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CALIBRATION OF A DIVIDED SCALE

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CALIBRATION OF A DIVIDED SCALE [1]

ABSTRACT

The calibration of a divided scale may, in most cases, be considered as, first, making a series of observations with a suitable length comparator, and then carrying out certain arithmetical processes which are indicated on a prepared computation form. A method for calibrating a scale in this manner is given in detail in this paper and references made to other methods which have previously been published.

CONTENTS

I. CALIBRATION OF A DIVIDED SCALE

Although the calibration of a divided scale is not by any means a difficult process, it is undoubtedly true that reliance is too often placed on the reputation of the maker of a scale, whereas a calibration should be made. This situation is due to several causes: The scarcity of information in the English language on this subject, the inaccessibility of the general literature in other languages, and a lack of sufficient definite information as to how to go about the making of a calibration.

This paper is written with a specific purpose in mind—namely, the brief description of a method for the calibration of a divided line standard or graduated scale. Reference is also made to more complete information. By limiting the scope to this specific purpose a treatment of the subject is obtained which is somewhat less complicated and also more consistent with the various conventions and methods usually employed in length measurements than is possible when dealing also with the determination of the progressive errors of a micrometer screw, the errors of graduation of a thermometer, and possibly even with the correction to a set of weights.

[1] By Lewis V. Judson, Associate Physicist.

Suppose, for example, that one desires to determine the corrections to the intervals 0 to 914 mm and 0 to 915 mm of a meter bar in order to standardize a yard bar. Since the legal ratio of the United States yard to the meter is $\frac{3600}{3937}$, the length of a yard bar at standard temperature, as determined from this ratio, should be 914.402 mm (approximately). The procedure for calibrating the meter bar in the case above cited would be first to calibrate the 10 dm of the meter bar, then to repeat this process with the centimeters of the tenth decimeter and lastly to do the same thing with the millimeters of the interval 910 mm to 920 mm. With the total length of the meter known, the corrections to the intervals 0 to 914 mm and 0 to 915 mm are thus readily obtained.

Several procedures are possible for the calibration of a length standard. The one to employ in any given case will depend upon the accuracy required. The simplest method consists in comparing with each other a group of intervals of the same nominal length; for example, in comparing with each other the 10 dm of a meter bar. This might give all the accuracy required for some work.

In the use of the method described in the preceding paragraph there is one fundamental weakness—namely, any error made in the measurement of one of the intervals affects them all. Hence, for precise work some systematic scheme must be employed wherein a group of observations are made involving not only the single intervals—the decimeters, for instance—but also the double intervals, the triple intervals, etc. These observations are reduced by the method of least squares. This simply means that, being given a larger number of observations than there are intervals whose length is to be determined, a special computation makes it possible to find the most probable values of the intervals.

While an understanding of the development of the mathematical theory of least squares reductions will greatly aid one in appreciating them and in applying them to those special and peculiar cases which are sure to arise at some time or other if any large amount of this work is undertaken, a lack of training in the theory of least squares need deter no one from undertaking the calibration of nearly any scale, if he has or can obtain the equipment and experience needed to make the length measurements required. The computational work can be regarded as simply the mechanically filling in of certain blank spaces on a prepared form with numerical data and the performance of the simple arithmetical processes indicated. A person doing much of the work in this routine way, however, would probably soon find it desirable to learn more of the theory involved, as some of the theory would be required in handling the more complicated problems which arise.

II. METHODS OF OBTAINING THE OBSERVATIONAL DATA

In the calibration of a bar by the method originally described by Hansen one must first decide as to the best way of dividing or grouping the intervals. For instance, if a 36-inch bar is to be calibrated in order to determine the corrections to each inch, it could be accomplished by dividing the intervals into 4 groups of 9 inches each, 9 groups of 4 inches each, or 6 groups of 6 inches each. This last grouping has been found by experience to be the most satisfactory. In general, the groups should be so chosen that there are as nearly as possible the same number of subintervals in each group as there are groups. A meter bar, for example, would be calibrated by continued decimal division.

For a calibration of this kind some sort of comparator is needed having a longitudinal motion; that is, a motion in the direction of the longitudinal axis of the scale being calibrated. Two types of comparators are possible—one with movable standards, the other with movable microscopes. To calibrate a meter bar one of the two microscopes should be set on the zero of the scale and the other on the 1 dm mark, the distance between the microscopes being then approximately a decimeter. Observations are made with the two microscopes. Either the carriage on which the scale rests or the carriage to which the microscopes are attached is moved, depending upon the type of comparator used, until the 1 dm mark is under the first microscope and the 2 dm mark is under the other. The individual microscopes must not be moved while comparisons are under way, as this would disturb their spacing. This method should be continued throughout the entire meter, and then a similar set of observations made moving in the opposite direction. Four readings on each microscope in each position are usually satisfactory and sufficient.

The microscopes are next set on the zero and the 2 dm lines and observations made as before on this and on each 2 dm interval, moving the carriage 1 dm each time. This method being continued throughout the entire meter, increasing the space between the two microscopes by a decimeter between the successive groups of observations, there are obtained the observed values necessary for the computation of the decimeter intervals.

