)}$$

$$\theta = \tan^{-1} \left(1/\gamma \right) = \tan^{-1} \left(\frac{a}{c} \right) \quad (\theta \text{ in radians}).$$

In case the hole is very shallow (d/c small), a better formula than (1.4) for the cylindrical case is

$$C/l = \frac{a}{2\pi c} - \frac{1}{2\pi^2} \left\{ \frac{d}{c} \log \frac{1}{q_1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left[\left(\frac{q_1}{r} \right)^n - (q_1 r)^n \right]}{n[1+q_1{}^{2n}]} \right\}, \tag{1.4'}$$

where r and q_1 may be computed by

$$\log r = -\frac{\pi a}{c} - \frac{\epsilon}{1+\epsilon \coth\frac{\pi a}{c}} \log \frac{\epsilon(1-\epsilon 2^{-1} \coth(ma/c))}{4 \sinh\ (ma/c)}$$

$$\log q_1 = \log \frac{\epsilon r \left(1-\frac{\epsilon}{2}\coth(ma/c)\right)}{2(1-r^2)}.$$

where $\epsilon = d/c$.

Equation (1.4), like (1.4′), is exact with slot of any depth. Both ignore clearance between electrode and its guard. To take account of this (to first order) let $2a_1$ denote the width of face of electrode; $2a_2$, the width of slot; and d, its depth. Then if

$$\bar{a} = (a_1 + a_2)/2,$$

$$C/l = \frac{\bar{a}}{2\pi c} + \frac{1}{2\pi^2}\left[\left(\frac{a_2-a_1}{c}\right)\tan^{-1}\left(\frac{d}{a_2-a_1}\right) - \frac{d}{c}\log\frac{4c \sinh\ \pi\bar{a}/c}{\sqrt{d^2+(a_2-a_1)^2}}\right] \qquad (1.4'')$$

neglecting terms of order

$$[d^2+(a_2-a_1)^2]\log[d^2+(a_2-a_1)^2].$$

1.2. Spheres or Cylinders

a. Concentric Case

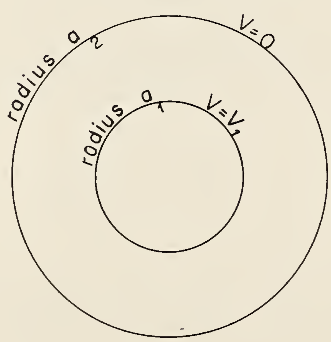


FIGURE 3

Spheres:

$$C = \frac{a_1 a_2}{a_2 - a_1} \qquad (1.5)$$

The capacity of one sphere alone ($a_2 \to \infty$) is $C = a_1$.

Cylinders:

$$C/l = \frac{1}{2\log\frac{a_2}{a_1}} \qquad (1.6)$$

Equations (1.5) and (1.6) are limiting cases of (1.11) and (1.12). The potential between the spheres is

$$V(r)=V_1\frac{1-a_2/r}{1-a_2/a_1};$$

that between the cylinders is

$$V(r)=V_1\frac{\log a_2/r}{\log a_2/a_1}.$$

b. Plane With Sphere or Cylinder

(This is a limiting case of equations 1.14 and 1.15.)

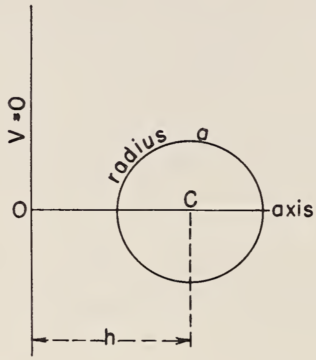


FIGURE 4

$$\gamma\equiv 2\log\left(\frac{h+\sqrt{h^2-a^2}}{a}\right). \tag{1.7}$$

Sphere:

$$C=2\sqrt{h^2-a^2}\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})\gamma}}{1-e^{-(n+\frac{1}{2})\gamma}} \tag{1.8}$$

Cylinder:

$$C/l=\frac{1}{\gamma}=\frac{1}{2\log\dfrac{h+\sqrt{h^2-a^2}}{a}}. \tag{1.9}$$

When $h\to\infty$, $\gamma\to\log(4h^2/a^2)$ and (1.8) gives $C\to a$, but (1.9) gives $C/l=0$, as it should, since the logarithmic potential becomes infinite at spatial infinity for any finite charge except zero.

c. **Eccentric Spheres or Cylinders** (Internal Case $0<b<a_2-a_1$)

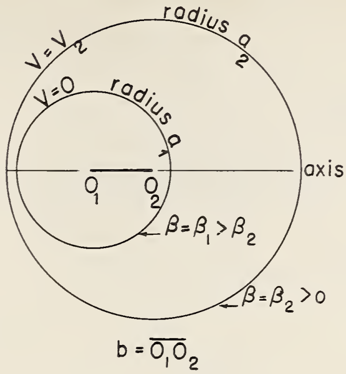


FIGURE 5

$\overline{O_1O_2}=b=$ the distance between centers (always positive)

$$\left.\begin{aligned}
2bc &\equiv \sqrt{[(a_2+a_1)^2-b^2][(a_2-a_1)^2-b^2]} && \text{(positive)}\\
\beta_1 &= \log \frac{a_2^2-a_1^2-b^2+2bc}{2a_1 b} && \text{(positive)}\\
\beta_2 &= \log \frac{a_2^2-a_1^2+b^2+2bc}{2a_2 b} && \text{(positive)}
\end{aligned}\right\} \tag{1.10}$$

Spheres:

$$C=2c\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\beta_1}}{1-e^{-(2n+1)(\beta_1-\beta_2)}} \tag{1.11}$$

Cylinders:

$$C/l=\frac{1}{2(\beta_1-\beta_2)}=1/\left[2\log\frac{a_1^2+a_2^2-b^2+2bc}{2a_1a_2}\right] \tag{1.12}$$

d. **Eccentric Spheres or Cylinders** (External Case, $b>a_1+a_2$)

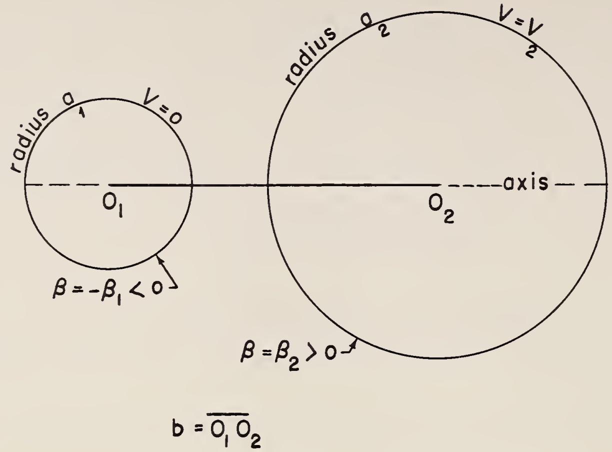


FIGURE 6

$b=\overline{O_1O_2}$

$$\left.\begin{aligned}2bc&=\sqrt{[b^2-(a_1+a_2)^2][b^2-(a_1-a_2)^2]} && \text{(positive)}\\ \beta_1&=\log\left[\frac{b^2+a_1^2-a_2^2+2bc}{2a_1b}\right] && \text{(positive)}\\ \beta_2&=\log\left[\frac{b^2-a_1^2+a_2^2+2bc}{2a_2b}\right] && \text{(positive)}\\ \gamma&=2(\beta_1+\beta_2)=2\log\left[\frac{b^2-a_1^2-a_2^2+2bc}{2a_1a_2}\right] && \text{(positive)}\end{aligned}\right\}. \tag{1.13}$$

Spheres:

$$C=2c\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})\gamma}}{1-e^{-(n+\frac{1}{2})\gamma}}=c\sum_{s=1}^{\infty}\frac{1}{\sinh\frac{s\gamma}{2}} \tag{1.14}$$

Cylinders or parallel wires:

$$C/l=\frac{1}{\gamma}=\frac{1}{2(\beta_1+\beta_2)}. \tag{1.15}$$

Placing $b=a_1+h$ and $a_1\to\infty$, eq (1.14) and (1.15) go into eq (1.8 and (1.9), respectively.

1.3. Spheroids

a. A Thin Circular Disk of Radius a

$$C=\frac{2a}{\pi}. \tag{1.16}$$

This is a limiting case $b=0$, of the following formula.

b. Oblate Spheroid

Major axis $2a$, minor axis $2b$:

$$C=\frac{\sqrt{a^2-b^2}}{\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)}. \tag{1.17}$$

c. Prolate Spheroid

Major axis $2a$, minor axis $2b$:

$$C=\frac{\sqrt{a^2-b^2}}{\log\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)}. \tag{1.18}$$

1.4. Toroidal Surface

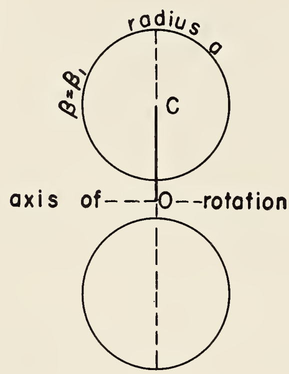


FIGURE 7

a=radius of generating circle

$A=\overline{OC}>a$

$$\cosh\beta_1=\frac{2}{k^2}-1=\frac{A}{a}$$

$$k^2=\frac{2a}{A+a} \quad \text{so } 0<k<1$$

$$C=\frac{4\sqrt{A^2-a^2}}{\pi}\sum_{n=0}^{\infty}\epsilon_n\frac{Q_{n-\frac{1}{2}}(\cosh\beta_1)}{P_{n-\frac{1}{2}}(\cosh\beta_1)}(\epsilon_0=\tfrac{1}{2},\epsilon_n=1 \text{ if } n\neq 0). \tag{1.19}$$

P and Q are the two Legendre functions with the same argument A/a. A method of finding these functions from tables of elliptic functions is given in section 4.

1.5. Conductor Bounded by Two Intersecting Spheres [14]

(Alone in space)

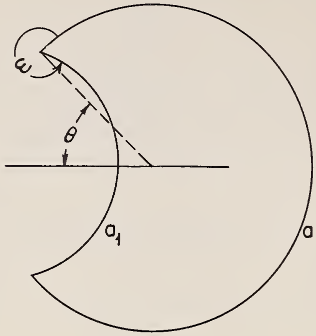


FIGURE 8.—*Axial section of intersecting spheres.*

a=radius of arc on the right, semiaperture=θ

$$a_1=\frac{a\sin\theta}{|\sin(w-\theta)|}=\text{radius of arc on the left}$$

w=angle at which the arcs intersect.

All figures may be obtained with the restrictions

$$0<\theta<\pi \text{ and } \theta<w<2\pi.$$

The capacitance for general w and θ is $C_w(\theta)$, where

$$C_w(\theta)=\frac{a\sin\theta}{w}\left\{\frac{\pi}{\sin\frac{\pi\theta}{w}}-8\sum_{n=1}^{\infty}\sin^2\frac{n\pi\theta}{w}\left[\psi\left(n+\frac{1}{2}\right)-\psi\left(\frac{n\pi}{w}+\frac{1}{2}\right)+\log\frac{\pi}{w}\right]\right\}, \tag{1.20}$$

where ψ denotes the psi-function, Γ'/Γ, whose values may be taken from the tables of H. T. Davis, "Tables of the higher mathematical functions" (Principia Press, Bloomington, Ind., 1933).

The series (1.20) converges like $\Sigma 1/n^2$. For a much more rapid series converging like Σn^{-14}, see reference [21], where the cases are considered that have finite terms for capacitance(w/π rational).

The simplest of these is the limiting case $\omega = 2\pi$, where the conductor is a thin shell with any aperture 2θ.

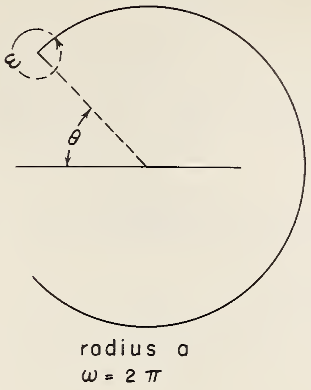


FIGURE 9

$$C_{2\pi}(\theta) = a - \frac{a}{\pi}(\theta - \sin\theta). \tag{1.21}$$

For $\theta = \pi/2$ this gives the capacitance of a hemispherical bowl

$$C_{2\pi}(\pi/2) = a\left(\frac{1}{2} + \frac{1}{\pi}\right) = .8183a.$$

Orthogonal Spheres: (External, $\omega = \pi/2$)

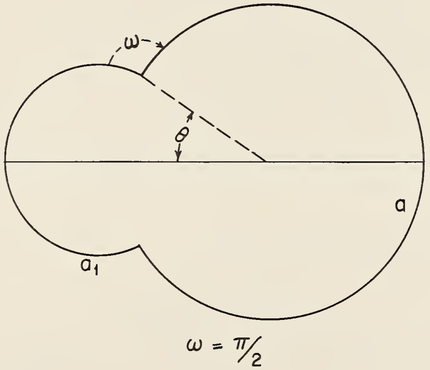


FIGURE 10

$$C_{\pi/2}(\theta) = a(1 + \tan\theta - \sin\theta) = a + a_1 - \frac{aa_1}{\sqrt{a^2 + a_1^2}}, \tag{1.22}$$

since $a_1 = a \tan\theta$.

Orthogonal Spheres: (Internal, $w=3\pi/2$)

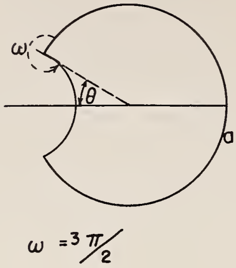
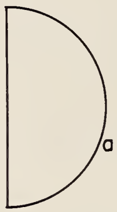


FIGURE 11

$$C_{3\pi/2}(\theta)=\frac{a\sin\theta}{\sqrt{3}}\left[\sqrt{3}-\frac{4}{3}+\frac{1}{2\sin\frac{\theta}{3}\left(\sin\frac{\theta}{3}+\sin\frac{\pi}{3}\right)}+\frac{1}{2\cos\frac{\theta}{3}\left(\cos\frac{\theta}{3}+\sin\frac{\pi}{3}\right)}\right], \tag{1.23}$$

where $a_1=a\tan\theta$ in this case also.

Hemisphere: ($w=3\pi/2$ and $\theta=\pi/2$)



A Hemisphere ($\omega=3\pi/2$ and $\theta=\pi/2$)

FIGURE 12

Placing $\theta=\pi/2$ in preceding case gives

$$C_{3\pi/2}(\pi/2)=2a\left(1-\frac{1}{\sqrt{3}}\right)=.8453a. \tag{1.24}$$

More generally, for $w=\pi/m$, where $m>1$,

$$C_{\pi/m}(\theta)=a+a\sin\theta\sum_{t=1}^{m-1}\left[\frac{1}{\sin\left(\frac{t\pi}{m}+\theta\right)}-\frac{1}{\sin\frac{t\pi}{m}}\right] \tag{1.25}$$

and for $w=2\pi/m$, where $m>2$.

$$C_{2\pi/m}(\theta)=a-\frac{a}{\pi}(\theta-\sin\theta)+a\sin\theta\sum_{t=1}^{m-1}\left[\frac{1-\left(\frac{2t}{m}+\frac{\theta}{\pi}\right)}{\sin\left(\frac{2t\pi}{m}+\theta\right)}-\frac{\left(1-\frac{2t}{m}\right)}{\sin\left(\frac{2\pi t}{m}\right)}\right] \tag{1.26}$$

$C_\pi(\theta)=a$ (complete sphere).

There are finite sums for capacitance when

$$w=\left(\frac{2n-1}{2m}\right)\pi,\ \text{where}\ 1<n\leqslant 2m,$$

and

$$w=\frac{2n}{2m-1}\pi,\ \text{where}\ 1<n<2m-1.$$

2. Inductance and Electromagnetic Force

2.1. General Formulation

If the unit of length is the centimeter and the permeability of the conductors and the surrounding media are unity, the formulas below give inductances in cgs electromagnetic units, that is in 10^{-9} henry. If the electric currents I_1 and I_2 are in cgs electromagnetic units, one of which is 10 amperes, the electromagnetic force is in dynes and torque in dyne-centimeters.

The vector B of magnetic induction is the curl of a vector potential A. If a wire of appreciable cross section, and in the form of a closed circuit, carries a unit current whose volume-density is the vector i_1, the integral over the volume of the wire

$$\iiint (i_1 \cdot A)\, dv_1$$

is a scalar quantity which is called either the self-inductance of this current distribution, or its mutual inductance with the field, according as A is produced by this distribution alone, or entirely by currents other than itself.

If the magnetic permeability μ is 1 everywhere, the vector potential A at any point P_1 due to a unit current in a wire No. 2, whose volume density of current at P_2 is the vector i_2, is

$$A=\iiint \frac{i_2\, dv_2}{\mathfrak{R}}$$

integrated over the volume of wire No. 2, where $\mathfrak{R}$ is the scalar distance from the fixed point P_1 to the point of integration in the volume-element dv_2.

The mutual inductance M between the two current distributions is the repeated volume integral

$$M=\iiint dv_1 \iiint (i_1 \cdot i_2)\frac{dv_2}{\mathfrak{R}}.$$

When the wires shrink to mathematical, closed curves this becomes Neumann's double line integral

$$M=\int ds_1\int \cos(ds_1, ds_2)\frac{ds_2}{\mathfrak{R}},$$

taken completely around both curves.

The self-inductance of a unit current distribution in a wire is

$$L=\iiint dv\iiint\frac{(i\cdot i')}{\mathfrak{R}}dv'.$$

In the case of long straight wires, the formulas below apply for the uniform current distribution.

In the case of wires in the form of circular turns, or in form of helices, the few very accurate formulas given below apply to the "natural" current distribution (current density inversely proportional to the distance from the axis of symmetry).

The distinction between uniform and natural distribution is only of interest for precision measurements.

In the formulas for L and M to be given for parts of a closed circuit (such as L for a long straight wire alone, or M for two parallel ones), these expressions must be understood to represent only such contributions to the multiple integrals for L and M as may be written without specifying the nature of the return circuit whose contribution is, of course, to be evaluated by the same type of integral.

The majority of formulas for L and M that are given below fall into one or other of two classes, in each of which the above volume integrals are reduced to surface integrals over a plane section of the conductor.

a. Axially Symmetric Configurations

The first class is that of axially symmetric conductors for which the surface integrals are taken over a cross section in a plane through the axis of symmetry, say the x-axis, where (x,ρ,ϕ) are cylindrical coordinates. The only component of the current density vector is $i_\phi=i(x,\rho)$, independent of longitude ϕ. The only component of the vector potential A is $A_\phi=A(x,\rho)$. The cylindrical components of the magnetic field are derived from A by $B=\mu H=\text{curl } A$, so

$$\mu H_x=\frac{1}{\rho}D_\rho[\rho A(x,\rho)]$$

$$\mu H_\rho=-D_x A(x,\rho) \text{ and } H_\phi\equiv 0.$$

With $\mu=1$ everywhere

(A) $$L \text{ or } M \text{ is } 2\pi\iint \rho' i(x',\rho')A(x',\rho')dx'd\rho',$$

where

$$\iint i(x,\rho)\,dx\,d\rho=1 \text{ (unit current).}$$

The surface integrals are taken over an axial section in the (x,ρ) half-plane. The symbol $\mathfrak{R}$ has been used to denote the distance (in space) between two points $P(x,\rho,\phi)$ and $P'(x',\rho',\phi')$. We may designate by R this distance when the points are in the same axial plane $(\phi=\phi')$

$$R^2=(x-x')^2+(\rho-\rho')^2$$

$$\mathfrak{R}^2=(x-x')^2+\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')=2\rho\rho'\left[1+\frac{R^2}{2\rho\rho'}-\cos(\phi-\phi')\right].$$

There is the known Fourier series

(B)
$$\frac{1}{\mathfrak{R}}=\frac{2}{\pi\sqrt{\rho\rho'}}\sum_{n=0}^{\infty}\epsilon_n Q_{n-\frac{1}{2}}\left(1+\frac{R^2}{2\rho\rho'}\right)\cos n(\phi-\phi'),$$

where $\epsilon_0=\frac{1}{2}$ and $\epsilon_n=1$ of $n>0$, and $Q_{n-\frac{1}{2}}$ is a Legendre function of the second kind with parameter $n-\frac{1}{2}$. Its reduction to elliptic integrals is given in section 4.

Equation (B) is equivalent to

(B′)
$$\int_{-\pi}^{\pi}\frac{\cos n(\phi-\phi')}{\mathfrak{R}}\,d\phi'=\frac{2}{\sqrt{\rho\rho'}}Q_{n-\frac{1}{2}}\left(1+\frac{R^2}{2\rho\rho'}\right).$$

Since the only cylindrical component of current density is $i_\phi=i(x,\rho)$ independent of ϕ, the only component of vector potential will be $A_\phi=A(x,\rho)$ independent of ϕ. Hence, it would be sufficient to evaluate A for $\phi=0$.

However, the volume integral defining A is the vector equation

$$A=\iiint\frac{i(x',\rho')\,dv'}{\mathfrak{R}}=\iint\rho'\,d\rho'\,dx'\int_{-\pi}^{\pi}\frac{i_\phi(x',\rho')\,d\phi'}{\mathfrak{R}}.$$

This integral is the sum of many vectors that are not parallel, so that we may use rectangular coordinates, $y=\rho\cos\phi$ and $z=\rho\sin\phi$, and write

$$i_y=-i_\phi(x',\rho')\sin\phi' \text{ and } i_z=i_\phi(x',\rho')\cos\phi',$$

so

$$A_y(x,\rho)=-\iint \rho' i_\phi(x',\rho')d\rho' dx' \int_{-\pi}^{\pi} \frac{\sin\phi' d\phi'}{\Re}$$

$$A_z(x,\rho)=\iint \rho' i_\phi(x',\rho')d\rho' dx' \int_{-\pi}^{\pi} \frac{\cos\phi' d\phi'}{\Re}$$

Hence

$$A_\phi=A(x,\rho)=-A_y(x,\rho)\sin\phi+A_z(x,\rho)\cos\phi=\iint \rho' i(x',\rho')d\rho' dx' \int_{-\pi}^{\pi} \frac{\cos(\phi-\phi')d\phi'}{\Re}$$

Consequently, by eq (B′) with $n=1$,

(C) $$A_\phi=A(x,\rho)=\frac{2}{\sqrt{\rho}}\iint_S \sqrt{\rho'}\, i(x',\rho')Q_{1/2}\left(1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}\right)dx' d\rho',$$

so that A satisfies

(C′) $$\left(D_x^2+D_\rho^2+\frac{1}{\rho}D_\rho-\frac{1}{\rho^2}\right)A=\begin{cases}-4\pi i(x,\rho) & \text{in } S\\ 0 & \text{outside } S\end{cases}$$

The integral in (C) is taken over any plane axial section S of the conductor, which may be of any shape. Its self-inductance L is therefore

(D) $$L=2\pi\iint_S i(x,\rho)\cdot\rho A(x,\rho)dx d\rho$$

$$=4\pi\iint_S \rho^{1/2} i(x,\rho)dx d\rho\iint_S \rho'^{1/2} i(x',\rho')Q_{1/2}\left(1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}\right)dx' d\rho',$$

where

$$\iint i(x,\rho)dx d\rho=1.$$

Also from (C), the mutual inductance M between two coaxial wires with any shapes or size of axial sections S_1 and S_2 is

(E) $$M=4\pi\iint \rho_1^{1/2} i_1(x_1,\rho_1)dx_1 d\rho_1\iint \rho_2^{1/2} i_2(x_2,\rho_2)Q_{1/2}\left(1+\frac{(x_1-x_2)^2+(\rho_1-\rho_2)^2}{2\rho_1\rho_2}\right)dx_2 d\rho_2,$$

where

$$\iint i_1(x_1,\rho_1)dx_1d\rho_1=\iint i_2(x_2,\rho_2)dx_2d\rho_2=1.$$

Letting both sections shrink to points gives

$$\text{(F)} \qquad M=4\pi\sqrt{a_1a_2}\,Q_{\frac{1}{2}}\left(1+\frac{(x_1-x_2)^2+(a_1-a_2)^2}{2a_1a_2}\right)$$

as the mutual inductance of two coaxial circular current filaments of radii a_1 and a_2, in the planes x_1 and x_2 (eq (2.1), page 1). See section 4 for the evaluation of the functions $Q_{n-\frac{1}{2}}$ in terms of elliptic integrals.

If the section S shrinks to a point, eq (C) gives the vector potential $A(x,\rho)$ at any point $P(x,\rho)$ in space, that is produced by unit circular current of radius a, in the plane $x=0$, and coaxial with the x-axis

$$A(x,\rho)=2\sqrt{\frac{a}{\rho}}\,Q_{\frac{1}{2}}\left(1+\frac{x^2+(\rho-a)^2}{2a\rho}\right)=2\sqrt{\frac{a}{\rho}}\left[2\frac{(K-E)}{k}-kK\right],$$

where

$$k^2=\frac{4a\rho}{x^2+(\rho+a)^2};$$

k is the modulus of the complete elliptic integrals K and E.

The cylindrical components of the magnetic field are given by $H_x(x,\rho)=1/\rho D_\rho(\rho A)$ and $H_\rho(x,\rho)=-D_xA$, so

$$H_x(x,\rho)=\frac{1}{\sqrt{x^2+(\rho+a)^2}}\left[K-E+\frac{2a(a-\rho)}{x^2+(a-\rho)^2}E\right]$$

$$H_\rho(x,\rho)=\frac{4ax}{[x^2+(\rho+a)^2]^{3/2}}\left[\frac{E}{1-k^2}-2\left(\frac{K-E}{k^2}\right)\right],$$

which apply for any point (x,ρ).

b. Cylindrical Configurations

The other class of formulas may be described (with rectangular coordinates x,y,z) as applying to a unit current with uniform current density $i_x=i_y=0$, $i_z=i(x,y)$ (independent of z). The total current $+1$ flows in the first conductor with cross section S, its density being $i_1=1/S$. The return current density in the second is $i_2=-1/S_2$. The self-inductance "per unit length of the line" is denoted by L/l, which means the self-inductance of two cylinders having the same two end planes, these planes being separated by one unit of length.

$$L/l = \frac{1}{S_1}\iint A\,dS_1 - \frac{1}{S_2}\iint A\,dS_2,$$

where A is the potential of both distributions. When $\mu = 1$ everywhere, the value of A at any point xy in the plane is (where A denotes A_z, the only component of A)

$$A(x,y) = \frac{-2}{S_1}\iint \log R\,dS_1 + \frac{2}{S_2}\iint \log R\,dS_2,$$

where R is the distance from a point of integration P_1 in dS_1, (or from P_2 in dS_2) to the general point $P(x,y)$ in the same xy plane. This gives

$$L/l = \frac{-2}{S_1^2}\iint dS_1 \iint \log R\,dS_1' - \frac{2}{S_2^2}\iint dS_2 \iint \log R\,dS_2' + \frac{4}{S_1 S_2}\iint dS_1 \iint \log R\,dS_2.$$

The first two integrals are frequently designated by L_1 and L_2, respectively, the third by $-2M_{12}$, but the separate integrals only have a meaning with reference to this equation of a "closed", or return, circuit. With this understanding, the self-inductance L/l per unit length of the line is written

(G) $$L/l = -2M_{12}/l + L_1/l + L_2/l = 2\left[2 \log D_{12} - \log D_{11} - \log D_{22}\right],$$

where

(H) $$\begin{cases} \log D_{12} = \dfrac{1}{S_1 S_2}\displaystyle\iint_{S_1} dS_1 \iint_{S_2} \log R\,dS_2 \\ \log D_{11} = \dfrac{1}{S_1{}^2}\displaystyle\iint_{S_1} dS_1 \iint_{S_1} \log R\,dS_1, \end{cases}$$

and similarly for D_{22}.

This D_{12} defined by the repeated surface integral over the two coplanar areas S_1 and S_2 is called the g.m.d., or geometric-mean-distance, of the area S_1 from S_2, and D_{11} the geometric-mean-distance of the area S_1 from itself. The definition is consistent with the extension to the g.m.d. of one curve from another or of the g.m.d. of the line from itself.

In the case of a return circuit of two parallel wires whose circular sections have radii a_1 and a_2, it is easily found that

(I) $$\log D_{11} = \log a_1 - 1/4, \quad \log D_{22} = \log a_2 - 1/4,$$

and

(J) $\log D_{12}=\log b$, where b is the distance between centers.

The general formula (G) for the inductance L/l per unit length of the line, $L/l=2[2\log D_{12}-\log D_{11}-\log D_{22}]$ gives eq (2.44).

For a tubular conductor whose cross section is an annular area with inner radius a_1 and outer A_1, it is found without difficulty that

(K) $$\log D_{11}=\log A_1-\frac{a_1^2}{2(A_1^2-a_1^2)}\left[\frac{2a_1^2}{A_1^2-a_1^2}\log\left(\frac{A_1}{a_1}\right)-1\right]-\tfrac{1}{4}$$

When $a_1\to A_1$, the tube becomes infinitely thin and this approaches the finite limit $\log D_{11}=\log A_1$, so that A_1 is the g.m.d. of the perimeter of the circle from itself.

When the current in tube No. 1 returns in a larger tube, No. 2, coaxial with it, whose annular section has inner radius a_2 and outer A_2, it is relatively simple to find that the g.m.d. between the two annule is given for $0<a_1<A_1<a_2<A_2$ by

(L) $$\log D_{12}=\frac{1}{A_2^2-a_2^2}[A_2^2\log A_2-a_2^2\log a_2]-\tfrac{1}{2}.$$

Formula (G), $L/l=2[2\log D_{12}-\log D_{11}-\log D_{22}]$ in this case leads to eq (2.46) by (K) and (L).

In checking for numerical errors, it may be noticed that the formula (G) for the self-inductance per unit length of a line (a return circuit) will be dimensionless. This is not true of the constituents $\log D_{11}$ and $\log D_{12}$, etc., as they do not involve logarithms of the ratio of two lengths.

In getting g.m.d., use may be made of any formal analogies to the logarithmic potentials of electrostatic distributions, for the same first integral occurs in both problems.

The logarithmic potential V of an endless cylinder of any cross section S with unit density per unit length perpendicular to the plane of S is

$$V(x,r)=-2\iint_S \log R\,dx'dy'.$$

For example, when the section S is a circular area, or an annulus, V is the same at outside points as if the charge were all concentrated at the center. At points inside, a simple law prevails. This may be used to check the g.m.d. given above for circles and annuli.

When the total charge Q per unit length in finite space is zero, the logarithmic potential (like L/l for a return circuit) is dimensionless as to length. Hence the capacitance per unit length of endless cylinders, as in eq (1.1) to (1.15), involves logarithms of the ratio of two lengths. But the potential of a circular cylinder with charge Q has the value $2Q\log r$ at outside points. This is not dimensionless.

A final example [15] may be quoted, the g.m.d. of a rectangular area from itself. If the width of section is w and breadth b,

(M) $$-4 \log D_{11}=-2 \log(w^2+b^2) \frac{w^2}{3b^2} \log\left(\frac{w^2+b^2}{w^2}\right)+\frac{b^2}{3w^2} \log\left(\frac{b^2+w^2}{b^2}\right)$$

$$-\frac{8}{3}\left(\frac{w}{b} \tan^{-1}\frac{b}{w}+\frac{b}{w} \tan^{-1} \frac{w}{b}\right)+\frac{25}{3}$$

$$=\frac{1}{3\delta^2} F(\delta)-\frac{4\pi}{3}\delta-4 \log w+\frac{25}{3},$$

where $\delta=b/w$, and

(N) $$F(x)=-2x^4 \log x+(1-6x^2+x^4)\log(1+x^2)-8x(1-x^2)\tan^{-1}x.$$

Since the expression for $\log D_{11}$ is symnmetric in b and w, this suggests the identity in x that is easily verified.

(O) $$F(x)=x^4F\left(\frac{1}{x}\right)-12x^2 \log x-4\pi x(1-x^2),$$

so that, from a power series for $F(x)$ in powers of x valid, when $x<1$, we get by this, the series for $x>1$. These equations, (M), (N), (O), are used in deriving eq (2.47).

If two or more long parallel, cylindrical conductors all carry current in the same direction with the same uniform current density, they are effectively one conductor of cross section S. If the sections are the coplanar, nonoverlapping areas S_1, S_2, S_3, etc., then $S=S_1+S_2+S_3\ldots$. For example, with three such sections the g.m.d. of the compound area S from itself is given by

(P) $$(S_1+S_2+S_3)^2 \log D_{11}=\iint dS \iint \log R dS'$$

$$=\iint dS_1 \iint \log R dS_1'+\iint dS_2 \iint \log R dS_2'+\iint dS_3 \iint \log R dS_3$$

$$+2\iint dS_1 \iint \log R dS_2+2\iint dS_2 \iint \log R dS_3+2\iint dS_3 \iint \log R dS_1$$

$$=\log(D_{11}^{S_1^2})\cdot(D_{22}^{S_2^2})(D_{33}^{S_3^2})(D_{12}^{2S_1S_2})(D_{23}^{2S_2S_3})(D_{31}^{2S_3S_1}).$$

The generalization of this is not difficult, when each conductor carries a uniform but different current density. Weighting factors are introduced.

2.2. Circular Filaments and Circular Turns of Wire

a. Coaxial Circular Filaments [16]

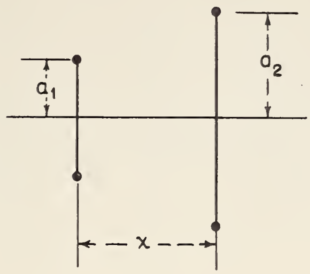


Figure 13

Their mutual inductance M is

$$M=4\pi\sqrt{a_1a_2}\left[\frac{2(K-E)}{k}-kK\right]=4\pi\sqrt{a_1a_2}\,Q_{1/2}\left(\frac{2}{k^2}-1\right), \tag{2.1}$$

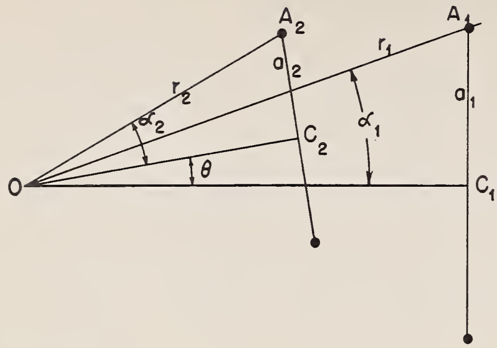
where the modulus k of the complete elliptic integrals is given by

$$k^2=\frac{4a_1a_2}{x^2+(a_1+a_2)^2}. \tag{2.2}$$

Their force of attraction is $X=-I_1I_2\partial M/\partial x$ or

$$X=\frac{I_1I_2xk}{\sqrt{a_1a_2}}\left[\frac{2+k^2}{1-k^2}E-2K\right]. \tag{2.3}$$

b. Circles Whose Axes Intersect [17]



$\mu_1 = \cos\alpha_1 \quad (0 \leq \alpha_1 \leq \pi)$
$\mu_2 = \cos\alpha_2 \quad (0 \leq \alpha_2 \leq \pi)$
$\mu = \cos\theta \quad (0 \leq \theta \leq \pi)$

FIGURE 14

$$\left.\begin{aligned} M &= 4\pi^2(1-\mu_1^2)(1-\mu_2^2)\sum_{n=1}^{\infty}\frac{r_2^{n+1}}{r_1^n}\frac{P_n'(\mu_1)P_n'(\mu_2)P_n(\mu)}{n(n+1)} \quad \text{if } r_2<r_1 \\ &= 4\pi^2(1-\mu_1^2)(1-\mu_2^2)\sum_{n=1}^{\infty}\frac{r_1^{n+1}}{r_2^n}\frac{P_n'(\mu_1)P_n'(\mu_2)P_n(\mu)}{n(n+1)} \quad \text{if } r_2>r_1 \end{aligned}\right\} \tag{2.4}$$

whatever the values of a_2/a_1. $P_n(\mu)$ is the Legendre polynomial and $P_n'(\mu)=(d/d\mu)P_n(\mu)$.