The subdivision of the decimeter intervals into centimeters requires a somewhat different method of calibration because of the fact that the microscopes can not be placed sufficiently close together to allow the short spacing necessary in the above method. It is found best, therefore, to use one of the decimeter intervals at one end of the bar against which the others may be compared. Considering the first decimeter interval as the one to be calibrated, the left microscope is set on zero and the right on the 90 cm line. After the observations have been recorded the carriage is moved a centimeter, and so on.

The first centimeter is thus compared with the centimeter between the 90 and 91 cm lines, the first double centimeter with the double centimeter between the 90 and 92 cm lines, etc. The relative position of the scale and the microscope is then changed so that the second centimeter can be compared with the interval 90 to 91 cm.

In the case of very short bars the above method is obviously not possible as it stands. The procedure must, therefore, be modified by using another bar as the comparison standard against which to compare the intervals being tested. In such a case care must be taken to maintain a constant relative position between the two bars. Sometimes a rearrangement of the comparisons may be necessary so that all of the unit intervals are compared first, then the double ones together, and so on.

III. COMPUTATIONS

Using the mean of the length values found for each interval in moving in the two directions, the relative lengths of the several intervals are determined. In order to compute the observed relative values of the several intervals from the microscope readings one must first determine to which of the three following classes the original observations belong and then follow the appropriate rule. These rules apply to any type of comparator with a longitudinal motion, and incidentally to practically any transverse comparator, it being assumed that micrometer microscopes of the usual type are used which indicate increasing readings as the cross hairs are moved from left to right across the field. In the first case one of the two microscopes is sighted at each end of the interval being measured; the bar may have its zero either toward the left or toward the right as the observer faces the machine. This is the case which would ordinarily occur in the calibration of a meter bar to decimeters, provided the two microscopes could be placed with an axial distance of a decimeter. In the second case the bar being tested is mounted on the comparator with its zero toward the observer's left and one microscope is sighted at points along the scale where corrections are desired and the second microscope is sighted at points along an auxiliary scale which may or not be another portion of the bar which is being calibrated. In the third case the bar being tested is mounted with the zero toward the observer's right, the other conditions being the same as in the second case.

For case one, if L_m is the mean of all of the left micrometer readings of a series of readings at the successive positions required for a constant distance between the microscopes and R_m is the mean right micrometer reading under the same conditions and if M_L and M_R are the screw values of the left and right microscopes, respectively, and if L_s and R_s are any pair of simultaneous readings, the observed relative length of the interval sighted on at the time the sth readings were taken is

$$(L_s - L_m)\ M_L + (R_m - R_s)\ M_R$$

It should be noted that L_m is the mean of all of the readings of the left micrometer of a series with constant separation of the microscope and not simply the mean of the observations on one point.

In the second case the rule is to subtract in succession each observation made on lines of the intervals being standardized from the observation on the point nearest the zero in the calibration being made and multiply these differences by the screw value of the micrometer used; then, for the readings made with the other microscope to subtract in succession the readings taken at that position at which this microscope is nearest the zero of the scale being tested, from each of the other readings and multiply these differences by the screw value; and, finally, to add corresponding pairs of values. Care must be taken to use the algebraic sign as indicated. Putting this in symbolical form let S_0, S_1, S_2 . . . . be the readings of the microscope at points being calibrated and M_S the screw value of this microscope and let the corresponding observations on the auxiliary scale (A) with the second microscope whose screw value is M_A be A_0, A_1, A_2, respectively, then the interval 0_0 to 2 on the S scale has an observed relative correction expressed as

$$(S_0 - S_2)\ M_s + (A_2 - A_0)\ M_A$$

Case three, which is similar to case two except that the zero of the scale being calibrated is toward the observer's right, has a rule for computation which is exactly the same as for case two except that the differences are taken in the reverse order; that is, the signs of the differences are reversed. The mathematical expression is

$$(S_2 - S_o)\ M_s + (A_0 - A_2)\ M_A$$

The following examples illustrate the method of computation:

Case I.—*Calibration of a meter bar to decimeters: Comparison of 6 dm intervals.*

Arrangement of scale |—|—|—|—|—|—|—|—|—|—|
0 1 2 3 4 5 6 7 8 9 10

Interval		Observation, revolutions	Difference M_L=100.25 μ		Observation, revolutions	Difference M_R=100.12 μ		Correction, microns
L Microscope	R Microscope		Revolutions	Microns		Revolutions	Microns	
0	6	L_0=15.785	−0.003	−0.3	R_6=18.742	+0.006	+0.6	+0.3
1	7	L_1=15.790	+.002	+.2	R_7=18.753	−.005	−.5	−.3
2	8	L_2=15.796	+.008	+.8	R_8=18.749	−.001	−.1	+.7
3	9	L_3=15.783	−.005	−.5	R_9=18.746	+.002	+.2	−.3
4	10	L_4=15.785	−.003	−.3	R_{10}=18.752	−.004	−.4	−.7
Mean		L_m=15.788			R_m=18.748			

CASE II.—*Calibration of interval 10 to 20 cm on a meter bar by comparison with interval 80 to 90 cm on the same bar.* (*Portion of set only.*)

Arrangement of scale 0 ..... 10 11 12 .... 80 81 82

Interval		Observation, revolutions	Difference $M_S = 46.25\ \mu$		Observation, revolutions	Difference $M_A = 48.50\ \mu$		Correction, microns
L Microscope	R Microscope		Revolutions	Microns		Revolutions	Microns	
10	80	$S_0=20.567$			$A_{80}=19.349$			
11	81	$S_1=20.578$	−0.011	−0.51	$A_{81}=19.354$	+0.006	+0.29	−0.21
12	82	$S_2=20.615$	−.048	−2.22	$A_{82}=19.388$	+.039	+1.81	−41

Case III. Same as Case II, but with the zero of the scale at the right.