In the special case where the center C_2 of circle 2 coincides with O, r_2 becomes a_2 and α_2 becomes $\pi/2$, and (2.4) reduces to

$$M=2\pi^{3/2}a_2(1-\mu_2^2)\sum_{n=1}^{\infty}(-1)^n\left(\frac{a_2}{r_1}\right)^{2n+1}\frac{\Gamma(n+\frac{1}{2})}{(n+1)!}P'_{2n+1}(\mu_1)P_{2n+1}(\mu). \tag{2.4'}$$

The torque T acting on either circle, tending to reduce the angle θ between their axes is

$$T=-\frac{\partial M}{\partial\theta}=\sin\theta\frac{\partial M}{\partial\mu}=2\pi^{3/2}a_2\sin\theta(1-\mu_1^2)\sum_{n=1}^{\infty}(-1)^n\left(\frac{a_2}{r_1}\right)^{2n+1}\frac{\Gamma(n+\frac{1}{2})}{(n+1)!}P'_{2n+1}(\mu_1)P'_{2n+1}(\mu). \tag{2.5}$$

These converge for all values of r_1, if $a_2<a_1$. M is positive when the currents circulate in the same sense around their axes $\overline{OC_1}$ and $\overline{OC_2}$.

c. Two Concentric Circles (Not Coaxial)

(Special case of eq (2.4$'$))

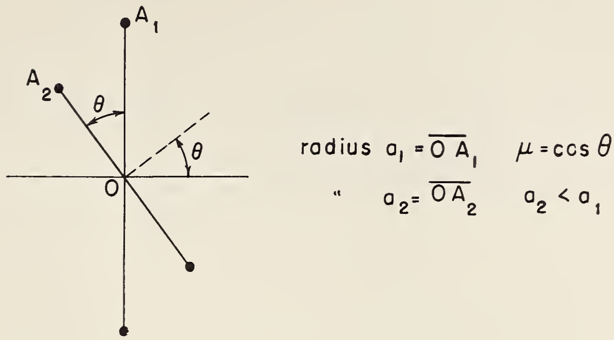


FIGURE 15

$$M=\frac{4\pi a_2^2}{a_1}\sum_{n=0}^{\infty}\left(\frac{a_2}{a_1}\right)^{2n}\frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!\,(n+1)!}P_{2n+1}(\mu). \tag{2.6}$$

$$T=\frac{4\pi I_1 I_2 a_2^2}{a_1}\sin\theta\sum_{n=0}^{\infty}\left(\frac{a_2}{a_1}\right)\frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!\,(n+1)!}P'_{2n+1}(\mu). \tag{2.7}$$

Axes Parallel ($\theta=0, \mu=1$), $T=0$

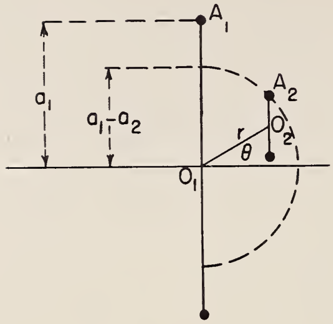
$$M=\frac{4\pi a_2^2}{a_1}\sum_{n=0}^{\infty}\left(\frac{a_2}{a_1}\right)^{2n}\frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!\,(n+1)!}. \tag{2.8}$$

Axes Perpendicular ($\theta=\pi/2, \mu=0$) $M=0$

$$T=\frac{8I_1 I_2\sqrt{\pi}a_2^2}{a_1}\sum_{n=0}^{\infty}\left(\frac{a_2}{a_1}\right)^{2n}\frac{\Gamma(n+\frac{1}{2})}{(n+1)!}\left[\frac{\Gamma(n+\frac{3}{2})}{n!}\right]^2. \tag{2.9}$$

d. Two parallel circles [18]

Case where $0<r<a_1-a_2$.



$\mu = \cos\theta$

$r = \overline{O_1O_2}$

$a_1 = \overline{O_1A_1}$

$a_2 = \overline{O_2A_2}$

FIGURE 16

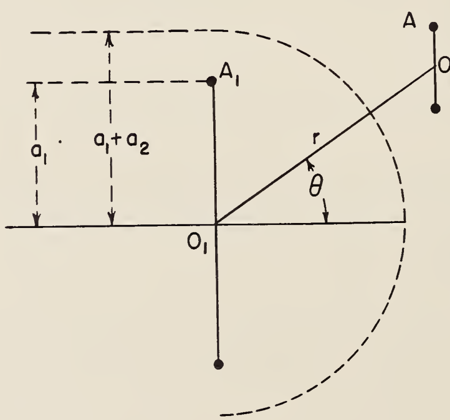
$$M=\frac{4\pi^{3/2}a_2^2}{a_1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{r}{a_1}\right)^{2n}\frac{\Gamma(n+3/2)}{n!}F_nP_{2n}(\mu), \tag{2.10}$$

where $F_n=F(n+1/2, n+3/2, 2;\ a^2{}_2/a^2{}_1)$ (hypergeometric series).

If the circles are coplanar, $(\theta=\pi/2, \mu=0)$ and

$$P_{2n}^{(0)}=(-1)^n\frac{\Gamma(n+1/2)}{n!}.$$

Case where $r>a_1+a_2$



$\mu = \cos\theta$

$r = \overline{O_1O_2}$

$a_1 = \overline{O_1A_1}$

$a_2 = \overline{O_2A_2}$

FIGURE 17

$$M=\frac{4\pi^{3/2}a_2^2}{a_1}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{a_1}{r}\right)^{2n+1}\frac{\Gamma(n+1/2)}{(n-1)!}F_nP_{2n}(\mu), \tag{2.11}$$

where $F_n=F(-n, 1-n, 2;\ a_2^2/a_1^2)$.

For equal radii $a_1=a_2=a$ and $r>2a$

$$M=2\pi a \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2a}{r}\right)^{2n+1} \frac{\Gamma(n+½)\Gamma(n+½)}{(n-1)!(n+1)!} P_{2n}(\mu). \tag{2.12}$$

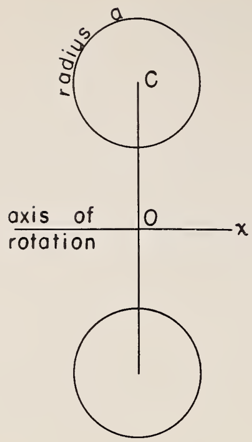
For two equal circles, coplanar and external, this becomes

$$M=-2\pi a \sum_{n=1}^{\infty} \left(\frac{2a}{r}\right)^{2n+1} \frac{\Gamma^3(n+½)}{(n-1)!\,n!\,(n+1)!} \tag{2.13}$$

(The plus sign would apply when the currents circulate in opposite senses with respect to the normal to their plane.) The torque is zero. The force of repulsium along their line of centers is $F=+I_1 I_2 \partial M/\partial r$, or

$$F=+2\pi I_1 I_2 \sum_{n=1}^{\infty} \left(\frac{2a}{r}\right)^{2n+2} \frac{\Gamma(n+{}^3/_2)\Gamma^2(n+½)}{(n-1)!\,n!\,(n+1)!} \tag{2.14}$$

e. **Self-Inductance of a Circular Turn of Wire**



ρ = distance of a point from x axis
a = radius of generating circle
A = $\overline{OC}$

FIGURE 18

Current density of the unit current is

$$i_\phi \doteq \frac{1}{\pi a^2 F}\left(\frac{\rho}{A}\right)^b \quad \text{(any } b\text{)},$$

where

$$F = F\left(\frac{-b}{2}, \frac{1-b}{2}, 2; \frac{a^2}{A^2}\right) \doteq 1 + \frac{b(b-1)}{8}\,\frac{a^2}{A^2} \text{ to 2d order}$$

$$L = 4\pi A\cdot\left[1+(2b+1)\frac{a^2}{8A^2}\right]\log\frac{8A}{a} - \frac{7}{4} + \frac{(b-1)(b-2/3)}{16}\,\frac{a^2}{A^2} + \text{zero}\left(\frac{a^3}{A^3}\log\frac{A}{a}\right). \tag{2.15}$$

For uniform current distribution $b=0$.
For "natural" current distribution $b=-1$.
For $b=-5/2$, the magnetic field outside the wire is exactly the same as if the unit current were concentrated in a circular filament of radius $\sqrt{A^2-a^2}$.

f. Self-Inductance of a Circular Turn of Wire Near a Magnetic Medium

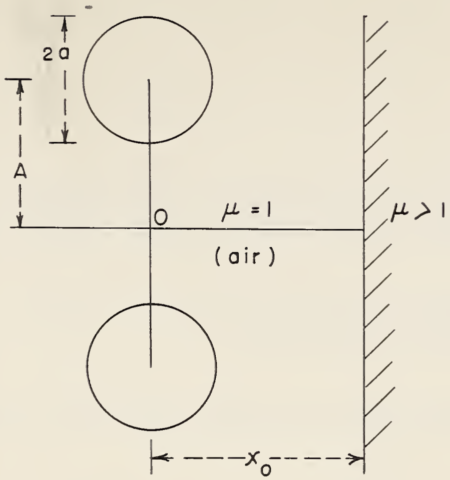


FIGURE 19

$$L = L_{air} + \frac{\mu-1}{\mu+1} 4\pi A Q_{1/2} \left(\frac{2}{k^2} - 1 \right)$$

$$= L_{air} + \frac{\mu-1}{\mu+1} 4\pi A \left[\frac{2(K-E)}{k} - kK \right]. \qquad (2.16)$$

(modulus k) where $k^2 = \dfrac{4Aa}{4x_0^2 + (A+a)^2}$.

L_{air} may be computed by preceding case, (2.15) on the basis $\mu=1$ everywhere.

In correcting the self- or mutual inductance of coils for the effect of thin lead-in wires, the diameter of the wire is important by affecting its self-inductance, but wires may be treated as linear conductors in estimating their mutual inductance.

g. Self-Inductance of a Wire

$$L = 2 \left[l \log \left(\frac{\sqrt{l^2+a^2}+a}{a} \right) - \sqrt{l^2+a^2} + \frac{l}{4} + a \right], \qquad (2.17)$$

where l is its length and a its radius.

h. Mutual Inductance of Two Parallel Wires Having the Same End-Planes

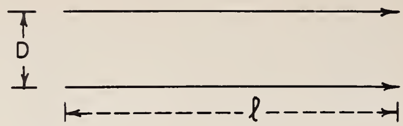


FIGURE 20

$$M=2\left[\, l\log\left(\frac{\sqrt{l^2+D^2}+l}{D}\right)-\sqrt{l^2+D^2}+D\,\right] \tag{2.18}$$

where l is their length, and D the distance between centers. If the currents are in opposite directions, the sign of M is reversed.

i. Mutual Inductance of Two Parallel Wires Not Co-terminous

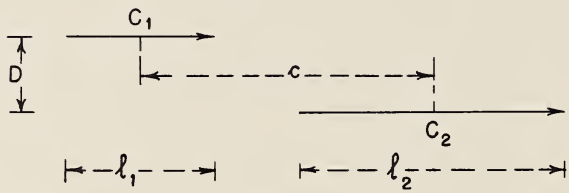


FIGURE 21

$c=\overline{C_1C_2}$ where C_1 and C_2 are centers of the wires

$$M=w\left(c+\frac{l_2+l_1}{2}\right)+w\left(c-\frac{l_2+l_1}{2}\right)-w\left(c+\frac{l_2-l_1}{2}\right)-w\left(c-\frac{l_2-l_1}{2}\right), \tag{2.19}$$

where

$$w(x)=|x|\log\left[\frac{\sqrt{x^2+D^2}+|x|}{D}\right]-\sqrt{x^2+D^2},$$

so that $w(x)$ is an even function of x.

This holds for collinear wires ($D=0$) if they do not overlap. The sign of M is reversed if the currents have opposite directions.

j. Mutual Inductance of Two Equal Rectangles Lying in Parallel Planes

(One is the perpendicular projection of the other.) The distance between their planes is d, the length and breath of each is a and b, respectively.

Neumann's formula is

$$\frac{M}{4}=a\log\left[\frac{(a+\sqrt{a^2+d^2})}{(a+\sqrt{a^2+b^2+d^2})}\,\frac{\sqrt{b^2+d^2}}{d}\right]+b\log\left[\frac{(b+\sqrt{b^2+d^2})}{(b+\sqrt{a^2+b^2+d^2})}\,\frac{\sqrt{a^2+d^2}}{d}\right]$$
$$+2\left[\sqrt{a^2+b^2+d^2}-\sqrt{a^2+d^2}-\sqrt{b^2+d^2}+d\right]. \qquad (2.20)$$

k. Self-Inductance of a Rectangle

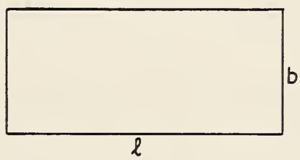


FIGURE 22

a = radius of wire, l = length of rectangle, b = breadth

$$L=4\left\{(b+l)\log\left[\frac{\sqrt{4(b+l)^2+a^2}+a}{a}\right]-b\cdot\log\left[\frac{\sqrt{b^2+l^2}+b}{l}\right]-l\log\left[\frac{\sqrt{b^2+l^2}+l}{b}\right]\right.$$
$$\left.+2\sqrt{b^2+l^2}+\frac{a}{2}-\frac{3}{4}(b+l)-\tfrac{1}{2}\sqrt{4(b+l)^2+a^2}\right\}. \qquad (2.21)$$

2.3. Concentric Solenoids (Current Sheets) [20]

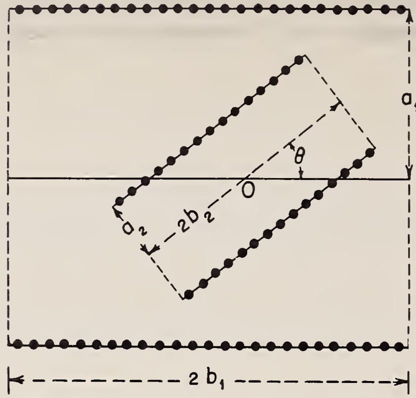


FIGURE 23

$r_1^2 = a_1^2 + b_1^2$ $\quad$ $r_2^2 = a_2^2 + b_2^2$

$\mu = b_1/r_1$ $\quad$ $\mu = b_2/r_2$

$\mu = \cos.\theta$

N_1 and N_2 are total numbers of turns

$$M=4\pi\left(\frac{N_1}{2b_1}\right)N_2\pi a_2^2\mu_1\left\{\mu-\frac{2(1-\mu_1^2)}{\mu_1\mu_2}\sum_{s=1}^{\infty}\left(\frac{r_2}{r_1}\right)^{2s}\frac{P_{2s+1}(\mu)P'_{2s}(\mu_1)P'_{2s+2}(\mu_2)}{2s(2s+1)(2s+2)(2s+3)}\right\}. \quad (2.22)$$

The series is relatively small compared to μ when $a_1/2b_1$ is small. $P_s(\mu)$ is Legendre polynomial and $P'_s(\mu)=d[P_s(\mu)]/d\mu$.

If the coils carry currents of strength I_1 and I_2 (in cgs electromagnetic units of current, the torque on either coil, tending to decrease θ, is $T=-I_1I_2\partial M/\partial\theta = I_1I_2\sin\theta\,\partial M/\partial\mu$, or

$$T=4\frac{N_1I_1}{2b_1}N_2I_2\pi a_2^2\mu_1\sin\theta\left\{1-2\frac{(1-\mu_1^2)}{\mu_1\mu_2}\sum_{s=1}^{\infty}\left(\frac{r_2}{r_1}\right)^{2s}\frac{P'_{2s+1}(\mu)P'_{2s}(\mu_1)P'_{2s+2}(\mu_2)}{2s(2s+1)(2s+2)(2s+3)}\right\}. \quad (2.23)$$

Case a. Axes perpendicular ($\theta=\pi/2$, $\mu=0$)

$M=0$

$$T=4\pi\left(\frac{N_1I_1}{2b_1}\right)N_2I_2\pi a_2^2\mu_1\left\{1+\frac{1-\mu_1^2}{\mu_1\mu_2\sqrt{\pi}}\sum_{s=1}^{\infty}(-1)^{s+1}\left(\frac{r_2}{r_1}\right)^{2s}\frac{\Gamma(s+\tfrac{1}{2})P'_{2s}(\mu_1)P'_{2s+2}(\mu_2)}{\Gamma(s+2)(2s+2)(2s+3)}\right\} \quad (2.24)$$

Case b. Axes parallel ($\theta=0$, $\mu=1$)

$T=0$

$$M=4\pi\frac{N_1N_2\pi a_2^2\mu_1}{2b_1}\left\{1-\frac{2(1-\mu_1^2)}{\mu_1\mu_2}\sum_{s=1}^{\infty}\left(\frac{r_2}{r_1}\right)^{2s}\frac{P'_{2s}(\mu_1)P'_{2s+2}(\mu_2)}{2s(2s+1)(2s+2)(2s+3)}\right\} \tag{2.25}$$

This may be computed in finite terms. (See coaxial coils, eq (2.30).)