With the observations as given in Case II, the corrections would here be $+0.51-0.29=+0.21\ \mu$ and $+2.22-1.81=+0.41\ \mu$.

The mean values thus computed are to be inserted in the lozenge-shaped spaces on the first sheet of the computation forms (fig. 1), as indicated. The q's are obtained by adding in one diagonal direction and the p's in the other. The s's are the sums of the vertical columns. The values of $t_0, t_1 \ldots . . t_n$, $\triangle_0 \ldots . \triangle_n$, x_2 and x_n are obtained as indicated and the work checked by the summations provided. Although the checks indicated are in most cases ample, attention is called to the fact that these do not guard against the possibility of the insertion of one or more incorrect values on the sheet in the first place. Two or three rather curious cases have also been noted of compensating errors not indicated by these checks.

Inserting on the second sheet the values thus computed and adding algebraically, the values of x_3 to x_{n-1} are obtained. The values x_2 to x_n in the case of the calibration of the third decimeter of a meter bar to tenths become x_2 to x_{10} and are the relative corrections to the intervals 20 to 21 cm, 20 to 22 cm, 20 to 23 cm, etc., on the basis of the correction to the interval 20 to 30 being zero. This will be true at some temperature. The actual correction to the intervals 20 to 30 cm at a stated temperature may be proportioned among the constituent intervals.

The computations on sheets one and two are merely a highly systematized mode of computing the most probable corrections to the scale intervals by the use of the method of least squares. The calculation of the probable error on sheet three and the distribution of the total length correction on sheet four are straightforward processes of computation.

Attention is called here to the fact that certain of the signs in the computations forms are not the same as those given in the International Bureau of Weights and Measures publications. The change in sign has been made so that the usual notation of sign used in other precision length work may be retained here. In this way the greater plus correction is to be assigned to the longer interval.

When two similar bars are to be calibrated for the same intervals, a simultaneous calibration, often called a cross calibration, is usually preferable. At the Bureau of Standards the method of double calibration of Dr. C. E. Guillaume has been used in several recent calibrations of this type.

IV. PRECAUTIONS NECESSARY IN PRECISION LENGTH OBSERVATIONS

Any least squares calibration implies precision; the application of the theory of the least squares does not correct errors of observation or mistakes caused by carelessness. It is simply a means of making the best possible use of the observation available. Some considerations of the precautions necessary in precise length observations are, therefore, desirable in connection with this subject of the calibration of linear scales.

In the first place a good focus is necessary; this may be tested by seeing that there is no parallax (relative movement of the observed line and the cross hair as the eye is moved sideways). There should be an even, symmetrical illumination produced by a vertical illuminator. Furthermore, the temperature should be fairly constant. For the most precise results when two standards are employed the longitudinal axes of these two standards should be colinear. The latter condition, however, is often not possible, and all that can be done is to have them parallel and fairly close together.

For the highest precision it is sometimes found necessary to compare directly each interval with the standard interval before going to another length of interval. This has already been mentioned.

V. LEAST SQUARES THEORY OF CALIBRATION OF DIVIDED SCALES

In a memoire published in 1874 Hansen gave a method for calibrating a divided scale. His method has been elaborated upon by various authors and rendered more convenient for numerical computation. These modifications of Hansen's method are listed in the bibliography which is a part of the present paper. Hansen's memoire contains the basic mathematical theory involved in the practical procedure already outlined.

Since the numerical values observed as described above are the constants in a set of observed equations, these equations may be written thus:

$$\begin{array}{ll} x_2 \qquad -\lambda_1 = {}_1a_2 & x_3 \qquad -\lambda_2 = {}_1a_3 \\ x_3 - x_2 - \lambda_1 = {}_2a_3 & x_4 - x_2 - \lambda_2 = {}_2a_4 \\ x_4 - x_3 - \lambda_1 = {}_3a_4 & x_5 - x_3 - \lambda_2 = {}_3a_5 \end{array}$$

where

x_2 = correction to the first interval of the scale,

x_3 = correction to the first two intervals of the scale,

x_4 = correction to the first three intervals of the scale, etc.,

and

λ_1 = correction to the arbitrary single-unit interval,

λ_2 = correction to the arbitrary double-unit interval, etc.

Normal equations are formed from these observational equations according to the theory of least squares and these are solved for the x's and the y's. Fortunately the symmetry of the equations is such that they can be solved by a more or less mechanical following out of a prearranged schedule of arithmetical processes as has already been stated.

The normal equations are

$$
\begin{aligned}
&x_{n+1} - n\lambda_1 = s_1 \\
-x_2 \quad &+ x_n + x_{n+1} - (n-1)\lambda_2 = s_2 \\
-x_2 - x_3 \ldots &+ x_{n-1} + x_n + x_{n+1} - (n-2)\lambda_3 = s_3 \\
&* \quad * \quad * \quad * \quad * \quad * \quad * \\
-x_2 \quad &+ x_n + x_{n-1} - 2\lambda_{n-1} = s_{n-1} \\
&x_{n+1} - \lambda_n = s_n
\end{aligned}
$$

and

$$
\begin{aligned}
&x_2 + x_3 + x_4 + x_5 \ldots + x_m \ldots + x_n + x_{n+1} - \lambda_1 - \lambda_2 - \\
&\quad \lambda_3 \ldots - \lambda_{n-2} - \lambda_{n-1} - \lambda_n = t_0 \\
-&nx_2 + x_3 + x_4 + x_5 \ldots + x_m \ldots + x_n + x_{n+1} \quad - \lambda_2 - \\
&\quad \lambda_3 \ldots - \lambda_{n-2} - \lambda_{n-1} = t_1 \\
&x_2 - nx_3 + x_4 + x_5 \ldots + x_m \ldots + x_n + x_{n+1} \quad - \\
&\quad \lambda_3 \ldots - \lambda_{n-2} = t_3 \\
&* \quad * \quad * \quad * \quad * \quad * \quad * \\
&x_2 + x_3 + x_4 + x_5 \ldots + x_m \ldots - nx_n + x_{n+1} \quad + \lambda_2 + \\
&\quad \lambda_3 \ldots + \lambda_{n-2} + \lambda_{n-1} = t_{n-1} \\
&x_2 + x_3 + x_4 + x_5 \ldots + x_m \ldots + x_n - nx_{n+1} + \lambda_1 + \lambda_2 + \\
&\quad \lambda_3 \ldots + \lambda_{n-2} + \lambda_{n-1} + \lambda_n = t_n
\end{aligned}
$$

These equations may be combined to form the following set of equations:

$$
\begin{aligned}
\triangle_0 &= (n+1)x_2 - \lambda_1 \\
\triangle_1 &= (n+1)x_2 + (n+1)x_3 - \lambda_2 - \lambda_{n-1} \\
&* \quad * \quad * \quad * \quad * \quad * \quad * \\
\triangle_{n-2} &= -(n+1)x_{n-1} + (n+1)x_n - \lambda_{n-1} - \lambda_2 \\
\triangle_{n-1} &= -(n+1)x_n + (n+1)x_{n+1} - \lambda_1
\end{aligned}
$$

and the required solutions obtained.

Since there are $(n-1)$ $(n-2)$ equations of condition, the probable error, r_x of any x, is given by the following equation where K is $\frac{(n-1)\ (n-2)}{2}$:

$$r_x=\sqrt{\frac{\Sigma v^2}{K}}$$

VI. BIBLIOGRAPHY

The following bibliography is given as a guide to one wishing further knowledge regarding the calibration of scales. In general, the more recent publications give methods better adapted to numerical computation and also much more suited to the modern requirements of precision. Certain articles regarding the calibration of thermometers are included in this bibliography; they are here not on account of any suitability for thermometer calibration, but solely because of the possible application to the calibration of length standards. This bibliography is by no means complete, as there are many references to textbooks on experimental physics and to journal articles in which some reference is made to scale calibration. The Bureau of Standards will be pleased to be of assistance at any time to anyone desiring further information as to the literature on this subject.

Benoit, J. René, Mesures de dilatation et comparaisons de règles métrique. 13 Thermomètres—Corrections de calibrage. Travaux et Mémoires du Bureau International des Poids et Mesures, **2**, pp. C35–C48; 1883. Method of Marek as used in the calibration of thermometers. This article is in French; Marek's original article was in German.

Bessel, Friedrich Wilhelm, Methode die Thermometer zu berichtigen. Pogg. Annalen der Physik, **6**, pp. 287–308; 1826. A method of calibration in which the various lengths used are not multiples of the smallest or unit interval.

Broch, Ole Jacob, Note sur l'étalonnage des sous-divisions d'une règle, sur l'étude des erreurs progressives d'une vis micrométrique, et sur le calibrage des thermomètres. Travaux et Mémoires du Bureau International des Poids et Mesures, **5**, pp. 1–82; 1886. Development of the method of Hansen for the calibration of a single scale. Computation forms. Applications to calibration of line standards, micrometer screws, and thermometers.

Brown, Charles C., Calibration of thermometer Green 5280 by Hansen's method. Van Nostrand's Eng. Mag., **29**, pp. 1–7; 1883. One of the few expositions in the English language of Hansen's method.

Dziobek, Über die Ermittelung der inneren Theilungsfehler zweier Massstäbe nach der Methode des Durchschiebens. Wissenschaftliche Abhandlungen der Kaisarlichen Normal-Aichungs-Kommission, **4**, pp. 1–56; 1903. A least squares solution for a double calibration with comparisons with the method of Thiesen; numerical examples.

Egen, P. N. C., Untersuchungen über das Thermometer: V. Die stereometrischen Verhältnisse der Glasröhre. Pogg. Annalen der Physik und Chemie, **11**, pp. 529–536; 1827. A method of calibration which is a slight modification of that of Bessel. Allusions are sometimes found in the literature to Egen's method without any journal reference being given.