2.4. Self-Inductance of a Cylindrical Current Sheet [21]

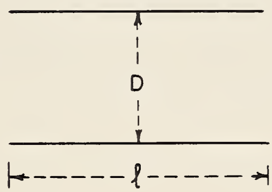


FIGURE 24

The sheet consists of N complete circular turns of thin tape without insulating space between them. Their diameter is D; the total length of the cylinder is l.

$$L_s=\frac{4\pi N^2}{3}\sqrt{l^2+D^2}\left[K-E+\frac{D^2}{l^2}(E-k)\right], \tag{2.26}$$

where $k=D/\sqrt{l^2+D^2}$ the modulus of the complete elliptic integral K and E. The complementary modulus is $k^1=l/\sqrt{l^2+D^2}$.

2.5. Self-Inductance of a Helical Wire [22]

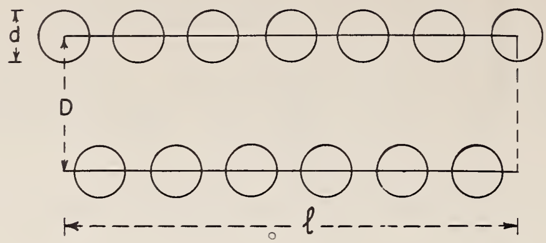


FIGURE 25

Centers of wire on a cylinder of diameter D
N complete turns; diameter of wire is d.

The length l of the equivalent sheet, is the distance from the center of the wire at start of first turn to center at end of last turn.

$$L=L_s+l\left[\log\left(\frac{1+k'}{1-k'}\right)+k'\log 4\right]$$

$$+\pi D\left\{2N\left[\frac{1}{4}-\log\left(\frac{N\pi d}{l}\right)\right]+\frac{1}{3}\log\left(\frac{N\pi D}{l}\right)-\frac{4}{\pi^2}\left(\frac{E}{k}-1\right)\left[1+\tfrac{1}{2}\left(\frac{N\pi d}{2l}\right)^2\right]\right.$$

$$\left.-\frac{2}{3}\left[\frac{K-E}{k}-\frac{kK}{2}\right]-\frac{k'}{2k}\left(1-\frac{k'}{k}\sin^{-1}k\right)\right\}. \qquad (2.27)$$

This takes account of the relatively small axial component of current. L_s is given by (2.26), moduli k, k', as in (2.26).

2.6. Bifilar Mutual Inductor [23]

Primary and secondary are helical wires identical in form, the turns of one midway between those of the other. Two cases are M_0 and M_π. M_0 is their mutual inductance when the second helix is displaced axially from the first by one-half the pitch. When the second is displaced 180° in azimuth from the first, but with its extremities in the same end-plane as the first, their mutual inductance is designated by M_π. The principal part of either is L_s given by (2.26), the self-inductance of the current sheet equivalent to primary or secondary. The moduli k and k' are the same as in (2.26).

$$M_0=L_s+l\left[\log\frac{1+k'}{1-k'}+k'\log 4\right]$$

$$-\pi D\left\{N\log 4+\frac{1}{6}\log\left(\frac{4ND}{l}\right)+\frac{4}{\pi^2}\left(\frac{E}{k}-1\right)\left[1+\tfrac{1}{2}\left(\frac{N\pi d}{2l}\right)^2\right]-\frac{1}{3}\left(\frac{K-E}{k}-\frac{kK}{2}\right)\right.$$

$$\left.+\frac{k'}{2k}\left(1-\frac{k'}{k}\sin^{-1}k\right)\right\}. \qquad (2.28)$$

$$M_\pi = L_s + l\left[\log\frac{1+k'}{1-k'} + k' \log 4\right]$$

$$-\pi D\left\{N \log 4 + \frac{1}{6}\log\left(\frac{4ND}{l}\right) + \frac{4}{\pi^2}\left(\frac{E}{k} - 1\right)\left[1 + {}^1\!/_2\left(\frac{N\pi d}{2l}\right)^2\right]\right.$$

$$\left. + \frac{2}{3}\left[\frac{K-E}{k} - \frac{kK}{2}\right] + \frac{1}{2k}\left[1 - \frac{1}{2k}\log\left(\frac{1+k}{1-k}\right)\right]\right\}. \qquad (2.29)$$

2.7. Coaxial Current Sheets [24]

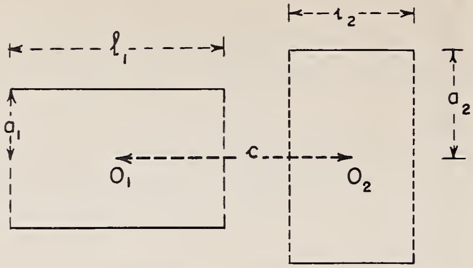


Figure 26

Total number of turns N_1 and N_2

$$M=\frac{2\pi N_1 N_2}{l_1 l_2}\left\{ w\left(c+\frac{l_2+l_1}{2}\right)+w\left(c-\frac{l_2+l_1}{2}\right)-w\left(c+\frac{l_2-l_1}{2}\right)-w\left(c-\frac{l_2-l_1}{2}\right)\right\}. \tag{2.30}$$

The force of attraction (in dynes) is $X=-I_1 I_2 \partial M/\partial c$

$$X=\frac{2\pi N_1 I_1 N_2 I_2}{l_1 l_2}\left\{ w'\left(c+\frac{l_2-l_1}{2}\right)+w'\left(c-\frac{l_2-l_1}{2}\right)-w'\left(c+\frac{l_2+l_1}{2}\right)-w'\left(c-\frac{l_2+l_1}{2}\right)\right\}, \tag{2.31}$$

where

$$w(x)=xw'(x)+\frac{8(a_1 a_2)^{3/2}}{3k}\left[K-\left(\frac{2}{k^2}-1\right)(K-E)\right] \tag{2.32}$$

This is an even function of x. Its derivatives $w'(x)$ is an odd function of x, vanishing with x, and given by

$$w'(x)=\frac{2x\sqrt{a_1 a_2}}{k}[K-E]\pm|a_1^2-a_2^2|\left[KE(\theta,k')-(K-E)F(\theta,k')-\frac{\pi}{2}\right] \tag{2.33}$$

the + sign is for x positive, -, for x negative.

The complete elliptic integrals K and E have modulus k, where

$$k^2=\frac{4a_1 a_2}{x^2+(a_1+a_2)^2}. \tag{2.34}$$

The incomplete integrals $F(\theta,k')$ and $E(\theta,k')$ have the complementary modulus $k'=\sqrt{1-k^2}$. Their amplitude θ is computed by

$$\sin\theta=\sqrt{\frac{1+\left(\frac{x}{a_1+a_2}\right)^2}{1+\left(\frac{x}{a_1\ a_2}\right)^2}},\quad \text{where}\quad 0<\theta<\frac{\pi}{2}.$$

The bracket with factor $\pm a_1^2 - a_2^2$ vanishes when $x=0$. If the coils are also concentric ($c=0$), the force vanishes, and M becomes

$$M=\frac{4\pi N_1 N_2}{l_1 l_2}\left[w\left(\frac{l_2+l_1}{2}\right) - w\left(\frac{l_2-l_1}{2}\right)\right] \tag{2.36}$$

Another special case is that in which the second sheet is replaced by a single turn of radius a_2 coaxial with the x-axis in the plane $x=c$. The mutual inductance between the circle and sheet 1 is

$$M=\frac{2\pi N_1}{l_1'}\left\{ w'\left(c+\frac{l_1}{2}\right) - w'\left(c-\frac{l_1}{2}\right)\right\} \tag{2.37}$$

$$X=\frac{2\pi N_1 I_1 I_2}{l_1}\left\{ w''\left(c-\frac{l_1}{2}\right) - w''\left(c+\frac{l_1}{2}\right)\right\} = \frac{N_1 I_1}{l_1} N_2 (M_1 - M_2), \tag{2.38}$$

where

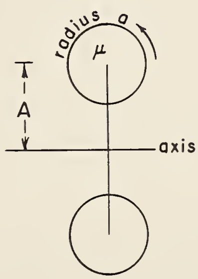
$$w''(x)=4\sqrt{a_1 a_2}\left[\frac{K-E}{k} - \frac{kK}{2}\right], \tag{2.39}$$

so that the circle has mutual inductance M_1 with the nearest circular turn of the sheet, and M_2 with the farthest.

2.8. Toroidal Current Sheets

Current in tape winding of N turns circulates around the core in planes through its axis of symmetry. Permeability of core is μ.

Case a. Core of Circular Section



$$L=\frac{4\pi a^2 \mu N^2}{A+\sqrt{A^2-a^2}} \tag{2.40}$$

FIGURE 27

Case b. Core of Rectangular Section

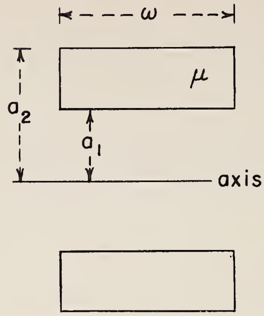


FIGURE 28

$$L = 2\mu\omega N^2 \log \frac{a_2}{a_1} \tag{2.41}$$

2.9. Endless Return-Circuits

(Self-Inductance)

a. Concentric Cable (Special Case of 2.46)

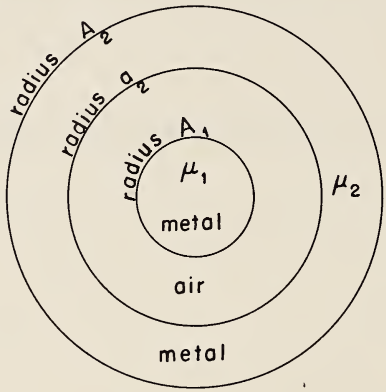


FIGURE 29

μ_1 and μ_2 are magnetic permeabilities. The current goes one way in the central wire and returns in the outer shell. The self-inductance of the line, per unit length, is, for low-frequency or direct current.

$$L/l = \frac{\mu_1}{2} + 2\log\frac{a_2}{A_1} + \frac{\mu_2 A_2^2}{A_2^2 - a_2^2}\left[\frac{2A_2^2}{A_2^2 - a_2^2}\log\left(\frac{A_2}{a_2}\right) - 1\right] - \frac{\mu_2}{2} \tag{2.42}$$

$$L/l = \frac{\mu_2}{2} + 2\log\left(\frac{a_2}{A_1}\right) \text{ if } A_2 = a_2 \tag{2.43}$$

$L/l \sim 2\log a_2/A_1$ for high frequency (see (3.1)).

b. Two Parallel Wires (Nonmagnetic)

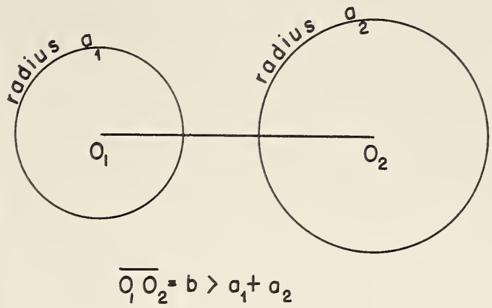


FIGURE 30

The self-inductance per unit length of the line when the unit current is in opposite directions in the two wires, and $\mu=1$ everywhere, is given by the exact formula

$$L/l=1+2\,\log\frac{b^2}{a_1 a_2}\,. \tag{2.44}$$

See (2.45) and (3.3). This is derived as in section 2, eq (I) and (J).

c. Two Parallel Wires of Magnetic Material

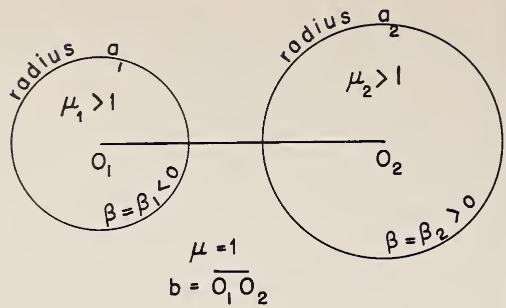


FIGURE 31

$$L/l=\frac{\mu_1+\mu_2}{2}+2\log\frac{b^2}{a_1a_2}$$

$$+2\sum_{n=1}^{\infty}\frac{1}{n(1-\epsilon_1\epsilon_2e^{-n\gamma})}\Big\{\Big[\epsilon_1+(\epsilon_2+2\epsilon_1\epsilon_2)e^{-n\gamma}\Big]e^{-2n\beta_1}$$

$$+\Big[\epsilon_2+(\epsilon_1+2\epsilon_1\epsilon_2)e^{-n\gamma}\Big]e^{-2n\beta_2}$$

$$-2\Big[\epsilon_1+\epsilon_2+\epsilon_1\epsilon_2+\epsilon_1\epsilon_2e^{-n\gamma}\Big]e^{-n\gamma}\Big\}, \qquad (2.45)$$

where

$$\epsilon_1=\frac{\mu_1-1}{\mu_1+1} \text{ and } \epsilon_2=\frac{\mu_2-1}{\mu_2+1} \text{ and } \gamma=2(\beta_1+\beta_2)$$

The positive constants β_1, β_2, and γ are defined in eq (1.13). This γ is the reciprocal of capacity (1.15). When $\mu_1=\mu_2=1$, eq (2.45) reduces to (2.44). (*See also* (3.3) for high frequency in this circuit.)

d. Two Coaxial Tubes

(μ=1) (See section 2, equations (*K*) and (*L*)

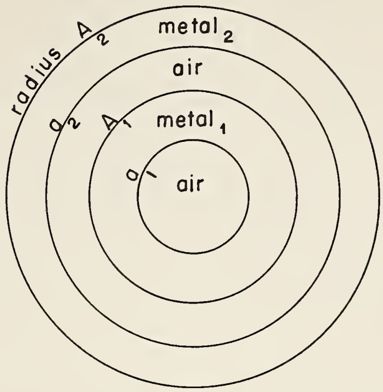


Figure 32

Current goes in opposite directions in the tubes. The inductance L/l per unit length of the line is

$$L/l=2\log\frac{a_2}{A_1}+\frac{A_2^2}{A_2^2-a_2^2}\left[\frac{2A_2^2}{A_2^2-a_2^2}\log\left(\frac{A_2}{a_2}\right)-1\right]+\frac{a_1^2}{A_1^2-a_1^2}\left[\frac{2a_1^2}{A_1^2-a_1^2}\log\left(\frac{A_1}{a_1}\right)-1\right]. \quad (2.46)$$

When $a_1\to 0$, the inner tube becomes a solid wire and this reduces to (2.42) with $\mu_1=\mu_2=1$.

When $a_2\to A_2$, $L/l=\tfrac{1}{2}+\log\frac{A_2}{A_1}+\frac{a_1^2}{A_1^2-a_1^2}\left[\frac{2a_1^2}{A_1^2-a_1^2}\log\left(\frac{A_1}{a_1}\right)-1\right].$

When $a_1\to A_1$, $L/l=-\tfrac{1}{2}+\log\frac{a_2}{A_1}+\frac{A_2^2}{A_2^2-a_2^2}\left[\frac{2A_2^2}{A_2^2-a_2^2}\log\left(\frac{A_2}{a_2}\right)-1\right].$

When $a_1\to A_1$ and $a_2\to A_2$, $L/l=\log\frac{A_2}{A_1}.$

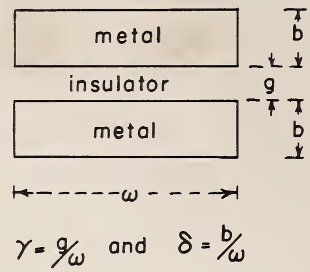


FIGURE 33

Since $D_{11}=D_{22}$ (given by eq (M)), the formula (G) for self-inductance L/l per unit length of line is $L/l=4\log D_{12}-4\log D_{11}$, which leads to

$$L/l=\frac{1}{3\delta^2}[F(\gamma+\delta)+F(\delta)-\tfrac{1}{2}F(\gamma+2\delta)-\tfrac{1}{2}F(\gamma)]+4\pi(\gamma+2\delta/3), \tag{2.47}$$

where as in equation (N)

$$F(x)=-2x^4\log x+(1-6x^2+x^4)\log(1+x^2)-8x(1-x^2)\tan^{-1}x. \tag{2.48}$$

This satisfies the identical relation in x (eq (O)).

$$F(x)=x^4F\left(\frac{1}{x}\right)-12x^2\log x-4\pi x(1-x^2). \tag{2.49}$$

When $x<1$, we find

$$F(x)=2x^4\log\frac{1}{x}-7x^2+\frac{25x^4}{6}+48\sum_{n=3}^{\infty}\frac{(-1)^{n+1}x^{2n}}{2n(2n-1)(2n-2)(2n-3)(2n-4)}. \tag{2.50}$$

This is obtained by use of the series

$$\log(1+x^2)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n} \quad\text{and}\quad x\tan^{-1}x=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{2n-1}.$$

Equation (2.50) might be preferable to (2.48) for computation with thin flat, strips as in the figure, where γ, or δ, or both, are small, that is g, or b, or both, are small compared to the width w.

In the other extreme (g or b or both large compared to w), the expansion of $F(x)$ is required for $x>1$. This is obtained from (2.50) by use of the identity (2.49) which gives for $x>1$

$$F(x)=2(1-6x^2)\log x-7x^2+\frac{25}{6}-4\pi x(1-x^2)+48x^4\sum_{n=3}^{\infty}\frac{(-1)^{n+1}x^{-2n}}{2n(2n-1)(2n-2)(2n-3)(2n-4)}. \tag{2.51}$$

The derivation of (2.47) depends in part upon eq (M), which is

$$-4\log D_{11}=\frac{1}{3\delta^2}F(\delta)-\frac{4\pi}{3}\delta-4\log w+\frac{25}{3}. \tag{2.52}$$

With the same function $F(x)$, it is found that Gray's [26] formula for $4\log D_{12}$ may be written

$$4\log D_{12}=\frac{1}{3\delta^2}[F(\gamma+\delta)-\tfrac{1}{2}F(\gamma+2\delta)-\tfrac{1}{2}F(\gamma)]+4\pi(\gamma+\delta)+4\log w-\frac{25}{3}, \tag{2.53}$$

which, with (2.52), gives (2.47).

The formula of Gray, corrected by Rosa [27] is

$$4w^2b^2\log D_{12}=-\frac{25}{3}w^2b^2+\left[(g+2b)^2\left\{w^2-\frac{(g+2b)^2}{6}\right\}-\frac{w^4}{6}\right]\log[w^2+(g+2b)^2]$$

$$-2\left[(g+b)^2\left\{w^2-\frac{(g+b)^2}{6}\right\}-\frac{w^4}{6}\right]\log[w^2+(g+b)^2]+\left[g^2\left\{w^2-\frac{g^2}{6}\right\}-\frac{w^4}{6}\right]\log[w^2+g^2]$$

minus the same terms with $w=0$,

$$+\frac{4w}{3}\left\{(g+2b)^3\tan^{-1}\frac{w}{g+2b}+w^2(g+2b)\tan^{-1}\frac{g+2b}{w}\right.$$

$$\left.-2\left[(g+b)^3\tan^{-1}\frac{w}{g+b}+w^2(g+b)\tan^{-1}\frac{g+b}{w}\right]+g^3\tan^{-1}\frac{w}{g}+w^2g\tan^{-1}\frac{g}{w}\right\}. \tag{2.54}$$

By use of $\log xy=\log x+\log y$, and $\tan^{-1}x=\pi/2-\tan^{-1}1/x$, this formula may be put in the form (2.53).

3. Frequency Effects [28]

3.1. Skin Effect in Concentric Cable

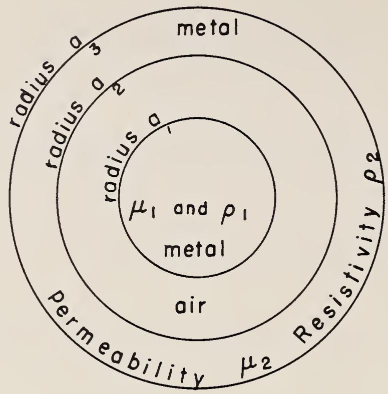


FIGURE 34

The resistivities ρ_1 and ρ_2 are in electromagnetic cgs units, one of which is equal to (10^{-9}) ohm-cm (for copper $1/\rho \sim 0.0006$ and for iron $1/\rho \sim 0.0001$).

For frequency f, let $R_{f/l}$ and $L_{f/l}$ denote resistance and self-inductance of the line per centimeter length, when the current flows one way in the central wire and returns through the outer shell. (See eq (2.42) and (2.46) for low frequency.) For very high frequency

$$\left.\begin{aligned} R_{\infty/l} &\sim \left[\frac{\sqrt{\mu_1\rho_1}}{a_1}+\frac{\sqrt{\mu_2\rho_2}}{a_2}\right]\sqrt{f} \\ L_{\infty/l} &\sim 2\log\frac{a_2}{a_1}+\frac{1}{2\pi}\left[\frac{\sqrt{\mu_1\rho_1}}{a_1}+\frac{\sqrt{\mu_2\rho_2}}{a_2}\right]\frac{1}{\sqrt{f}} \end{aligned}\right\} \tag{3.1}$$

For any frequency f, the resistance and inductance may be computed by use of certain tabulated functions, which are the real and imaginary parts of Bessels' (and Hankels' function of the first kind), these having parameters 0 and 1 and argument $x\sqrt{i}$, where x is a positive real.

The resistance $R_{f/l}$ and inductance $L_{f/l}$ are obtained by equating real and imaginary components in the complex equation

$$L_{f/l}+i\left[\frac{R_{f/l}}{2\pi f}\right]=2\log\frac{a_2}{a_1}+\frac{2\mu_2}{x_2}\mathfrak{g}-\frac{2\mu_1}{x_1}\frac{J_0(x_1\sqrt{i})}{[\sqrt{i}J_1(x_1\sqrt{i})]}, \tag{3.2}$$

where

$$x_1=2\pi a_1\sqrt{\frac{2\mu_1}{\rho_1}f}, \quad x_2=2\pi a_2\sqrt{\frac{2\mu_2}{\rho_2}f} \text{ and } x_3=2\pi a_3\sqrt{\frac{2\mu_2}{\rho_2}f}$$

$$g=\frac{J_0(x_2\sqrt{i})[\sqrt{i}H_1^{(1)}(x_3\sqrt{i})]-[\sqrt{i}J_1(x_3\sqrt{i}]H_0^{(1)}(x_2\sqrt{i})}{[\sqrt{i}J_1(x_2\sqrt{i})]\cdot[\sqrt{i}H_1^{(1)}(x_3\sqrt{i})]-[\sqrt{i}J_1(x_3\sqrt{i})]\cdot[\sqrt{i}H_1^{(1)}(x_2\sqrt{i})]},$$

where

$$J_0(x\sqrt{i})=u_0(x)+iv_0(x) \text{ and } \sqrt{i}J_1(x\sqrt{i})=u_1(x)+iv_1(x)$$

$$H_0^{(1)}(x\sqrt{i})=U_0(x)+iV_0(x) \text{ and } \sqrt{i}H_1^{(1)}(x\sqrt{i})=U_1(x)+iV_1(x)$$

The eight real functions u_n, v_n, U_n, V_n, $(n=0,1)$ are tabulated in Jahnke-Emde's "Tables of functions," pages 246-258 (fourth edition, 1945) for values of x from 0 to 5.99. For larger values the asymptotic expansions may be used and lead to the high-frequency formulas given above.

3.2. Proximity Effect in Parallel Wires [29]

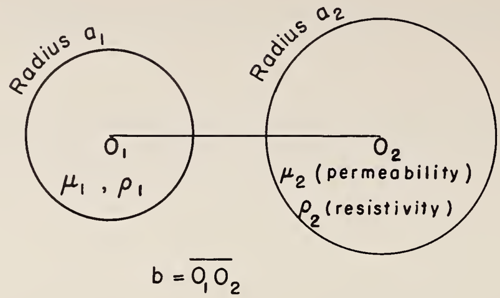


FIGURE 35

Current goes one way in one of the wires and returns in the other. See (2.44) and (2.45) for low frequency.

The resistance R/l and self-inductance L/l per unit length of the line (of both wires) for high frequency f, are given by

$$\left.\begin{aligned} R/l &\sim \left[\left(1+2\frac{a_1}{a_2}e^{-\gamma/2}+e^{-\gamma}\right)\frac{\sqrt{\mu_1\rho_1}}{a_1}+\left(1+2\frac{a_2}{a_1}e^{-\gamma/2}+e^{-\gamma}\right)\frac{\sqrt{\mu_2\rho_2}}{a_2}\right]\frac{\sqrt{f}}{1-e^{-\gamma}} \\ L/l &\sim \gamma+\frac{R/l}{2\pi f}, \end{aligned}\right\} \tag{3.3}$$

where γ is the reciprocal of the capacitance per unit length, so that by (1.15)

$$\frac{1}{C/l}=\gamma=2\log\left[\frac{b^2-a_1^2-a_2^2+\sqrt{[b^2-(a_1+a_2)^2][b^2-(a_1-a_2)^2]}}{2a_1a_2}\right]$$

$$=2\log\left[\frac{2a_1a_2}{b^2-a_1^2-a_2^2-\sqrt{[b^2-(a_1+a_2)^2][b^2-(a_1-a_2)^2]}}\right]$$

For equal wires of the same material, these become

$$\left.\begin{aligned} R/l &\sim \frac{2b}{a}\sqrt{\frac{\mu\rho f}{b^2-4a^2}} \\ L/l &\sim 4\log\left(\frac{b+\sqrt{b^2-a^2}}{2a}\right)+\frac{R/l}{2\pi f}. \end{aligned}\right\} \tag{3.4}$$

3.3. Single Wire Parallel to the Earth

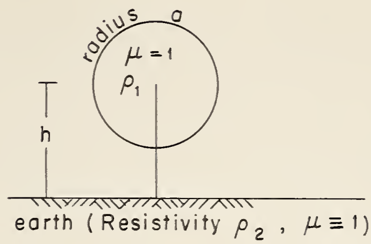


FIGURE 36

$$\left.\begin{aligned} R/l &\sim \left(\frac{\sqrt{\rho_1}}{a}+\frac{\sqrt{\rho_2}}{h}\right)\sqrt{f} \\ L/l &\sim 2\log\left(\frac{h+\sqrt{h^2-a^2}}{a}\right)+\frac{R/l}{2\pi f}. \end{aligned}\right\} \tag{3.5}$$

4. Legendre Functions That Occur in the Formulas

The Legendre polynomials $P_n(x)$ and their derivatives $P_n'(x)$, (where n is a positive integer or zero), occur in formulas (2.4) to (2.12) and in (2.22) to (2.25). These satisfy the recurrence relations

$$(2n+1)xP_n(x)=nP_{n-1}(x)+(n+1)P_{n+1}(x), \tag{4.1}$$

$$(2n+1)(1-x^2)P_n'(x)=n(n+1)[P_{n-1}(x)-P_{n+1}(x)]. \tag{4.2}$$

They are even or odd functions of x, according as n is an even or odd integer

$$P_n(x)=\sum_{s=0}^{n}(-1)^s\left(\frac{1-x}{2}\right)^s\frac{(s+n)!}{s!\,s!\,(n-s)!} \tag{4.3}$$

$$P_n'(x)=\tfrac{1}{2}\sum_{s=0}^{n-1}(-1)^s\left(\frac{1-x}{2}\right)^s\frac{(s+n+1)!}{s!\,(s+1)!\,(n-1-s)!} \tag{4.4}$$

or in powers of x,

$$P_{2n}(x)=(-1)^n\sum_{s=0}^{n}\frac{(-1)^s x^{2s}\Gamma(s+\frac{1}{2}+n)}{s!\,(n-s)!\,\Gamma(s+\frac{1}{2})} \tag{4.5}$$

$$P_{2n+1}(x)=(-1)^n x\sum_{s=0}^{n}\frac{(-1)^s x^{2s}\Gamma(s+\frac{3}{2}+n)}{s!\,(n-s)!\,\Gamma(s+\frac{3}{2})} \tag{4.6}$$

The last two equations give

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\tfrac{1}{2}(3x^2-1)$$

$$P_3(x)=\tfrac{1}{2}(5x^3-3x)$$

$$P_4(x)=\tfrac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x)=\tfrac{1}{8}(63x^5-70x^3+15x)$$

$$P_6(x)=\tfrac{1}{16}(231x^6-315x^4+105x^2-5)$$

$$P_7(x)=\tfrac{1}{16}(429x^7-693x^5+315x^3-35x)$$

$$P_8(x)=\tfrac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35).$$

(See references [5] and [11].)

The Legendre functions $Q_{n-\frac{1}{2}}$ and $P_{n-\frac{1}{2}}$ occur in the capacitance formula (1.19). The function $Q_{\frac{1}{2}}$ appears in the general inductance formula (B) of section 2 and is the origin of the elliptic functions in (2.1) (2.16) (2.26) to (2.29). These are infinite series that occur frequently for real argument greater than 1, sometimes written cosh β, where β is a positive real quantity.

$$Q_{\nu-\frac{1}{2}}(\cosh\beta)=\frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}e^{-(\nu+\frac{1}{2})\beta}F(\tfrac{1}{2},\nu+\tfrac{1}{2},\nu+1;e^{-2\beta})$$

$$=e^{-(\nu+\frac{1}{2})\beta}\sum_{s=0}^{\infty}e^{-2s\beta}\frac{\Gamma(s+\frac{1}{2})\Gamma(s+\nu+\frac{1}{2})}{s!\,\Gamma(s+\nu+1)} \qquad (4.7)$$

$$P_{\nu-\frac{1}{2}}(\cosh\beta)=P_{-\nu-\frac{1}{2}}(\cosh\beta)=F(\tfrac{1}{2},\tfrac{1}{2}+\nu,1;1-e^{-2\beta})$$

$$=\frac{e^{-(\nu+\frac{1}{2})\beta}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{s=0}^{\infty}\frac{(1-e^{-2\beta})^s\Gamma(s+\frac{1}{2})\Gamma(s+\nu+\frac{1}{2})}{s!\,.\,s!} \qquad (4.8)$$

$$\sinh^2\beta\,[Q_{\nu-\frac{1}{2}}(\cosh\beta)P'_{\nu-\frac{1}{2}}(\cosh\beta)-Q'_{\nu-\frac{1}{2}}(\cosh\beta)P_{\nu-\frac{1}{2}}(\cosh\beta)]=1, \qquad (4.9)$$

where $P'(z)$ denotes $dP(z)/dz$, etc.

The hypergeometric functions in (4.7) (4.8) may be transformed, leading in the following equivalent expressions;

$$Q_{\nu-\frac{1}{2}}(\cosh\beta) = \sqrt{\pi}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\left(\tfrac{1}{2}\operatorname{sech}\frac{\beta}{2}\right)^{2\nu+1} F(\nu+\tfrac{1}{2}, \nu+\tfrac{1}{2}, 2\nu+1; \operatorname{sech}^2\frac{\beta}{2}) \qquad (4.10)$$

$$P_{\nu-\frac{1}{2}}(\cosh\beta) = (\operatorname{sech}\beta/2)^{1-2\nu} F(\tfrac{1}{2}-\nu, \tfrac{1}{2}-\nu, 1; \tanh^2\beta/2)$$

$$= (\operatorname{sech}\beta/2)^{1+2\nu} F(\tfrac{1}{2}+\nu, \tfrac{1}{2}+\nu, 1; \tanh^2\beta/2) \qquad (4.11)$$

The identity

$$F(\alpha,\beta,\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma; z) \qquad (4.12)$$

shows that $P_{\nu-\frac{1}{2}}$ is an even function of ν.

The recurrence relations (4.1) and (4.2) become

$$2\nu\cosh\beta P_{\nu-\frac{1}{2}}(\cosh\beta) = (\nu+\tfrac{1}{2}) P_{\nu+\frac{1}{2}}(\cosh\beta) + (\nu-\tfrac{1}{2}) P_{\nu-1-\frac{1}{2}}(\cosh\beta) \qquad (4.13)$$

$$2\nu\sinh^2\beta P'_{\nu-\frac{1}{2}}(\cosh\beta) = (\nu^2-\tfrac{1}{4}) [P_{\nu+\frac{1}{2}}(\cosh\beta) - P_{\nu-1-\frac{1}{2}}(\cosh\beta)]. \qquad (4.14)$$

The same formulas are satisfied by $Q_{\nu-\frac{1}{2}}$.

In all these expressions ν may be replaced by any integer n. The functions $P_{\nu-\frac{1}{2}}$ are even functions of ν, but the functions $Q_{\nu-\frac{1}{2}}$ are not, except when $\nu=n$. In that case eq (4.7) gives $Q_{-n-\frac{1}{2}}(z) = Q_{n-\frac{1}{2}}(z)$, where n is any integer.

For the formulas given above ν is an integer n, so that it is not necessary to compute these functions by the series (4.7) or (4.8) in view of the many excellent tables of elliptic functions. If we find the two functions for $n=0$ and $n=1$, namely, $Q_{-\frac{1}{2}}$ and $Q_{\frac{1}{2}}$, any other $Q_{n-\frac{1}{2}}$ may be computed by (4.13).

Similarly, if $P_{-\frac{1}{2}}$ and $P_{\frac{1}{2}}$ are known, the recurrence relation (4.13) gives the $P_{n-\frac{1}{2}}$ for $n>1$.

The complete elliptic integrals $K(k)$ and $E(k)$ with modulus k, where $0<k<1$, are given by

$$K = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1; K^2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \qquad (4.15)$$

$$E = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta\, d\theta}. \qquad (4.16)$$

The same functions with complementary modulus $k' = \sqrt{1-k^2}$ are denoted by K' and E', respectively. (These E', K', etc. are not derivatives.)

Legendre's relation between the four is

$$KE'+K'E-KK'=\frac{\pi}{2}. \tag{4.17}$$

Let

$$\left.\begin{aligned}\cosh\beta&=\frac{2}{k^2}-1,\\ \sinh\beta&=\frac{2k'}{k^2}\end{aligned}\right\}\text{ so that }\left\{\begin{aligned}k^2&=\operatorname{sech}^2\beta/2\\ k'^2&=\tanh^2\beta/2\\ e^{-\beta}&=\frac{1-k'}{1+k'}.\end{aligned}\right\} \tag{4.18}$$

The equations to be derived are

$$Q_{-1/2}(\cosh\beta)=kK. \tag{4.19}$$

$$Q_{1/2}(\cosh\beta)=2\left(\frac{K-E}{k}\right)-kK. \tag{4.20}$$

$$\frac{\pi}{2}P_{-1/2}(\cosh\beta)=kK'. \tag{4.21}$$

$$\frac{\pi}{2}P_{1/2}(\cosh\beta)=\frac{2}{k}E'-kK'. \tag{4.22}$$

Placing $\nu=0$ in eq (4.10) gives by reference to (4.15) and (4.18)

$$Q_{-1/2}(\cosh\beta)=Q_{-1/2}\left(\frac{2}{k^2}-1\right)=\frac{\pi}{2}kF(1/2,1/2,1;k^2)=kK,$$

which proves eq (4.19).