Gay-Lussac, ——, ——. Gay-Lussac is credited by very many writers on this subject with having given the first method of calibration. Some have, however, given evidence to prove that he did not originate the method. An outline of the method of Gay-Lussac is given in many treatises on experimental physics and in books on thermometry but without reference to Gay-Lussac's description of his method.

Gill, Sir David, On the determination of errors of graduation without cumulative errors and the application of the method to the scales of the Cape heliometer. Monthly notices of the Royal Astronom. Soc., **49**, pp. 105–118; 1889. (See also the errata on p. 241 of the same volume.) A method is outlined of calibrating two scales simultaneously. This method is essentially that of Lorentzen. No general formulas are given and no computation system outlined. It is to be noted that Gill considered all intervals to be calibrated with the same weight, whereas, as Jacoby pointed out, weight factors given by Lorentzen must be used.

Guillaume, Charles Edouard, Traité de la thermomètrie de precision, pp. 43–99; 1889. Methods of calibration of thermometers with illustrative examples: Gay-Lussac's method; Neumann-Thiesen method; Complete least squares method.

Guillaume, Charles Edouard, L'etalonnage des echelles divisees. Travaux et Mémoires du Bureau International des Poids et Mesures, **13**, pp. 1–54; 1907. Both a method of single and one of double calibration are given, each with examples.

Hallström, Gustav Gabriel, Anmërkningar angående thermometrars fȯrtfårdigande och Bruk. Åbo, 1823, Akad. Dissert. A very early method of calibration, concerning which there seems to be considerable difference of opinion, although the Hallström method is often referred to in the literature. In a footnote to an article by Hallström in the Annalen der Physik und Chemie in in 1836, Poggendorf states that, as the method is the same as that already published by Bessel, to save space it is not repeated, although included in Hallström's manuscript of the article. Rudberg later pointed out some slight differences. The committee of the British Association (see under Stewart) put the method in an entirely different classification from that of Bessel. The original dissertation unfortunately has not been available for consultation in the preparation of this bibliography.

Hansen, P. A., Bestimmung der Theilungsfehler eines gradlinigen Maassstabes. Abhandlungen der Mathematisch-Physikalischen Classe der Königlich Sächsischen Gesellschaft der Wissenschaften, **15**, pp. 525–667; 1879. The basic memoir on which present-day methods of calibration of scales are founded. It discusses the calibration of a single scale. Methods of double calibration are a later development.

Hennert, Traité de Thermomètre, p. 184; 1758. Chwolson in his Traité de Physique, **3**, p. 42, states that the method generally ascribed to Gay-Lussac was first given by Hennert.

Jacoby, Harold, Note on the division errors of a standard scale. Astronomische Nachrichten, **137** (No. 3285), pp. 357–360; 1895. Jacoby points out the advance made by Lorentzen over Hansen and also the practical identity of Lorentzen's method with that used and described by Sir David Gill. The fact that Lorentzen gives a weight factor for the several intervals is emphasized. Jacoby suggests repeating the observations on the first and last comparisons of the two scales in the usual order of measurement four times. The final results would then be more uniformly weighed. The possibility of computing the number of times to repeat each separate observation in order to produce a weight of unity for all points is suggested by Jacoby as of some possible theoretical importance.

Jacoby, Harold, On the determination of the division of a straight scale, Am. J. of Sci., **151**, pp. 333–347; 1896. A method of calibration of two scales in which the various observations are repeated such number of times as to secure a unit weight for the final result for the correction to each interval. An application to the calibration of screws is included.

Leman, A., Über die gleichseitige Bestimmung der Theilungsfehler zweier Maszstabe durch die Methode des Durchschiebens. Wiss. Abh. der k. Normal-Aichungs-Kommission, **6**, pp. 1–75; 1906. A method of double calibration.

Lorentzen, G., Über die Untersuchung der Scalen eines Heliometers I. Astron. Nachr., **131** (No. 3134), pp. 217–238; 1893. This is a method of double calibration which differs only slightly from a least squares solution.

Lorentzen, G., Über die Untersuchung der Scalen eines Heliometers II. Astronomische Nachrichten, **135** (No. 3236), pp. 353–366; 1894. This is an application of the method of double calibration described by the same author in a previous paper to the calibration of two heliometer scales, each a decimeter in length.

Marek, W. J., Über die Anwendung der Methode der kleinsten Quadrate auf die Kalibrirung der Thermometer. Carl's Repertorium für Experimental-Physik, **15**, pp. 300–319; 1879. Method of calibration of a thermometer (applicable also to a scale) for cases of division into 2, 3, 4, 5, and 6 parts with a brief statement of the general case. Corrections to the intervals are obtained by the solution of the normal equations obtained by least squares methods. These equations are given a form especially applicable for ready numerical solution. Numerical examples exemplifying the method in the calibration of thermometers are given, together with formulas for obtaining the probable error of the results. See also reference to article by Benoit, above.

Pérard, Albert, Note sur le calcul des etalonnages de grandeurs en progression arithmetrique. Travaux et Mémoires du Bureau International des Poids et Mesures, **16**, pp. 1–77; 1917. A method of single calibration and a method of double calibration. The method given in this circular is a variation of the method of single calibration described by Pérard.