Similarly, taking $\nu=0$ in (4.11) gives

$$\frac{\pi}{2}P_{-1/2}(\cosh\beta)=kF(1/2,1/2,1,k'^2)=kK',$$

which proves (4.21).

The proof of the remaining two equations, (4.20) and (4.22), is not so simple. For this we may place the notation of (4.18) in eq (B) and (B′) of section 2, so that

$$\left.\begin{aligned}\cosh\beta&=1+\frac{R^2}{2\rho\rho'}=1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}=\frac{2}{k^2}-1,\\ k^2&=\frac{4\rho\rho'}{(x-x')^2+(\rho+\rho')^2}.\end{aligned}\right\} \tag{4.23}$$

Equations (B) and (B′) become (with $\alpha=\phi-\phi'$)

$$\frac{1}{\sqrt{2(\cosh\beta-\cos\alpha)}}=\frac{k}{2\sqrt{1-k^2\cos^2\frac{\alpha}{2}}}=\frac{2}{\pi}\sum_{n=0}^{\infty}\epsilon_n Q_{n-\frac{1}{2}}(\cosh\beta)\cos n\alpha \tag{4.24}$$

$$Q_{n-\frac{1}{2}}(\cosh\beta)=Q_{n-\frac{1}{2}}\left(\frac{2}{k^2}-1\right)=\frac{1}{2}\int_0^{\pi/2}\frac{\cos n\alpha d\alpha}{\sqrt{2(\cosh\beta-\cos\alpha)}}=(-1)^n k\int_0^{\pi/2}\frac{\cos 2n\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta. \tag{4.25}$$

Taking $n=1$ in (4.25) gives

$$Q_{\frac{1}{2}}\left(\frac{2}{k^2}-1\right)=k\int_0^{\pi/2}\frac{(2\sin^2\theta-1)}{\sqrt{1-k^2\sin^2\theta}}d\theta$$

$$=\frac{2}{k}\int_0^{\pi/2}\frac{1-(1-k^2\sin^2\theta)}{\sqrt{1-k^2\sin^2\theta}}d\theta-k\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$=2\left(\frac{K-E}{k}\right)-kK,$$

which proves (4.20).

For the remaining eq (4.22) take $\nu=1$ in (4.11). This gives

$$\frac{\pi}{2}P_{\frac{1}{2}}(\cosh\beta)=\frac{\pi}{2}k^3F\left(\frac{3}{2}\frac{3}{2}1;k'^2\right)=\frac{\pi}{2k}F(-\tfrac{1}{2},-\tfrac{1}{2},1;k'^2)\text{ by (4.12).}$$

By writing out the series for E' and K' it is readily found that

$$2E'-k^2K'=\frac{\pi}{2}F(-\tfrac{1}{2},-\tfrac{1}{2},1;k'^2). \tag{4.26}$$

Hence $(\pi/2)P_{\frac{1}{2}}(\cosh\beta)=2E'/k-kK'$, which is eq (4.22) to be proved. Hence the function $Q_{n-\frac{1}{2}}(\cosh\beta)$ and $P_{n-\frac{1}{2}}(\cosh\beta)$ may be evaluated by use of any of the tables referred to in section 6 that give the complete elliptic integrals K and E as functions of the modulus k. This would apply to eq (1.19).

In case of the mutual inductance M between two coaxial circles, the formula (2.2) gives $M/\sqrt{a_1a_2}=4\pi Q_{\frac{1}{2}}(2/k^2-1)$, and this is tabulated against k^2 in table 2 of Nagoaoka and Sakurai [7].

It is found that the functions

$$Q=Q_{n-\frac{1}{2}}\left(1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}\right)\text{and } P_{n-\frac{1}{2}}\left(1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}\right)$$

satisfy the partial differential equation

$$\left(D_x{}^2+D_\rho^2+\frac{\frac{1}{4}-n^2}{\rho^2}\right)Q=0 \tag{4.27}$$

in the cylindrical coordinates (x,ρ), and also in (x',ρ'). The canonical expansions in various systems of coordinates of $Q_{n-\frac{1}{2}}$ with this argument are obtained in reference [12].

From (4.27) it is found that if

$$U_n(x,\rho)=\frac{2}{\sqrt{\rho}}\int_S\int\sqrt{\rho'}\,f(x',\rho')\,Q_{n-\frac{1}{2}}\left(1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}\right)dS', \tag{4.28}$$

then

$$\left.\begin{aligned}\left(D_x^2+D_\rho^2+\frac{\frac{1}{4}-n^2}{\rho^2}\right)\left(\rho^{\frac{1}{2}}U_n\right)&=0, \text{ where } (x,\rho) \text{ is outside } S\\ &=4\pi\rho^{\frac{1}{2}}f(x,\rho), \text{ where } (x,\rho) \text{ is inside } S\end{aligned}\right\} \tag{4.29}$$

which may be written

$$\left.\begin{aligned}\left(D_x^2+D_\rho^2+\frac{1}{\rho}D_\rho-\frac{n^2}{\rho^2}\right)U_n&=0 \text{ when } (x,\rho) \text{ is outside } S\\ &=-4\pi f(x,\rho) \text{ when } (x,\rho) \text{ is inside } S\end{aligned}\right\} \tag{4.30}$$

For the case $n=1$, $U_1=A_\phi=$ the ϕ-component of vector potential of a current distribution whose ϕ-component of current density is $i_\phi=f(x,\rho)$. This is eq (C′) of section 2.

For the case $n=0$, $U_0(x,\rho)$ is the axially symmetric potential V, of a ring distribution of charge whose density is $f(x,\rho)$ in the ring of section S. Hence

$$V(x,\rho)=\frac{2}{\sqrt{\rho}}\iint_S\sqrt{\rho'}\,f(x',\rho')\,Q_{-\frac{1}{2}}\left(1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}\right)dx'\,d\rho' \tag{4.31}$$

$$\left.\begin{aligned}\nabla^2\ V=(D_x{}^2+D_\rho^2+\frac{1}{\rho}D_\rho)V&=0 \quad \text{outside } S\\ &=-4\pi f(x,\rho) \quad \text{inside } S\end{aligned}\right\} \tag{4.32}$$

Hence the potential at (x,ρ) due to a circular line charge M in the plane x' with radius ρ' and coaxial with the x-axis is

$$V(x,\rho)=\frac{M}{\pi\sqrt{\rho\rho'}}\,Q_{-\frac{1}{2}}\left(1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}\right) \tag{4.33}$$

5. Derivation of Some Formulas

5.1. Eccentric Spheres and Cylinders (Internal) Equations (1.11) and (1.12)

Equations (1.11) and (1.12) are derived by use of biaxial coordinates α and β, defined by the transformation

$$x+iy=ic\cot\left(\frac{\alpha+i\beta}{2}\right) \text{ where } c>0$$

or

$$x=\frac{c\sinh\beta}{\cosh\beta-\cos\alpha} \tag{5.1}$$

$$y=\frac{c\sin\alpha}{\cosh\beta-\cos\alpha} \tag{5.2}$$

$$r\equiv\sqrt{x^2+y^2}=c\sqrt{\frac{\cosh\beta+\cos\alpha}{\cosh\beta-\cos\alpha}} \tag{5.3}$$

$$\sqrt{dx^2+dy^2}=\frac{c\sqrt{d\alpha^2+d\beta^2}}{\cosh\beta-\cos\alpha}\;. \tag{5.4}$$

The family of circles, $\beta=$ constant, has the equation

$$(x-c\coth\beta)^2+y^2=\left(\frac{c}{\sinh\beta}\right)^2, \text{ or } \coth\beta=\frac{x^2+y^2+c^2}{2cx}. \tag{5.5}$$

The orthogonal family of circular arcs, $\alpha=$ constant, is

$$x^2+(y-c\cot\alpha)^2=\left(\frac{c}{\sin\alpha}\right)^2, \text{ or } \cot\alpha=\frac{x^2+y^2-c^2}{2cy}. \tag{5.6}$$

The two-dimensional potential satifys

$$(D_x{}^2+D_y^2)V=\left(\frac{\cosh\beta-\cos\alpha}{c}\right)^2(D_\alpha{}^2+D_\beta{}^2)V=0. \tag{5.7}$$

For the axially symmetric potential, Laplace's equation with cylindrical coordinates $\left(D_x^2+D_\rho^2+\frac{1}{\rho}D_\rho\right)V=0$ becomes

$$\left(D_\alpha{}^2+D_\beta^2+\frac{1}{4\sin^2\alpha}\right)(\rho^{1/2}V)=0. \tag{5.8}$$

The correspondence of the (x, y) half-plane $(y>0)$ and the (α,β) strip $(0<\alpha<\pi)$, $(-\infty<\beta<\infty)$ is shown by the lettering in figure 37.

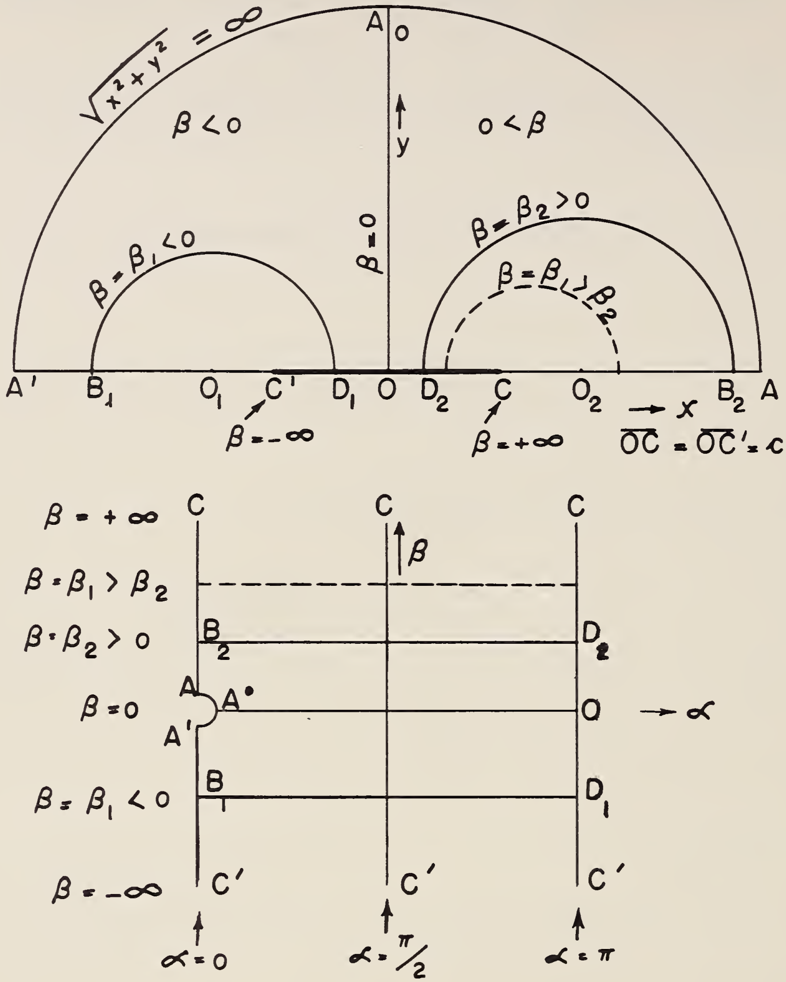


FIGURE 37

The three constants c, β_1, and β_2 are determined by eq (5.5) in terms of the given radii $a_1=c/\sinh\beta_1$, $a_2=c/\sinh\beta_2$ and the distance between centers $b=c(\coth\beta_2-\coth\beta_1)$. The solution of these three equations for the case of internal circles as in figure 5 is given in (1.10). The inner circle β_1 of figure 5 is the dotted semicircle of figure 37.

In the case of cylinders the two-dimensional potential between these cylinders is

$$V(\beta)=\left[\frac{\beta_1-\beta}{\beta_1-\beta_2}\right]V_2 \text{ for } \beta_2\leq\beta\leq\beta_1. \tag{5.9}$$

The positive charge per unit length on cylinder 2 is $Q_2>0$; the negative charge on 1 is Q_1, where

$$-Q_1=Q_2=\frac{-1}{4\pi}\int_{-\pi}^{\pi}\left(\frac{\partial V}{\partial\beta}\right)_{\beta=\beta_1}d\alpha=\frac{1}{2(\beta_1-\beta_2)},$$

so that

$$\frac{Q_2}{V_2}=C/\text{cm}=\frac{1}{2(\beta_1-\beta_2)},$$

which is eq (1.12).

To derive (1.11) for eccentric spheres, one within the other, we find the axially symmetric potential between the spheres, satisfying Laplace's equation in the form (5.8) for $\beta_2<\beta<\beta_1$

$$V(\alpha,\beta)=V_2\sqrt{2(\cosh\beta-\cos\alpha)}\sum_{n=0}^{\infty}\frac{e^{-(n+1/2)\beta}\sinh(n+1/2)(\beta_1-\beta)}{\sinh(n+1/2)(\beta_1-\beta_2)}P_n(\mu), \tag{5.10}$$

where $\mu=\cos\alpha$ and $P_n(\mu)$ is the Legendre polynomial. This potential vanishes on the inner sphere $\beta=\beta_1$. To show that it has the constant value V_2 on the outer sphere where $\beta=\beta_2$, the normal series

$$f(\mu)=\sum_{n=0}^{\infty}(n+1/2)P_n(\mu)\int_{-1}^{1}f(\mu')P_n(\mu')d\mu' \text{ for } -1<\mu<1$$

may be used. Since $\mu=\cos\alpha$, we find for $0<\beta$,

$$\int_{-1}^{1}\frac{P_n(\mu)\,d\mu}{\sqrt{2(\cosh\beta-\mu)}}=\frac{e^{-(n+1/2)\beta}}{(n+1/2)}, \tag{5.11}$$

which gives the normal series

$$\frac{1}{\sqrt{2(\cosh\beta-\cos\alpha)}}=\sum_{n=0}^{\infty}e^{-(n+1/2)\beta}P_n(\mu) \text{ for } -1<\mu<1. \tag{5.12}$$

(Equation (4.24) is the Fourier series for this same function.)

Taking $\beta=\beta_2$ in (5.12) shows that $V(\alpha,\beta_2)=V_2$. There is a positive charge Q_2 on sphere No. 2 and a negative charge Q_1 on No. 1, where (since y is now replaced by the cylindrical coordinate ρ)

$$-Q_1=Q_2=\frac{-1}{4\pi}\int_0^{\pi}2\pi\rho_1\left(\frac{\partial V}{\partial\beta}\right)_{\beta=\beta_1}d\alpha=-\frac{c}{2}\int_0^{\pi}\frac{\sin\alpha}{(\cosh\beta_1-\cos\alpha)}\left(\frac{\partial V}{\partial\beta}\right)_{\beta_1}d\alpha=\frac{-c}{2}\int_{-1}^{1}\left(\frac{\partial V}{\partial\beta}\right)_{\beta=\beta_1}\frac{d\mu}{(\cosh\beta_1-\mu)}$$

This gives by use of (5.10)

$$Q_2=V_2c\sum_{n=0}^{\infty}\frac{e^{-(n+1/2)\beta_2}}{\sinh(n+1/2)(\beta_1\ \beta_2)}(n+1/2)\int_{-1}^{1}\frac{P_n(\mu)\,d\mu}{\sqrt{2(\cosh\beta_1-\mu)}},$$

or by (5.11) with $\beta=\beta_1$

$$\frac{Q_2}{V_2}=c\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})(\beta_1+\beta_2)}}{\sinh(n+\frac{1}{2})(\beta_1-\beta_2)}=2c\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\beta_1}}{1-e^{(2n+1)(\beta_1-\beta_2)}},$$

where $0\leqslant\beta_2\leqslant\beta_1$, which proves eq (1.11).

When the circles become coaxial $b\to 0$, but $c\to\infty$, and $2bc\to a_2^2-a_1^2$. Hence eq (1.11) reduces to the coaxial case (1.5) and eq (1.12) reduces to (1.6).

5.2. Eccentric Spheres and Cylinders (External) Equations (1.14) and (1.15)

In this case the circles are external. The circle No. 1 on the left is $\beta=\beta_1<0$, and the derivation is made with β_1 negative. At the end we then replace β_1 by $-\beta_1$, so that in figure 37 circle No. 1 is $\beta=-\beta_1$, where $\beta_1>0$. This is done to keep the three constants c, β_1, and β_2 all positive, as stated in the three equations (1.13), which have been determined by use of (5.5).

Hence with β_1 negative, the potential between the cylinders is

$$V(\beta)=\left[\frac{\beta-\beta_1}{\beta_2-\beta_1}\right]V_2 \text{ for } 0>\beta_1<\beta\leqslant\beta_2<0. \tag{5.13}$$

As before,

$$-Q_1=Q_2=\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\frac{\partial V}{\partial\beta}\right)_{\beta=\beta_1}d\alpha=\frac{1}{2(\beta_2-\beta_1)}$$

so

$$Q_2/V_2=C/\mathrm{cm}=\frac{1}{2(\beta_2-\beta_1)},$$

which becomes (1.15) on replacing $-\beta_1$ by β_1.