Rogers, William Augustus, The Cumulative Errors of a Graduated Scale. Proc. Am. Soc. of Mech. Eng., **15**, pp. 127–146; 1893. The method of Broch and the method of summing the corrections found for the unit lengths are each discussed and illustrated. Rather undue emphasis is placed on the possibilities of simply summing the corrections found for the individual intervals; Mr. Rogers was an experienced and careful observer; repetition of length measurements to a certain precision is no proof of their accuracy to that degree.

Rudberg, F., Über die Construction des Thermometers. Pogg. Annalen der Physik, **40**, pp. 562–582; 1837. A method of repeated subdivision; for instance, first divide the total length in half, then compare the three-thirds of the total length, then divide the total into twelfths, etc.

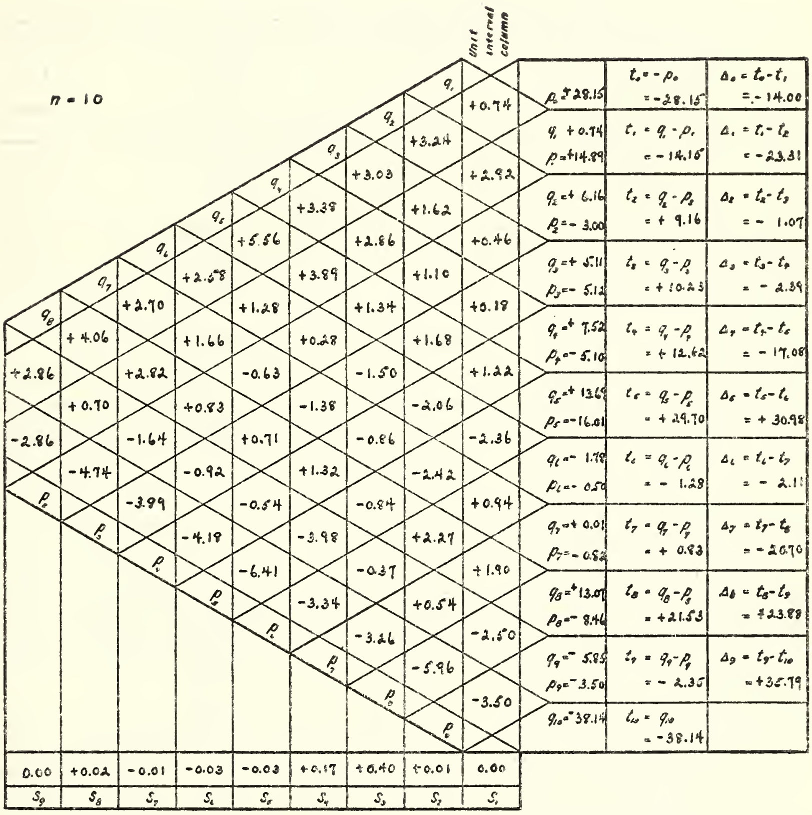
Stewart, Balfour; Rücker; Thorpe, T. E., Report of the committee appointed for the purpose of reporting on the methods employed in the calibration of mercurial thermometers. Report of the British Assoc. for 1882, pp. 145–204. Discussion with illustrative examples using the following methods: Gay-Lussac, Hallström, Rudberg, Thiesen, Marek, Bessel (as modified by von Ottingen and improved by Thorpe and Rücker). These methods are also applicable, with certain modifications to the calibration of divided scales. Four classes of methods are distinguished by this committee. (1) "Step by step," example, Gay-Lussac's method; (2) "Principal points," examples, Hallström, Thiesen, Marek; (3) "Repeated division," example, Rudberg; (4) "Distributed points," examples, Bessel, and the modification by von Ottingen.

Thiesen, Max, Über das Kalibriren von Thermometern. Carl's Repertorium für Experimental-Physik, **15**, pp. 285–299; 1879. The author points out that Lambert in his "Pyrometrie" published in 1779 had used (pp. 31, 43) the method of calibration commonly attributed to Gay-Lussac, and that it was known even earlier. An application to thermometry of this method is briefly outlined by Thiesen. It is stated that Rudberg's method is only a special case of Hallström's, but that it has the disadvantage that the corrections to the divisions obtained in the component parts of the calibration are not well coordinated to make a unified system of corrections.

Thiesen, Max, Über das Kalibriren von Thermometern, inbesondere über die wahrscheinlichen Fehler der Kalibercorrection. Carl's Repertorium für Experimental-Physik, **15**, pp. 677–681; 1879. By a consideration of the probable errors, the feasibility of an abbreviated method of calibration is shown. There is a saving of time compared with the rigorous least squares method proposed by Marek. The artifice used by Thiesen had been proposed by F. E. Neumann in the case of another set of equations similar to those involved in this work, and the method of calibration is sometimes known as the Neumann-Thiesen method.

Weinstein Handbuch der Physikalischen Maasbestimmung, **2**, pp. 274–282; 1888. Method of Hansen for single scales and method of Thiesen for a double calibration.

WASHINGTON, February 24, 1927.