For the case of spheres

$$V(\alpha,\beta)=V_2\sqrt{2(\cosh\beta-\cos\alpha)}\sum_{n=0}^{\infty}e^{-(n+\frac{1}{2})\beta_2}\frac{\sinh(n+\frac{1}{2})(\beta-\beta_1)}{\sinh(n+\frac{1}{2})(\beta_2-\beta_1)}P_n(\mu). \tag{5.14}$$

We now find

$$-Q_1=Q_2=+\frac{c}{2}\int_{-1}^{1}\left(\frac{\partial V}{\partial\beta}\right)_{\beta_1}\frac{d\mu}{(\cosh\beta_1-\mu)},$$

or

$$\frac{Q_2}{V_2}=C=c\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})\beta_2}}{\sinh(n+\frac{1}{2})(\beta_2-\beta_1)}(n+1/2)\int_{-1}^{1}\frac{P_n(\mu)\,d\mu}{\sqrt{2(\cosh\beta_1-\mu)}}.$$

Since β_1 is here negative, we must write eq (5.11)

$$(n+\tfrac{1}{2})\int_{-1}^{1}\frac{P_n(\mu)\,d\mu}{\sqrt{2(\cosh\beta_1-\mu)}}=e^{-(n+\frac{1}{2})|\beta_1|}=e^{+(n+\frac{1}{2})\beta_1}\quad\text{for } \beta_1<0,$$

so that

$$C=c\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})(\beta_2-\beta_1)}}{\sinh(n+\frac{1}{2})(\beta_2-\beta_1)}=2c\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})\gamma}}{1-e^{-(n+\frac{1}{2})\gamma}},$$

where $\gamma=2(\beta_2-\beta_1)$ when $\beta_1<0$ and $\beta_2>0$.

On reversing the sign of β_1, this gives eq (1.14). For the limiting cases $\beta_1\to 0$, in which the sphere or cylinder on the left of figure 6 or figure 37 has an infinite radius, we may place $b=a_1+h$ and $a_2=a_1$. When $a\to\infty$; $c\to\sqrt{h^2-a^2}$, $\beta_2=\log(h+\sqrt{h^2-a^2})/a$, so eq (1.14) and (1.15) become (1.8) and (1.9), respectively, The potential between cylinder and plane $\beta=0$ (fig. 4) is

$$V(\beta)=\frac{\beta}{\beta_2}V_2\quad\text{for } 0<\beta<\beta_2=\log\frac{h+\sqrt{h^2-a^2}}{a}.$$

Between the sphere and plane the potential is eq (5.14) with $\beta_1=0$.

5.3. Derivation of Equations (1.17) and (1.16) for Oblate Spheroid and Circular Disk

With oblate spheroidal coordinates (α,β) the (x,ρ) half-plane is represented on the (α,β) strip $\begin{pmatrix}0<\alpha<\pi\\0<\beta<\infty\end{pmatrix}$ by

$$x+i\rho=ic\,\sin(\alpha+i\beta)\quad\text{where } c>0$$

or

$$x=-c\cos\alpha\sinh\beta\ \text{and}\ \rho=c\sin\alpha\cosh\beta$$

so

$$r=\sqrt{x^2+\rho^2}=c\sqrt{\sinh^2\beta+\sin^2\alpha}$$

or

$$\frac{x^2}{c^2\sinh^2\beta}+\frac{\rho^2}{c^2\cosh^2\beta}=1\ \text{(confocal ellipses)}$$

$$\frac{-x^2}{c^2\cos^2\alpha}+\frac{\rho^2}{c^2\sin^2\alpha}=1\ \text{(confocal hyperbolas)}.$$

The equation for the axially symmetric potential

$$\left(D_\alpha^2+D_\beta^2+\frac{1}{4\sin^2\alpha}-\frac{1}{4\cosh^2\beta}\right)(\rho^{1/2}V)=0$$

has solutions $V=Q_n(i\sinh\beta)P_n(\cos\alpha)$. The oblate spheroid $\beta=\beta_1$ has semiaxes a and $b<a$, where

$$c=\sqrt{a^2-b^2},\ \sinh\beta_1=b/c \text{ and } \cosh\beta_1=a/c$$

$$iQ_0(i\sinh\beta)=\frac{i}{2}\log\frac{i\sinh\beta+1}{i\sinh\beta-1}=\sin^{-1}(\text{sech}\,\beta),$$

so

$$iQ_0(i\sinh\beta_1)=\sin^{-1}(c/a).$$

The potential outside the conducting spheroid β_1 at potential V_1 with charge M_1 is (for $\beta_1<\beta<\infty$)

$$V(\beta)=V_1\frac{iQ_0(i\sinh\beta)}{iQ_0(i\sinh\beta_1)}=V_1\frac{\sin^{-1}(\text{sech}\,(\beta))}{\sin^{-1}(c/a)},\tag{5.15}$$

$$\text{and } M_1=\underset{r\to\infty}{\text{limit}}[rV(\beta)]=\lim_{\beta\to\infty}[c\sinh\beta V(\beta)]=\frac{cV_1}{\sin c\ a},$$

which is eq (1.17).

5.4. Derivation of Equation (1.18) for Prolate Spheroid

With prolate spheroidal coordinates the (x,ρ) half-plane is represented on the (α,β) strip $(0<\alpha<\pi)$, $(0<\beta<\infty)$ by

$$x+i\rho=-c\cos(\alpha+i\beta),\ \text{where } c>0,\ \text{or}$$

$$x=-c\cos\alpha\cosh\beta \text{ and } \rho=c\sin\alpha\sinh\beta,\ \text{so}$$

$$r=\sqrt{x^2+\rho^2}=c\sqrt{\sinh^2\beta+\cos^2\alpha}.$$

Hence

$$\frac{x^2}{c^2\cosh^2\beta}+\frac{\rho^2}{c^2\sinh^2\beta}=1 \text{ (ellipses)}$$

$$\frac{x^2}{c^2\cos^2\alpha}-\frac{\rho^2}{c^2\sin^2\alpha}=1 \text{ (hyperbolas).}$$

The equation for the axially symmetric potential

$$\left(D_\alpha^2+D_\beta^2+\frac{1}{4\sin^2\alpha}+\frac{1}{4\sinh^2\beta}\right)(\rho^{1/2}V)=0$$

has solutions $V=Q_n(\cosh\beta)P_n(\cos\alpha)$.

For the prolate spheroid $\beta=\beta_1$ with semiaxes a and $b<a$

$$c=\sqrt{a^2-b^2} \text{ and } \cosh\beta_1=a/c,$$

$$Q_0(\cosh\beta)=\tfrac{1}{2}\log\frac{\cosh\beta+1}{\cosh\beta-1}=\log\coth\frac{\beta}{2}=\log\left(\frac{1+e^{-\beta}}{1-e^{-\beta}}\right),$$

$$Q_0(\cosh\beta_1)=\log(a+c)/b.$$

The potential outside the conducting spheroid β_1 at potential V_1 with charge M_1 is (for $\beta_1<\beta<\infty$)

$$V(\beta)=V_1\frac{Q_0(\cosh\beta)}{Q_0(\cosh\beta_1)}=V_1\frac{\log\coth\beta/2}{\log(a+c)/b}, \tag{5.16}$$

and

$$M_1=\lim_{r\to\infty}[rV(\beta)]=\lim_{\beta\to\infty}[c\sqrt{\sinh^2\beta+\cos^2\alpha}\,V(\beta)]$$

$$=\frac{cV_1}{\log(a+c)/b},$$

which is eq (1.18).

5.5. Derivation of Equation (1.19) for a Toroid

The strip $(-\pi<\alpha<\pi)$, $(0<\beta<\infty)$ of the toroidal (or "ring") coordinates, represents the (x,ρ) half-plane, if this is cut from zero to c along the ρ-axis. The equation $x+i\rho=-c\cot(\alpha+i\beta)/2$ gives

$$\left.\begin{aligned} &x=\frac{-c\sin\alpha}{\cosh\beta-\cos\alpha} \text{ and } \rho=\frac{c\sinh\beta}{\cosh\beta-\cos\alpha} \\ &r=\sqrt{x^2+\rho^2}=c\sqrt{\frac{\cosh\beta+\cos\alpha}{\cosh\beta-\cos\alpha}} \end{aligned}\right\} \tag{5.17}$$

$$\sqrt{dx^2+dy^2}=\frac{c\sqrt{d\alpha^2+d\beta^2}}{\cosh\beta-\cos\alpha}. \tag{5.18}$$

The family of circles β= constant, each member of which generates a toroidal surface by rotation around the x-axis, belongs to the equation

$$x^2+(\rho-c\coth\beta)^2=\frac{c^2}{\sinh^2\beta}. \tag{5.19}$$

The equation of the family of circular arcs, orthogonal to these circles, is

$$(x+c\cot\alpha)^2+\rho^2=\frac{c^2}{\sin^2\alpha}. \tag{5.20}$$

The equation for the axially symmetric potential

$$\left(D_\alpha^2+D_\beta^2+\frac{1}{4\sinh^2\beta}\right)(\rho^{1/2}V)=0 \tag{5.21}$$

has solutions of the form

$$V=\sqrt{2(\cosh\beta-\cos\alpha)}(A\cos n\alpha+B\sin n\alpha)(CP_{n-1/2}(\cosh\beta)+DQ_{n-1/2}(\cosh\beta)).$$

The third of eq (5.17) shows that spatial infinity, ($r=\infty$) corresponds to the point $\alpha=\beta=0$. The first two of these equations show that $\beta=+\infty$ corresponds to $x=0$ and $\rho=c=$ the radius of the focal circle.

If the generating circle of figure 7 has the equation $\beta=\beta_1$, it is evident from (5.19) that

$$c=\sqrt{A^2-a^2} \text{ and } \cosh\beta_1=\frac{A}{a}. \tag{5.22}$$

If the toroidal surface has a constant potential V_1 and charge M_1, the Newtonian potential at outside points where $0<\beta<\beta_1$ is

$$V(\alpha,\beta)=\frac{2V_1}{\pi}\sqrt{2(\cosh\beta-\cos\alpha)}\sum_{n=0}^{\infty}\epsilon_n\frac{Q_{n-1/2}(\cosh\beta_1)}{P_{n-1/2}(\cosh\beta_1)}P_{n-1/2}(\cosh\beta)\cos n\alpha, \tag{5.23}$$

where $\epsilon_0=1/2$, $\epsilon_n=1$ for $n\neq 0$.

This vanishes at $r=\infty$ (i.e., when $\alpha=\beta=0$). On the surface β_1 it becomes

$$V(\alpha,\beta_1)=V_1\sqrt{2(\cosh\beta_1-\cos\alpha)}\cdot\frac{2}{\pi}\sum_{n=0}^{\infty}\epsilon_n Q_{n-1/2}(\cosh\beta_1)\cos n\alpha$$

$$=V_1=\text{constant by eq (4.24)}.$$

By a fundamental property of Newtonian potentials, the charge M_1 on the torus is

$$M_1=\lim_{r\to\infty}(rV)=\lim_{\alpha\to\beta\to 0} c\sqrt{\frac{\cosh\beta+\cos\alpha}{\cosh\beta-\cos\alpha}}V(\alpha,\beta)$$

$$=\frac{4cV_1}{\pi}\lim_{\alpha=\beta=0}\sum_{n=0}^{\infty}\epsilon_n\frac{Q_{n-\frac{1}{2}}(\cosh\beta_1)}{P_{n-\frac{1}{2}}(\cosh\beta_1)}P_{n-\frac{1}{2}}(\cosh\beta)\cos n\alpha.$$

Since $P_{n-\frac{1}{2}}(1)=1$, this gives

$$\frac{M_1}{V_1}=\frac{4c}{\pi}\sum_{n=0}^{\infty}\epsilon_n\frac{Q_{n-\frac{1}{2}}(\cosh\beta_1)}{P_{n-\frac{1}{2}}(\cosh\beta_1)},$$

which is eq (1.19), since $c=\sqrt{A^2-a^2}$ and $\cosh\beta_1=A/a$. The evaluation of these functions by elliptic integrals is discussed in section 4.

5.6. Self-Inductance of a Single Turn of Wire Equation (2.15)

With cylindrical coordinates (x,ρ) the vector potential $A_\phi=A(x,\rho)$ at any point (x,ρ) in space is by eq (*C*) of section 2.

$$\rho^{\frac{1}{2}}A(x,\rho)=2\iint\rho'^{\frac{1}{2}}i(\rho')Q_{\frac{1}{2}}\left(1+\frac{D^2}{2\rho\rho'}\right)dS' \tag{5.24}$$

where $D^2=(x-x')^2+(\rho-\rho')^2$, and the integration is taken with respect to (x',ρ') over the upper circle of radius a in figure 18.

Also by eq (*D*) of section 2 the self-inductance is given by

$$L=2\pi\iint\rho^{\frac{1}{2}}i(\rho)\cdot\rho^{\frac{1}{2}}A(x,\rho)\,dS \tag{5.25}$$

integrated with respect to (x,ρ) over the same circular section. Since (x,ρ) and (x',ρ') are both points in this circle, we may use the expansion

$$Q_{n-\frac{1}{2}}\left(1+\frac{D^2}{2\rho\rho'}\right)$$

$$=\frac{(-1)^{n+1}}{2\pi}\sum_{s=0}^{\infty}\left(\frac{-D^2}{4\rho\rho'}\right)^s\frac{\Gamma(s+\frac{1}{2}+n)\Gamma(s+\frac{1}{2}-n)}{s!\qquad s!}\left[\log\frac{D^2}{4\rho\rho'}+\psi(s+\tfrac{1}{2}+n)+\psi(s+\tfrac{1}{2}-n)-2\psi(s+1)\right]. \tag{5.26}$$

This is valid if $D^2/4\rho\rho'<1$, which will be true for all positions of the points $P(x,\rho)$ and $P'(x',\rho')$ both within the circle, provided that $a<A/2$, which will be true here,

since it is assumed that a/A is so small that terms smaller than $a^2/8A^2 \log 8A/a$ may be neglected in comparison with 1.

Hence for $n=1$, eq (5.26) gives, to this approximation,

$$-2Q_{\frac{1}{2}}\left(+\frac{D^2}{2\rho\rho'}\right)=\left[1+\frac{3}{4}\left(\frac{D^2}{4\rho\rho'}\right)\right]\log\frac{D^2}{4\rho\rho'}+4(1-\log 2)+\tfrac{1}{2}(1-6\log 2)\frac{D^2}{4\rho\rho'} \tag{5.27}$$

Let $y=\rho-A$ and $y'=\rho'-A$, so that x and y are rectangular coordinates with origin at the center of the circle. Then, to the second order in a/A,

$$-2Q_{\frac{1}{2}}\left(1+\frac{D^2}{2\rho\rho'}\right)=-\left[4+2\log\frac{D}{8A}-\frac{y+y'}{A}+\frac{y^2+y'^2}{2A^2}+\frac{D^2}{8A^2}\left(1+3\log\frac{D}{8A}\right)\right] \tag{5.28}$$

For the assumed current density

$$i=\frac{1}{\pi a^2 F}\left(\frac{\rho}{A}\right)^b, \tag{5.29}$$

the total current is 1, which gives

$$F=F\left(-\frac{b}{2},\frac{1-b}{2},2;\frac{a^2}{A^2}\right)=1+b(b-1)\frac{a^2}{8A^2}+\dots\ . \tag{5.30}$$

Then

$$\rho^{\frac{1}{2}}i(\rho)=\frac{A^{\frac{1}{2}}}{\pi a^2 F}\left(1+\frac{y}{A}\right)^{b+\frac{1}{2}}=\frac{A^{\frac{1}{2}}}{\pi a^2 F}\left[1+C_1\frac{y}{A}+C_2\frac{y^2}{A^2}\right], \tag{5.31}$$

where

$$C_1=b+\tfrac{1}{2} \text{ and } C_2=\tfrac{1}{2}\left(b^2-\frac{1}{4}\right). \tag{5.32}$$

This gives

$$2\rho'^{\frac{1}{2}}i(\rho')Q_{\frac{1}{2}}\left(1+\frac{D^2}{2\rho\rho'}\right)=-\frac{A^{\frac{1}{2}}}{\pi a^2 F}\left\{\left[1+C_1\frac{y'}{A}+C_2\frac{y'^2}{A^2}\right]\left[4+2\log\frac{D}{8A}\right]\right.$$

$$\left.-\left(\frac{y+y'}{A}\right)+\frac{y^2}{2A^2}-C_1\frac{yy'}{A^2}+(\tfrac{1}{2}-C_1)\frac{y'^2}{A^2}+\frac{D^2}{8A^2}\left(1+3\log\frac{D}{8A}\right)\right\}. \tag{5.33}$$

To use this expression in the integral (5.24) it is better to use polar coordinates, placing $x=r\cos\theta$ and $y=r\sin\theta$, so that

$$D^2=r^2-2rr'\cos(\theta-\theta')+r'^2, \tag{5.34}$$

and

$$\text{Log } D = \log r - \sum_{n=1}^{\infty} \left(\frac{r'}{r}\right)^n \frac{\cos n(\theta-\theta')}{n} \quad \text{if } r' \leqslant r$$

$$= \log r' - \sum_{1}^{\infty} \left(\frac{r}{r'}\right)^n \frac{\cos n(\theta-\theta')}{n} \quad \text{if } r' \geqslant r. \tag{5.35}$$

The result of integrating eq (5.24) is

$$\rho^{1/2} A(x,\rho) = -\frac{A^{1/2}}{F}[f_0(r) + f_1(r,\theta) + f_2(r,\theta)] \tag{5.36}$$

where

$$f_0(r) = 3 - 2\left(1 + \frac{a^2 C_2}{4A^2}\right) \log\left(\frac{8A}{a}\right) + \frac{r^2}{a^2} \tag{5.37}$$

$$f_1(r,\theta) = -\left[1 + C_1 - \frac{C_1 r^2}{2a^2}\right] \frac{r \sin\theta}{A} \tag{5.38}$$

$$f_2(r,\theta) = \frac{a^2}{8A^2}\left[\frac{9}{8} - 2C_1 + 7C_2 - \frac{3}{2}\log\frac{8A}{a}\right]$$

$$+ \frac{r^2}{8A^2}\left[\frac{5}{2} + C_2 - 3\log\frac{8A}{a} + (4 - 2C_2)\sin^2\theta\right] + \frac{r^4}{24a^2A^2}\left[\frac{9}{8} + C_2 + 4C_2\sin^2\theta\right] \tag{5.39}$$

With this, the integral (5.25) gives

$$L = \frac{4\pi A}{F^2}\left\{\left[1 + \left(4C_2 + \frac{3}{2}\right)\frac{a^2}{8A^2}\right]\log\left(\frac{8A}{a}\right) - \frac{7}{4} + \frac{a^2}{8A^2}\left[2C_1\left(1 + \frac{C_1}{3}\right) - \frac{22}{3}C_2 - \frac{7}{4}\right]\right\} \tag{5.40}$$

Finally, multiplying by the factor,

$$\frac{1}{F^2} = 1 - 2b(b-1)\frac{a^2}{8A^2} = 1 - C_0\frac{a^2}{8A^2}, \tag{5.41}$$

where $C_0 = 2b(b-1)$, gives

$$L = 4\pi A\left\{\left[1 + \left(4C_2 - C_0 + \frac{3}{2}\right)\frac{a^2}{8A^2}\right]\log\left(\frac{8A}{a}\right) - \frac{7}{4} + \frac{a^2}{8A^2}\left[\frac{7}{4}(C_0 - 1) + 2C_1\left(1 + \frac{C_1}{3}\right) - \frac{22}{3}C_2\right]\right\}. \tag{5.42}$$

On substituting the expressions given above for C_0, C_1, and C_2, it is found that

$$L = 4\pi A\left\{\left[1 + (2b+1)\frac{a^2}{8A^2}\right]\log\left(\frac{8A}{a}\right) - \frac{7}{4} + \frac{(b-1)(b-2/3)}{16}\left(\frac{a}{A}\right)^2\right\}, \tag{5.43}$$

which is eq (2.15).

Exact expressions for the magnetic field and inductance of any toroid with this current distribution may be found as normal series of ring functions, using the toroidal coordinates of reference [20]. It is thus found that for $b=-5/2$ the external magnetic field is the same as if the total current were concentrated in the focal circle. This is true for the more general case $i_\phi=C\rho^{-5/2}f(\beta)$.

To get an expression for the potential $A(x,\rho)$ when the point $P(x,\rho)$ is outside the circle of figure 18, the approximation (5.27) based on (5.26) cannot be used unless the distance of P from the center C is small compared to A. When this distance is of the order of magnitude of A or greater, while $P(x'\rho')$ remains in the circle, it is sufficient to use Taylor's series, with (x,ρ) fixed and the variables x'/A and $y'/A(=(\rho'-A)/A)$ small. For brevity, let

$$\mathscr{g}=1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}=\frac{(x-x')^2+\rho^2+\rho'^2}{2\rho\rho'}$$

$$\mathscr{g}_0=\frac{x^2+\rho^2+A^2}{2A\rho}$$

so $\mathscr{g}\to\mathscr{g}_0$ when $x'\to y'\to 0$. Then, to the second order

$$Q_{1/2}(\mathscr{g})=Q_{1/2}(\mathscr{g}_0)+(x'Q_x+y'Q_y)+{}^1\!/_2(x'^2Q_{xx}+y'^2Q_{yy}+2x'y'Q_{xy}), \qquad (5.44)$$

where Q_{xx} is the value of $D_{x'}^2Q_{1/2}(\mathscr{g})$ when $\mathscr{g}=\mathscr{g}_0(x'=y'=0)$ and similarly, Q_{xy} and $Q_{xy}=D_xD_yQ_{1/2}=D_xD_\rho Q_{1/2}$.

From eq (4.27), with $n=1$ and variables x',ρ', we find an exact expression when $x'=0$ and $\rho'=A$.

$$Q_{xx}+Q_{yy}=\frac{3}{4A^2}Q_{1/2}(\mathscr{g}_0). \qquad (5.45)$$

By use of (5.44) with the current in (5.31) in eq (5.24), it is found that

$$A(x,\rho)=2\sqrt{\frac{A}{\rho}}\left\{\left[1+(b+{}^1\!/_2)\frac{a^2}{8A^2}\right]Q_{1/2}(\mathscr{g}_0)+\frac{a^2}{8A^2}(b+{}^1\!/_2)2AQ_y\right\}, \qquad (5.46)$$

where

$$2AQ_y=-2\left(\mathscr{g}_0-\frac{A}{\rho}\right)Q'_{1/2}(\mathscr{g}_0)=-\left(\frac{\mathscr{g}_0-\frac{A}{\rho}}{\mathscr{g}_0^2-1}\right)[\mathscr{g}_0Q_{1/2}(\mathscr{g}_0)-Q_{-1/2}(\mathscr{g}_0)], \qquad (5.47)$$

and

$$\mathscr{g}_0=1+\frac{x^2+(\rho-A)^2}{2A\rho}. \qquad (5.48)$$

Equation (5.46) is valid when g_0-1 is not small. Hence there remains a gap, not here considered, between the ranges of validity of the two equations (5.46) and (5.36), which could only be bridged by an equation more complicated than either. Applications of (5.46) that would require the retention of the second-order terms are exceedingly rare. It is generally sufficient to consider the total current concentrated in a filament with trace at center of the circular section of the wire.

5.7. Derivation of Equation (2.16) for Self-Inductance of a Single Turn of Wire Near a Magnetic Medium

Referring to figure 19 let $A_a(x,\rho)$ denote the value at any point $P(x,\rho)$ in space due to any axially symmetric distribution of currents when $\mu=1$ everywhere. These currents are all to the left of the boundary plane $x=x_0$.

Similarly, let $A_2(x,\rho)$ denote the potential at any point to the left of $x=x_0$ that would be produced (with $\mu=1$ everywhere) by a fictitious distribution of currents that is the image of the existing distribution by reflection in the plane $x=x_0$.

Then the potential $A(x,\rho)$ due to the actual currents in the presence of the magnetic material with $\mu\neq 1$, where $x_0<x$, is in the air, where $-\infty<x\leqslant x_0$,

$$A(x,\rho)=A_a(x,\rho)+\frac{\mu-1}{\mu+1}A_2(x,\rho), \tag{5.49}$$

and in the material, where $x_0\leqslant x<+\infty$,

$$A(x,\rho)=\frac{2\mu}{\mu+1}A_a(x,\rho). \tag{5.50}$$

By this definition of $A_a(x,\rho)$ and $A_2(x,\rho)$ it is evident that at the plane $x=x_0$, $A_a=A_2$, and $D_xA_2=-D_xA_a$ identically in ρ.

Consequently, A is continuous, which makes B_x continuous. Also the continuity of H_ρ is assured by that of D_xA/μ.

The inductance of the turn of wire near the material as figure 19 is by (5.49)

$$L=L_{air}+\frac{\mu-1}{\mu+1}2\pi\iint i(x,\rho)\rho A(x,\rho)dS \tag{5.51}$$

integrated over a circular section of the wire. This integration could be effected for the form of current in eq (5.31) by use of eq (5.46), assuming that $2x_0$ is not small compared to A. Formula (2.16) assumes that the fictitious current producing $A_2(x,\rho)$ is a filament of radius a coaxial with the x-axis in the plane $x=2x_0$.

For this approximation, we place in (5.51)

$$\rho A(x,\rho)=2AQ_{1/2}\left(1+\frac{4x_0^2+(A-a)^2}{2Aa}\right)=2AQ_{1/2}\left(\frac{2}{k^2}-1\right)=2A\left[\frac{2(K-E)}{k}-kK\right], \tag{5.52}$$

where

$$k^2=\frac{4Aa}{4x_0^2+(A+a)^2}. \tag{5.53}$$

Since $\iint i\,dS=1$, this gives

$$L=L_{air}+\frac{\mu-1}{\mu+1}4\pi A\left[\frac{2(K-E)}{k}-kK\right], \tag{5.54}$$

where L_{air} is given by 5.43).

5.8. Derivation of Equations (2.40) and (2.41) for the Self-Inductance of Toroidal Current Sheets (Tape Winding)

With ideal tape windings the current circulates as indicated by the arrow in figure 27. There is no external field and the internal field of the unit current is $H_\phi=2N/\rho$, where N is the number of turns, and ρ is the distance of a point from the axis of revolution. The inductance L is equal to twice the integral defining total electrokinetic energy T.

$$T=1/8\pi\iint \mu H^2 dv$$

integrated over all space. Hence

$$L=\frac{\mu}{4\pi}\iint H^2 dv=2\mu N^2\iint\frac{dS}{\rho}$$

integrated over the axial section. For circular and rectangular axial sections shown in figure 27 and 28, this results in eq (2.40) and (2.41), respectively.

5.9. Derivation of Equation (2.45) for Self-Inductance per Unit Length of Two Parallel Wires of Magnetic Material

Referring to figure 31, the current +1 flows upward perpendicular to paper with uniform current density $i_1=1/\pi a_1{}^2$ in cylinder No. 1. The current density in cylinder No. 2 is $i_2=-1/\pi a_2^2$.

The only components of current density and of vector potential are the z-components where the z-axis is upward perpendicular to the paper. The general field equations $B=\mu H=\text{curl } A$ and $\text{curl } H=4\pi i$ give

$$B_x=D_yA,\quad B_y=-D_xA,\quad B_z=0,$$

where $A(x,y)=A_z$. Hence

$$\left.\begin{aligned}(D_x{}^2+D_y{}^2)A&=-\frac{4\mu_1}{a_1{}^2}\text{ in cylinder 1}\\ &=+\frac{4\mu_2}{a_2{}^2}\text{ in cylinder 2}\\ &=0\text{ in the air between them.}\end{aligned}\right\} \tag{5.55}$$

The boundary conditions at the surface of each wire are:

A is continuous (continuity of normal component of B), (5.56)

$\frac{1}{\mu}\frac{\partial A}{\partial n}$ is continuous (continuity of tangential H). (5.57)

With plane polar coordinates (r_1, θ_1) with origin at center O_1 of wire No. 1.

$$(D_x^{\,2}+D_y^{\,2})A=\frac{1}{r_1}D_{r_1}\left(r_1 D_{r_1} A\right)+\frac{1}{r_1^2}D_{\theta_1}^2 A.$$

Similarly, with polar coordinates (r_2, θ_2) with center at O_2

$$(D_x^{\,2}+D_y^{\,2})A=\frac{1}{r_2}D_{r_2}\left(r_2 D_{r_2} A\right)+\frac{1}{r_2^2}D_{\theta_2}^2 A.$$

Hence let

$$\left.\begin{aligned} A&=U-\mu_1\left(\frac{r_1^2}{a_1^2}\right) && \text{in wire No. 1}\\ &=U+\mu_2\left(\frac{r_2^2}{a_2^2}\right) && \text{in wire No. 2}\\ &=U && \text{in the air}\end{aligned}\right\} \tag{5.58}$$

Then $(D_x^{\,2}+D_y^{\,2})U=0$ everywhere. (5.59)

At $r_1=a_1$,

$$U_0=U_i-\mu_1 \text{ and } D_{r_1}U_0=\frac{1}{\mu_1}D_{r_1}U_i-\frac{2}{a_1}. \tag{5.60}$$

At $r_2=a_2$

$$U_0=U_i+\mu_2 \text{ and } D_{r_2}U_0=\frac{1}{\mu_2}D_{r_2}U_i+\frac{2}{a_2}, \tag{5.61}$$

where U_0 means outside, U_i inside the wire.

The self-inductance of the line per centimeter length

$$L/\text{cm}=\frac{1}{\pi a_1^2}\iint A\,dS_1-\frac{1}{\pi a_2^2}\iint A\,dS_2$$

$$=-\tfrac{1}{2}(\mu_1+\mu_2)+\frac{1}{\pi a_1^2}\iint U\,dS_1-\frac{1}{\pi a_2^2}\iint U\,dS_2. \tag{5.62}$$

The biaxial coordinates α,β, see reference [16], are suitable for constructing the harmonic function $U(\alpha,\beta)$ that satisfies the four boundary conditions in (5.56) and (5.57). We follow the procedure adopted in deriving the potential V in eq (5.14), that is, we take as the equation of circle No. 1 of figure 37 the equation $\beta=\beta_1$, where $\beta_1<0$. In the end result we change the sign of β_1 to make all the constants c,β_1, and β_2 positive, as given in the three eq (1.13).

By (5.4) the surface element dS for integrating over a circular area bound by the circle $\beta=$ constant is

$$dS=\frac{c^2\,d\alpha\,d\beta}{(\cosh\beta-\cos\alpha)^2},$$

where $d\alpha$, $d\beta>0$. Now $\sinh\beta_1=-c/a_1$ and $\sinh\beta_2=c/a_2$. Hence after an expression for $U(\alpha,\beta)$ is found, eq (5.62) becomes

$$L/\text{cm}=-\left(\frac{\mu_1+\mu_2}{2}\right)+\frac{2\sinh^2\beta_1}{\pi}\int_{-\infty}^{\beta_1<0}d\beta\int_0^{\pi}\frac{U(\alpha,\beta)\,d\alpha}{(\cosh\beta-\cos\alpha)^2}$$

$$-\frac{2\sinh^2\beta_2}{\pi}\int_{\beta_2>0}^{+\infty}d\beta\int_0^{\pi}\frac{U(\alpha,\beta)\,d\alpha}{(\cosh\beta-\cos\alpha)^2}. \tag{5.63}$$

$U(\alpha,\beta)$ will be found as a series in $\cos n\alpha$, so the following integrals will be required.

$$\frac{1}{\pi}\int_0^{\pi}\frac{\cos n\alpha\,d\alpha}{(\cosh x-\cos\alpha)^2}=\frac{e^{-nx}(n+\coth x)}{\sinh^2 x}=4e^{nx}\sum_{s=1}^{\infty}s(s+n)e^{-2(s+n)x} \tag{5.64}$$

if $0<x$.

From this we find, when $0<\beta$,

$$2\sinh^2\beta\int_{\beta}^{+\infty}\frac{e^{-2nx}(n+\coth x)}{\sinh^2 x}dx=e^{-2n\beta}. \tag{5.65}$$

The function $U(\alpha,\beta)$ that satisfies the four boundary conditions in (5.60) and (5.61) is

In wire No. 1, where $-\infty<\beta\leqslant\beta_1\leqslant 0$:

$$U(\alpha,\beta)=\mu_1-C_0-2\beta_1+\sum_{n=1}^{\infty} A_n e^{n(\beta-\beta_1)}\cos n\alpha. \tag{5.66}$$

In wire No. 2, where $0<\beta_2<\beta\leqslant+\infty$:

$$U(\alpha,\beta)=-\mu_2-C_0-2\beta_2+\sum_{n=1}^{\infty} B_n e^{-n(\beta-\beta_2)}\cos n\alpha. \tag{5.67}$$

In the air between them, where $\beta_1<\beta<\beta_2$:

$$U(\alpha,\beta)=-C_0-2\beta+\sum_{n=1}^{\infty}\left[\frac{A_n\sinh n(\beta_2-\beta)+B_n\sinh n(\beta-\beta_1)}{\sinh n(\beta_2-\beta_1)}\right]\cos n\alpha, \tag{5.68}$$

where

$$C_0=\sum_{n=1}^{\infty}\frac{A_n\sinh n\beta_2-B_n\sinh n\beta_1}{\sinh n(\beta_2-\beta_1)}. \tag{5.69}$$

This makes $U=A$ vanish at spatial infinity ($\alpha=\beta=0$).

The boundary conditions require

$$A_n e^{n\beta_1}=\frac{2(1+\epsilon_1)}{n(1-\epsilon_1\epsilon_2 e^{-n\gamma})}[(1+\epsilon_2 e^{-n\gamma})e^{2n\beta_1}-(1+\epsilon_2)e^{-n\gamma}] \tag{5.70}$$

$$B_n e^{-n\beta_1}=\frac{-2(1+\epsilon_2)}{n(1-\epsilon_1\epsilon_2 e^{-n\gamma})}[(1+\epsilon_1 e^{-n\gamma})e^{-2n\beta_2}-(1+e^{-n\gamma})], \tag{5.71}$$

where

$$\epsilon_1=\frac{\mu_1-1}{\mu_1+1},\ \epsilon_2=\frac{\mu_2-1}{\mu_2+1}\ \text{and}\ \gamma=2(\beta_2-\beta_1). \tag{5.72}$$

Performing the integrations in (5.63) by use of (5.68) and (5.69) gives

$$\left.\begin{aligned}
L/\text{cm}&=+\frac{\mu_1+\mu_2}{2}+2(\beta_2-\beta_1)+\sum_{1}^{\infty}(A_n e^{n\beta_1}-B_n e^{-n\beta_2}\\
&=\frac{\mu_1+\mu_2}{2}+2(\beta_2-\beta_1)\\
&+2\sum_{n=1}^{\infty}\frac{1}{n(1-\epsilon_1\epsilon_2 e^{-n\gamma})}[(1+\epsilon_1)(1+\epsilon_2 e^{-n\gamma})e^{2n\beta_1}\\
&\qquad+(1+\epsilon_2)(1+\epsilon_1 e^{-n\gamma})e^{-2n\beta_2}\\
&\qquad-2(1+\epsilon_1)(1+\epsilon_2)e^{-n\gamma}]
\end{aligned}\right\} \tag{5.73}$$

To obtain positive constants for computing we next reverse the sign of β_1, so that $\gamma=2(\beta_1+\beta_2)$, as in eq (1.13), where β_1,β_2 and c are all positive. After this change we find that when $\mu_1=\mu_2=1$ the formula reduces to the known correct expression, say L_0, that is given in (2.44), where

$$L_0-1=2\log\frac{b^2}{a_1a_2}=2(\beta_1+\beta_2)+2\sum_{n=1}^{\infty}\frac{1}{n}(e^{-2n\beta_1}+e^{-2n\beta_2}-2e^{-n\gamma}). \tag{5.74}$$

Subtracting this from the expression for L (with positive β_1) gives the eq (2.45).

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