$\lambda = -\frac{s_1}{n} = 0.00$ $\qquad$ $+\frac{s_1}{n} = 0.00$

$-\Delta_0 = +14.00$ $\qquad$ $+\Delta_{n-1} = +35.79$

$-\frac{s_1}{n} - \Delta_0 = +14.00 = (n+1)X_2$ $\qquad$ $\frac{s_1}{n} + \Delta_{n-1} = +35.79 = (n+1)X_n$

$\therefore X_2 = +1.27$ $\qquad$ $\therefore X_n = +3.25$

Check

$\Sigma p = \Sigma q = \Sigma s$ ✓

$\Sigma t = 0$ ✓

$\Sigma \Delta = -p_0 - q_{10}$ ✓

FIG. 1.—*Calculation of the p's, q's, s's, t's, and Δ's*

		M_2	+ 1.98	M_3	+ 3.79	M_4	+ 3.32
x_n	+ 3.25	x_{n-1}	+ 5.31	x_{n-2}	+ 3.28	x_{n-3}	+ 2.97
$-x_2$	− 1.27	$-x_3$	− 3.50	$-x_4$	− 3.75	$-x_5$	− 4.09
M_2	+ 1.98	M_3	+ 3.79	M_4	+ 3.32	M_5	+ 2.20
$-s_2$	− 0.01	$-s_3$	− 0.40	$-s_4$	− 0.17	$-s_5$	+ 0.03
$(n-1)\lambda_2$	+ 1.97	$(n-2)\lambda_3$	+ 3.39	$(n-3)\lambda_4$	+ 3.15	$(n-4)\lambda_5$	+ 2.23
λ_2	+ 0.22	λ_3	+ 0.42	λ_4	+ 0.45	λ_5	+ 0.37
M_2	+ 1.98	M_3	+ 3.79	M_4	+ 3.32	M_5	+ 2.20
$-s_{n-1}$	0.00	$-s_{n-2}$	− 0.02	$-s_{n-3}$	+ 0.01	$-s_{n-4}$	+ 0.03
$2\lambda_{n-1}$	+ 1.98	$3\lambda_{n-2}$	+ 3.77	$4\lambda_{n-3}$	+ 3.33	$5\lambda_{n-4}$	+ 2.23
λ_{n-1}	+ 0.99	λ_{n-2}	+ 1.26	λ_{n-3}	+ 0.83	λ_{n-4}	+ 0.45
$\lambda_2+\lambda_{n-1}$	+ 1.21	$\lambda_3+\lambda_{n-2}$	+ 1.68	$\lambda_4+\lambda_{n-3}$	+ 1.28	$\lambda_5+\lambda_{n-4}$	+ 0.82
$-\Delta_1$	+ 23.31	$-\Delta_2$	+ 1.07	$-\Delta_3$	+ 2.39	$-\Delta_4$	+ 17.08
$(n+1)x_2$	+ 14.00	$(n+1)x_3$	+ 38.52	$(n+1)x_4$	+ 41.27	$(n+1)x_5$	+ 44.94
$(n+1)x_3$	+ 38.52	$(n+1)x_4$	+ 41.27	$(n+1)x_5$	+ 44.94	$(n+1)x_6$	+ 62.84
x_3	+ 3.50	x_4	+ 3.75	x_5	+ 4.09	x_6	+ 5.71
$-(\lambda_2+\lambda_{n-1})$	− 1.21	$-(\lambda_3+\lambda_{n-2})$	− 1.68	$-(\lambda_4+\lambda_{n-3})$	− 1.28	$-(\lambda_5+\lambda_{n-4})$	− 0.82
$+\Delta_{n-2}$	+ 23.88	$+\Delta_{n-3}$	− 20.70	$+\Delta_{n-4}$	− 2.11	$+\Delta_{n-5}$	+ 30.98
$(n+1)x_n$	+ 35.79	$(n+1)x_{n-1}$	+ 58.46	$(n+1)x_{n-2}$	+ 36.08	$(n+1)x_{n-3}$	+ 32.69
$(n+1)x_{n-1}$	+ 58.46	$(n+1)x_{n-2}$	+ 36.08	$(n+1)x_{n-3}$	+ 32.69	$(n+1)x_{n-4}$	+ 62.85
x_{n-1}	+ 5.31	x_{n-2}	+ 3.28	x_{n-3}	+ 2.97	x_{n-4}	+ 5.71

FIG. 2.—*Calculation of the corrections*

0.00	−0.22	−0.42	−0.45	−0.37	−0.45	−0.83	−1.26	−0.99
x_2-x_1 + 1.27	x_3-x_1 + 3.50	x_4-x_1 +3.75	x_5-x_1 + 4.09	x_6-x_1 +5.71	x_7-x_1 +2.97	x_8-x_1 +3.28	x_9-x_1 + 5.31	$x_{10}-x_1$ + 3.25
x_3-x_2 +2.23	x_4-x_2 +2.48	x_5-x_2 +2.82	x_6-x_2 + 4.44	x_7-x_2 + 1.70	x_8-x_2 +2.01	x_9-x_2 +4.04	$x_{10}-x_2$ + 1.98	$x_{11}-x_2$ − 1.27
x_4-x_3 + 0.25	x_5-x_3 +0.59	x_6-x_3 +2.21	x_7-x_3 − 0.53	x_8-x_3 − 0.22	x_9-x_3 + 1.81	$x_{10}-x_3$ −0.25	$x_{11}-x_3$ −3.60	
x_5-x_4 + 0.34	x_6-x_4 + 1.96	x_7-x_4 −0.78	x_8-x_4 −0.47	x_9-x_4 +1.56	$x_{10}-x_4$ −0.50	$x_{11}-x_4$ −3.75		
x_6-x_5 + 1.62	x_7-x_5 − 1.12	x_8-x_5 −0.81	x_9-x_5 +1.22	$x_{10}-x_5$ −0.84	$x_{11}-x_5$ −4.09			
x_7-x_6 −2.74	x_8-x_6 −2.43	x_9-x_6 −0.40	$x_{10}-x_6$ −2.46	$x_{11}-x_6$ −5.71				
x_8-x_7 + 0.31	x_9-x_7 +2.34	$x_{10}-x_7$ +0.28	$x_{11}-x_7$ −2.97					
x_9-x_8 + 2.03	$x_{10}-x_8$ −0.03	$x_{11}-x_8$ −3.28						
$x_{10}-x_9$ − 2.06	$x_{11}-x_9$ −5.31							
$x_{11}-x_{10}$ −3.25								

Obsvd.	Comptd.	v	v^2	Obsvd.	Comptd.	v		Obsvd.	Comptd.	v	v^2
+0.74	+1.27	−0.53	0.2809	+3.24	+ 3.28	−0.04	0.0016	+3.03	+3.33	−0.30	0.0900
+2.92	+2.23	+0.69	0.4761	+ 1.62	+ 2.26	−0.64	0.4096	+2.86	+2.40	+0.46	0.2116
+0.46	+0.25	+0.21	0.0441	+ 1.10	+ 0.37	+0.73	0.5329	+ 1.34	+1.79	−0.45	0.2025
+0.18	+0.34	−0.16	0.0256	+ 1.68	+ 1.74	−0.06	0.0036	−1.50	−1.20	−0.30	0.0900
+1.22	+1.62	−0.40	0.1600	− 2.06	− 1.34	−0.72	0.5184	−0.86	−1.23	+0.37	0.1369
−2.36	−2.74	+0.38	0.1444	−2.42	−2.65	+0.23	0.0529	−0.84	−0.82	−0.02	0.0004
+0.94	+0.31	+0.63	0.3969	+2.27	+2.12	+0.15	0.0225	−0.37	−0.14	−0.23	0.0529
+1.90	+2.03	−0.13	0.0169	+0.54	−0.25	+0.79	0.6241	−3.26	−3.70	+0.44	0.1936
−2.50	−2.06	−0.44	0.1936	−5.96	−5.53	−0.43	0.1849				
−3.50	−3.25	−0.25	0.0625					+3.38	+3.64	−0.26	0.0676
				+5.56	+5.34	+0.22	0.0484	+3.89	+3.99	−0.10	0.0100
+2.58	+2.52	+0.06	0.0036	+1.28	+1.33	−0.05	0.0025	+0.28	−0.98	+1.26	1.5876
+1.66	+1.56	+0.10	0.0100	−0.63	−0.59	−0.04	0.0016	−1.38	−0.92	−0.46	0.2116
+0.83	+1.36	−0.53	0.2809	+0.71	+1.19	−0.48	0.2304	+1.32	+0.77	+0.55	0.3025
−0.92	−0.95	+0.03	0.0009	−0.54	−1.21	+0.67	0.4489	−3.98	−2.91	−1.07	1.1449
−4.18	−4.54	+0.36	0.1296	−6.41	−6.08	−0.33	0.1089	−3.34	−3.42	+0.08	0.0064
+2.70	+2.45	+0.25	0.0625	+4.06	−4.05	+0.01	0.0001	+2.86	+2.26	+0.60	0.3600
+2.82	+3.21	−0.39	0.1521	+0.70	+0.72	−0.02	0.0004	−2.86	−2.26	−0.60	0.3600
−1.64	−1.08	−0.56	0.3136	−4.74	−4.76	+0.02	0.0004				
−3.89	−4.58	+0.69	0.4761								

v=Obsvd.−Comptd.

$\Sigma v^2 = 11.4509$

$r = .6745\sqrt{\frac{\Sigma v^2}{K}}$

$= .6745\sqrt{\frac{11.4509}{36}}$

$= .6745\sqrt{0.318081}$

$= .6745\,(0.564\ )$

$= 0.38\,\mu$

$r(x_2)=r(x_n) = 0.416 \cdot r = 0.16\,\mu$

$r(x_3)=r(x_{n-1}) = 0.401 \cdot r = 0.15$

$r(x_4)=r(x_{n-2}) = 0.387 \cdot r = 0.15$

$r(x_5)=r(x_{n-3}) = 0.376 \cdot r = 0.14$

$r(x_6) = 0.369 \cdot r = 0.14$

FACTORS

$n=$	10	9	8	7	6	5	4
K	36	28	21	15	10	6	3
$r(x_2)$	.416	.435	.456	.482	.509	.538	.592
$r(x_3)$	.401	.417	.436	.458	.483	.512	.539
$r(x_4)$	.387	.402	.419	.439	.463		
$r(x_5)$	.376	.391	.408				
$r(x_6)$	.369						

FIG. 3.—*Calculation of the residuals and probable errors*